Tutorial Sheet 1, Topology 2013

1. Prove that the finite complement topology, which was defined in lecture, is indeed a topology.

   Solution: Just check this satisfies the definition of topology: 1) contains the empty set and the entire space; 2) contains the intersection of two sets in the topology; and 3) contains arbitrary unions of sets in the topology. For 1), note that $\emptyset \in \tau$ by definition. Also, the complement of X is $\emptyset$, which contains finitely many elements, so $X \in \tau$. For 2), suppose $U, V \in \tau$. One can check that $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$. (You should make sure you understand and can explain why this is true.) Since the latter two sets are finite, so is their union. Hence, $U \cap V \in \tau$. For 3), suppose $U_\alpha \in \tau$ for all α. One can check that $X \setminus (\cup_\alpha U_\alpha) = \cap_\alpha (X \setminus U_\alpha)$. Since each of the individual complements is finite, the intersection must be as well. Hence, $\cup_\alpha U_\alpha \in \tau$.

2. Let X be any set and $p \in X$ be some point in X. Define τ to be the collection of all subsets of X that do not contain p, plus X itself. Prove that τ is a topology on X. (It is called the "excluded point topology.")

   Solution: Similar to question 1, just check this satisfies the definition of topology: contains the empty set, entire space, unions, and finite complements.

3. Consider the following metrics on $\mathbb{R}^2$ (which are not the usual metric): $d_1(x, y) = \max_{i=1,2} |x_i - y_i|$ and $d_2(x, y) = |x_1 - y_1| + |x_2 - y_2|$. Draw a picture of what $B_\epsilon(x)$ looks like for each of these metrics.

   Solution: For the first metric, the ϵ-neighborhood at the origin is a square with sides length 2ϵ, not including its edges, which are parallel to the axes, and centered at the origin. For the second metric, we also get a square but it has been rotated so that its vertices now lie on the axes, at points $(\pm\epsilon, 0)$ and $(0, \pm\epsilon)$.

4. Prove that the metric topology is indeed a topology.

   Solution: Similar to question 1, just check this satisfies the definition of topology: contains the empty set, entire space, unions, and finite complements.

5. Let (X, d) be a metric space containing at least two points. Prove that the metric topology is not the trivial topology. (Recall the trivial topology was defined in lecture.)

   Solution: We just need to find an open set other than $\emptyset$ or X, which implies the topology cannot be trivial. Take $x, y \in X$ such that $x \neq y$. Then $d(x, y) = \epsilon > \delta > 0$ and so $B_\delta(x)$ is the desired open set, since $x \in B_\delta(x)$ but $y \notin B_\delta(x)$.

6. Prove that the subspace topology is indeed a topology.

   Solution: Similar to question 1, just check this satisfies the definition of topology: contains the empty set, entire space, unions, and finite complements.