Practice Test, Topology, Autumn 2013

Instructions: Answer two of the following three questions. Each question is worth 10 marks. This test will be marked, to provide you with feedback regarding how well you are understanding the material, but it will not counts towards your assessment for the course.

Question 1

(i) Carefully state the definition of the finite complement topology. [**2 marks**]

Solution: A set is open if it is either empty or if its complement is finite. (This definition was given in lecture.)

(ii) Prove that the finite complement topology is, in fact, a topology. [**4 marks**]

Solution: Given a set X, the finite complement topology is $\tau = \{U \subset X : X \setminus U \text{ if finite, or } U = \emptyset\}$. By definition, $\emptyset \in \tau$, and $X \in \tau$ because its complement is empty, hence finite. If U and V are open, $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$, which is finite because each individual complement is finite. Hence, the intersection is open. If $\{U_\alpha\}$ is a collection of open sets, the complement of the union $\cup_\alpha U_\alpha$ is the intersection of the individual complements: $\cap_\alpha (X \setminus U_\alpha)$. Since each of these is finite, the intersection is finite, so the union is open. (This is Tutorial Sheet 1, Question 1.)

(iii) Is $\mathbb{R}$, equipped with the finite complement topology, compact? Provide an argument to support your answer. [**4 marks**]

Solution: This set is compact. Let $\{U_\alpha\}$ be an open cover. Pick any single element of the cover, U_{α_0}. Its complement has finitely many elements, so there are only finitely many elements of $\mathbb{R}$, $x_1, \ldots, x_n$, that are not in this set. For each one, find a U_{α_i} containing x_i. Then $\{U_{\alpha_i}\}_{i=0}^n$ is a finite subcover. (This is essentially Tutorial Sheet 4, Question 3.)

Question 2

(i) Carefully state the definition of a closed set. [**2 marks**]

Solution: A set is closed if its complement is open. (This definition was given in lecture.)

(ii) Consider the set $A = \mathbb{Q}$, where $\mathbb{Q}$ denotes the rational numbers, as a subset of the real line with the usual topology. Is A closed? Provide an argument to support your answer. [**2 marks**]

Solution: No, this set is not closed. We showed in lecture that $\overline{\mathbb{Q}} = \mathbb{R}$. Thus, the set A does not contain all of its limit points, so it isn't closed. An example of a limit point that isn't in A is π. Any open ball containing π will also contain a rational number, because the rationals are dense. (This was discussed as an example in lecture, and is also in Tutorial Sheet 2, Question 2.)

(iii) Is the set $A = [0, 1]$, considered as a subset of $\mathbb{R}$ with the trivial topology, closed? Provide an argument to support your answer. [**2 marks**]

Solution: No. The only open sets in this topology are the empty set and all of $\mathbb{R}$. Since $A \setminus [0,1]$ is neither of those sets, the complement of A is not open, so A is not closed.

(iv) Consider the space $X = \mathbb{R}$, with the particular point topology for the particular point zero. What is the closure of $(-1,1)$, considered as a subset of X? Provide an argument to support your answer. [**4 marks**]

Solution: The closure of this set is the entire real line. This is because if $x \in \mathbb{R}$ and U is an open set containing x, then $0 \in U$. Hence, $U \cap [(-1,1) \setminus x] \neq \emptyset$. Thus, every point (except zero) is a limit point of this set.

Question 3

(i) Carefully state the definition of a continuous function. [**2 marks**]

Solution: Let $f : X \to Y$, where X and Y are topological spaces. f is continuous if for any set U that is open in Y the inverse image $f^{-1}(U)$ is open in X. (This definition was given in lecture.)

(ii) Consider the function $f : X \to Y$, where X is the real line with the discrete topology and Y is the real line with the usual topology, defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

Is f continuous? Provide an argument to support your answer. [**2 marks**]

Solution: Yes, f is continuous. If U is open in Y, then $f^{-1}(U)$ is open in X because any subset is open in the discrete topology. Note this proof doesn't rely on the definition of f – as discussed in lecture, any function whose domain has the discrete topology is continuous. (This example was discussed in lecture.)

(iii) Recall the fact that, if $f : X \to Y$ is continuous and $C \subset X$ is compact, then $f(C)$ is compact. Use this fact to prove that compactness is a topological invariant. [**2 marks**]

Solution: A topological invariant is a property that is necessarily shared between homeomorphic spaces. If X is compact and $f : X \to Y$ is a homeomorphism, then $f(X) = Y$ and the image of X must be compact by the given fact. Similarly, if Y is compact we can make the same argument using f^{-1}, which is continuous because f is a homeomorphism. Hence X is compact if and only if Y is compact. (This example was discussed in lecture.)

(iv) Suppose $f : X \to Y$ is one-to-one and continuous and Y is Hausdorff. Prove that X is Hausdorff. [**4 marks**]

Solution: Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Since f is one-to-one, $f(x_1) \neq f(x_2)$, and since Y is Hausdorff there exist two nonempty disjoint open sets, U_1, U_2, such that $f(x_1) \in U_1$ and $f(x_2) \in U_2$. Since f is continuous, $f^{-1}(U_1)$ is open, as is $f^{-1}(U_2)$. They're both nonempty, because they contain x_1 and x_2, respectively. We must show they're disjoint. Suppose $z \in f^{-1}(U_1) \cap f^{-1}(U_2)$. This implies $f(z) \in U_1$ and $f(z) \in U_2$, which can't happen because $U_1 \cap U_2 = \emptyset$. Hence, $f^{-1}(U_1) \cap f^{-1}(U_2) = \emptyset$, and so X is Hausdorff.