Tutorial Sheet 3, Topology 2013

1. Prove that $f : X \to Y$ is continuous if and only if $C \subset Y$ being closed implies $f^{-1}(C) \subset X$ is also closed.

   Solution: Assume that f is continuous. Then if C is closed, $Y \setminus C$ is open, so $f^{-1}(Y \setminus C)$ is open, and so $X \setminus f^{-1}(Y \setminus C) = f^{-1}(C)$ is closed. Suppose now that inverse images of closed sets are closed. Let O be open, so $Y \setminus O$ is closed. The proof now follows as before.

2. Suppose that $f : X \to Y$ is continuous, $A \subset X$, and p is a limit point of A. Prove that $f(p) \in \overline{f(A)}$.

   Solution: If $f(p) \in f(A)$ the result is clear, so assume $f(p) \ /f(A)$. Let U be any open subset containing $f(p)$. Since $f^{-1}(U)$ is open and $p \in f^{-1}(U)$, we know that there is a $y \in f^{-1}(U) \cap A \setminus \{p\}$. This implies that $f(y) \in U \cap f(A)$. Hence, the intersection is not empty, so $f(p)$ is a limit point of A, and therefore in $\overline{f(A)}$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ (with the usual topologies) be continuous and define $g : \mathbb{R} \to \mathbb{R}^2$ (with the usual topologies) to be $g(x) = (x, f(x))$. Prove that g is continuous.

   Solution: Let U be any open set in $\mathbb{R}^2$ and take any $x \in g^{-1}(U)$. Then $B_\epsilon((x, f(x))) \subset U$ for some $\epsilon > 0$. Since f is continuous, there is a $\delta > 0$ such that for all $\tilde{x}$ with $d(\tilde{x}, x) < \delta$, we have $d(f(x), f(\tilde{x})) < \epsilon$. Let $\gamma = \min\{\epsilon, \delta\}$. Then $B_\gamma(x) \subset g^{-1}(U)$. Hence we've found a neighborhood for an arbitrary point of $g^{-1}(U)$, so it must be open, and g must be continuous.

4. Let $X = \mathbb{R}$ with the finite complement topology and let $Y = \mathbb{R}$ with the usual topology. Let $f : X \to Y$, $f(x) = x$. Is f continuous? If f^{-1} continuous? Justify your answers.

   Solution: f is not continuous. Take $U = (0, 1)$ which is open in Y. But $f^{-1}(0, 1) = (0, 1)$ is not open in X because its complement is not finite. On the other hand, f^{-1} is continuous. Take any open set $V \in X$. $f(V) = V$. We'll show V must also be open Y. Since its complement is finite, $X \setminus V = \{x_1, x_2, \ldots, x_n\}$ for some points $x_i \in \mathbb{R}$, and so $V = \mathbb{R} \setminus \{x_1, x_2, \ldots, x_n\}$. We can assume we've listed them so that $x_1 < x_2 < \cdots < x_n$. Thus, $V = (-\infty, x_1) \cup (x_1, x_2) \cup \cdots \cup (x_n, \infty)$, which is a union of open sets, so it must be open.

5. Prove that any two open intervals in the real line (with the usual subspace topology) of the form (a, b) and (c, d) are homeomorphic.

   Solution: Suppose the intervals are given by (a, b) and (c, d). Use the linear homeomorphism $f(x) = (d - c)(x - a)/(b - a) + c$. Prove this is continuous, one-to-one, onto, and has continuous inverse.

6. Prove that a disc and an ellipse (both with the usual subspace topology) are homeomorphic. Recall that

   $$D = \{(x, y) : x^2 + y^2 \leq R^2\}, \qquad E = \left\{(x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\}$$

   for some $R, a, b > 0$.

   Solution: This proof is very similar to the one from lecture, where we showed that two (open) discs of equal size were homeomorphic. You could use, for example, the function

   $$f : D \to E, \qquad f(x, y) = (ax/R, by/R).$$