Tutorial Sheet 2, Topology 2013

1. Find a countable basis (ie a basis with countably many elements) for the usual topology on $\mathbb{R}$.

   Solution: A countable basis for the real line with the usual topology is the collection of all open intervals centered at rational numbers whose endpoints are rational. (You should convince yourself that this is in fact a basis.)

2. Prove that, on the real line with the usual topology, every point is a limit point of the rationals.

   Solution: Take any $x \in \mathbb{R}$ and any open set O containing x. Pick $B_\epsilon(x) \subset O$. Define N so that $1/N < \epsilon$. Then the points of the set $A = \{p/N : p \in \mathbb{Z}\} \subset \mathbb{Q}$ divide the real line up into subintervals of length strictly less than ϵ. Hence, $B_\epsilon(x) \cap (A \setminus \{x\}) \neq \emptyset$, and so $B_\epsilon(x) \cap (\mathbb{Q} \setminus \{x\}) \neq \emptyset$.

3. Find all the limit points of the following subsets of the real line (with the usual topology). Explain why such points are limit points and why there are no others (but you don't need to give a formal proof).

   (a) $\{(1/m) + (1/n) : n, m = 1, 2, 3, \dots\}$

   (b) $\{(1/n) \sin n : n = 1, 2, 3, \dots\}$

   Solution: a) Notice that, similar to the example from lecture, $1 + (1/n)$ limis to 1 only, $1/2 + (1/n)$ limits to $1/2$ only, etc. Hence, the limit points are the set $\{1/n\}$, as well as zero. b) This is also similar to the example from lecture, in the sense that for any $x \neq 0$ we can take the infimum of the distance to points in this set, to show x is not a limit point. Also, since $\lim_{x \to \infty} (1/x) \sin(x) = 0$, one can show that zero is a limit point. Hence, zero is the only limit point.

4. Let X be the real line equipped with the finite complement topology. Prove that if A is an infinite set, then every point is a limit point of A. In addition, prove that if A is a finite set, then it has no limit points.

   Solution: In the first case, let U be open and contain x. Then the complement of U is finite. As a result, because A is infinite, there must be some point in A (other than x) that's also in U. Hence, x is a limit point. Conversely, if A is finite, then consider the open set $U = X \setminus A$. (Put x back in if $x \in A$). This is an open set containing x that doesn't intersect $A \setminus \{x\}$, so x cannot be a limit point.

5. Find a family of closed subsets of the real line whose union is not closed.

   Solution: $C_n = [1/n, 1]$, so that $\cup_n C_n = (0, 1]$. (Alternatively, take any non-closed set, for example $(0, 1)$. Let $C_x = \{x\}$ for each x in your set. This also works.)

6. Verify the following for arbitrary subsets A and B of a topological space X: $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$. Show that equality need not hold.

   Solution: First note that if $x \in A \cap B$, then clearly $x \in \bar{A} \cap \bar{B}$. So assume $x \in \overline{A \cap B} \setminus (A \cap B)$, which means that x is a limit point of $A \cap B$. Then for any open set O containing x, $O \cap ((A \cap B) \setminus \{x\}) \neq \emptyset$. But this implies that there is a $y \in O \cap ((A \cap B) \setminus \{x\})$, and so $y \in O \cap (A \setminus \{x\})$ and $y \in O \cap (B \setminus \{x\})$. Since O was an arbitrary open set containing x, this implies $x \in \bar{A}$ and $x \in \bar{B}$. Hence $x \in \bar{A} \cap \bar{B}$.

A simple example where equality doesn't hold is $A = (0,1)$ and $B = (1,2)$. Another one is

$$A = \{p/q \in \mathbb{Q} : q = 2^n, n \in \mathbb{N}\}, \qquad B = \{p/q \in \mathbb{Q} : q = 3^n, n \in \mathbb{N}\}.$$

We have $\overline{A \cap B} = \overline{\emptyset} = \emptyset$, but $\bar{A} \cap \bar{B} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$.

7. Determine the interior, closure, and frontier of each of the following sets.

   (a) The plane with both axes removed.

   (b) $\mathbb{R}^2 \setminus \{(x, \sin(1/x)) : x > 0\}$

   Solution: a) The closure is the entire plane, the interior is the set itself, and the frontier is the axes. b) Denote $A = \mathbb{R}^2 \setminus \{(x, \sin(1/x)) : x > 0\}$. Then $\mathrm{int}(A) = A \setminus \{(0,y) : -1 \le y \le 1\}$, $\mathrm{cl}(A) = \mathbb{R}^2$, $\mathrm{front}(A) = \mathbb{R}^2 \setminus \mathrm{int}(A) = \{(x, \sin(1/x)) : x > 0\} \cup \{(0,y) : -1 \le y \le 1\}$.