Solutions to Exercises on Topological Groups, Topology 2013

1. (4.3 #13) Let G_1 and G_2 be topological groups. Since they are both Hausdorff, $G_1 \times G_2$ is a Hausdorff topological space under the product topology. Also, $i : G_1 \times G_2 \to G_1 \times G_2$ is continuous because $p_{1,2} \circ i$ is continuous for each projection function. Similarly, $m : (G_1 \times G_2) \times (G_1 \times G_2) \to G_1 \times G_2$ because its compositions with the projection functions are continuous.

2. (4.3 #15) Suppose $m : G \times G \to G$ is continuous. Let C be closed. We must show that $i^{-1}(C)$ is closed. (By a previous tutorial exercise this will imply that i is continuous.) Since G is Hausdorff, this will follow if we can show that $i^{-1}(C)$ is compact. Let $\{V_\alpha\}$ be an open cover of $i^{-1}(C)$. Note that, since G is Hausdorff, the set $\{e\}$ is closed. It is closed because, if $g \in G \setminus e$, since $g \neq e$ there exists disjoint open sets U_g and V_e containing those points. But then $U_g \subset (G \setminus e)$, so $G \setminus e$ is open. Thus, $m^{-1}(e)$ is closed, and hence compact. Note that $G \times C$ is closed and compact, because it is the product of compact spaces, and so

   $$K := m^{-1}(e) \cap (G \times C) = \{(g^{-1}, g) : g^{-1} \in C\}$$

   is the intersection of two closed sets, so it is closed, and hence compact. Note that $\{G \times V_\alpha\}$ is an open cover of K, and so there is a finite subcover $\{G \times V_{\alpha_i}\}_{i=1}^n$. But then $\{V_{\alpha_i}\}_{i=1}^n$ is a finite subcover of $i^{-1}(C)$.

3. (4.3 #17) Since $m : G \times G \to G$ is continuous and $A \times B$ is compact, $m(A \times B) = AB$ is compact. (Continuous images of compact sets are compact.)

4. (4.3 #18) Since $e \in U$ and U is a neighborhood, there is an open set $\tilde{V}$ such that $e \in \tilde{V} \subset U$. This implies that $\tilde{V}^{-1} = i^{-1}(\tilde{V})$ is open. Also, $m^{-1}(\tilde{V})$ is open and contains (e, e), so $m^{-1}(\tilde{V}) \cap (\tilde{V} \times \tilde{V}^{-1})$ is open and nonempty. This implies it contains a basis element $(e, e) \in (V_1 \times V_2)$, and so we can set $V := i^{-1}(V_1) \cap i^{-1}(V_2) \cap V_1 \cap V_2$, which is open.