Tutorial Sheet 4, Topology 2013

1. Find an open cover of $\mathbb{R}$ that does not contain a finite subcover. Do the same for $(0,1)$ considered as a subset of $\mathbb{R}$.

   Solution: For the real line, take $B_1(n)$ for all $n \in \mathbb{Z}$. (This is essentially the same as the example in class regarding $\mathbb{R}^2$.) For $(0,1)$ take $(1/n, 1-1/n)$, for $n \in \mathbb{N}$.

2. Are either of the following sets compact?

   (a) The rational numbers, considered as a subset of the real line.

   (b) $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, with finitely many points removed, considered as a subset of $\mathbb{R}^{n+1}$.

   Solution: We use the Heine-Borel Theorem. The first is not compact because it is not closed or bounded. The second is not compact because it is not closed (although it is bounded).

3. Let X be an infinite set with the finite complement topology.

   (a) Prove that X is not Hausdorff.

   (b) Prove that every subset of X is compact.

   (c) Find an example of a subset of X that is not closed, which is therefore an example of a compact set that is not closed.

   Solution: a) We'll prove that no two open sets are disjoint, which implies the result. BWOC, let U, V be disjoint open sets. This implies $V \subset (X \setminus U)$, and $X \setminus U$ is a finite set, which implies V is finite. But then $X \setminus V$ is infinite, contradicting the fact that V is open. b) Let $A \subset X$ and let $\{U_\alpha\}$ be an open cover. Pick any single element of the cover, U_{α^*}. Its complement has finitely many elements, so there are only finitely many elements of A, $x_1, \ldots, x_n$, that are not in this set. For each one, find a U_{α_i} containing x_i. Then $U_{\alpha^*}, \{U_{\alpha_i}\}_{i=1}^n$ is a finite subcover. c) Any open set other than X or $\emptyset$ is a set that is compact, but not closed.

4. Find an example of a function $f : X \to Y$, where X is compact and f is a continuos bijection, but such that f is not a homeomorphism. (Note: based on the theorem from class this implies Y cannot be Hausdorff.)

   Solution: Let $X = [a,b]$ with the usual topology and Y be $[a,b]$ with the finite complement topology. Above we showed that any infinite space with the finite complement topology is not Hausdorff. Let $f(x) = x$, and check it is continuous, but its inverse isn't. (It is clearly one to one and onto.) This is related to question 4 of tutorial sheet 3.

5. Suppose $f : X \to \mathbb{R}$ where X is compact and f is continuous. Prove that f is bounded and attains its bounds. (This means you should prove that $f(X) \subseteq [a,b]$ for some $a, b \in \mathbb{R}$ and $\exists\ x, y \in X$ such that $f(x) = a$ and $f(y) = b$. Hint: use the Heine-Borel theorem.)

   Solution: Since X is compact, so is $f(X)$, and so by Heine-Borel it is closed and bounded. Consider $a = \inf_{x \in X} f(x)$. Since $f(X)$ is closed, $a \in f(X)$, so there must be some $x \in X$ such that $f(x) = a$. A similar argument works for $b = \sup_{x \in X} f(x)$.