Tutorial Sheet 5, Topology 2013

1. Let X be the real line with the particular point topology for the point 0, and let Y be the real line with the particular point topology for the point 1. What is the product topology on $X \times Y$?

   Solution: The product topology on $X \times Y$ can be described as follows. Consider a basis element $U \times V$ where U is open in X and V is open in Y. Since $0 \in U$ and $1 \in V$, $(0,1) \in U \times V$. Hence, any basis element contains $(0,1)$ and so any open set contains $(0,1)$. Now, suppose that O is an open set that contains some point $(x,y) \neq (0,1)$. Since $O = \cup_\alpha (U_\alpha \times V_\alpha)$, we have $(x,y) \in U_\alpha \times V_\alpha$ for some α. Thus, $\{0,x\} \subset U_\alpha$ and $\{1,y\} \subset V_\alpha$. Hence, $\{(0,1),(0,y),(x,1),(x,y)\} \subset U \times V$. Hence, open sets are built up of the point $(0,1)$ and sets of 4 points of the form $\{(0,1),(0,y),(x,1),(x,y)\}$ (although this set actually contains fewer points if $x = 0$ or $y = 1$).

2. Consider the diagonal map $\Delta : X \to X \times X$, $\Delta(x) = (x,x)$, where X is some topological space and $X \times X$ has the product topology. a) Prove that Δ is continuous. b) Prove that X is Hausdorff if and only if $\Delta(X)$ is closed in $X \times X$.

   Solution: a) To see that it is continuous, let U_1 and U_2 be open in X, and note that $\Delta^{-1}(U_1 \times U_2) = U_1 \cap U_2$, which is open. By the lemma from lectures, this implies that the function is continuous, because we've showed that the inverse image of any basis element is open.

   b) Assume that X is Hausdorff. Suppose that (p_1,p_2) is a limit point of $\Delta(X)$ but $p_1 \neq p_2$, so $(p_1,p_2) \notin \Delta(X)$. Let U_1 and U_2 be disjoint open sets containing p_1 and p_2. But then $U_1 \times U_2$ is an open set in $X \times X$ containing (p_1,p_2) and such that $(U_1 \times U_2) \cap \Delta(X) = \emptyset$, which is a contradiction.

   Finally, assume that $\Delta(X)$ is closed and let $x_1 \neq x_2$. Then (x_1,x_2) is not a limit point of $\Delta(X)$, so I can find an open set $O \in X \times X$ that contains this point and is disjoint from $\Delta(X)$. Note that $O = \cup_\alpha U_\alpha \times V_\alpha$ for open sets U_α and V_α in X. O being disjoint from $\Delta(X)$ implies that $U_{\alpha_i} \cap V_{\alpha_j} = \emptyset$, for all $\alpha_{i,j}$, and there must be some $\alpha_{1,2}$ so that $x_1 \in U_{\alpha_1}$ and $x_2 \in V_{\alpha_2}$, which shows that X is Hausdorff.

3. We know that the projection maps send open sets to open sets. Do they send closed sets to closed sets?

   Solution: No. Consider $\mathbb{R} \times \mathbb{R}$ and define $C = \cup_{n=1}^{\infty} \{(1/n, n)\}$ (this is a union of sets consisting of single points in the product space). Note that $p_1(C) = \{1/n\}_{n=1}^{\infty}$, which is not closed because 0 is a limit point not in this set.

   I claim that C is a closed set in the product space. Suppose that (p_1,p_2) is a limit point not in the set. If $p_2 \neq n$ for any n, then let N be the closest natural number to p_2 and let $\epsilon = \min(p_2 - N, N+1-p_2)/2$. Then $U \times B_\epsilon(p_2)$ for any open set U containing p_1 is an open set containing this point that is disjoint from C. A similar argument works if $p_1 \neq 1/n$ and $p_1 \neq 0$. If $p_1 = 0$, let N be the closest natural number to p_2. Take $\delta < 1/(2(N+1))$ and $\epsilon = |p_2 - N|/2$ (or $\epsilon = 1/2$, if this difference is zero). Then $B_\delta(0) \times B_\epsilon(p_2)$ is an open set containing $(0,p_2)$ but disjoint from C.

4. Prove that $X \times Y$ is Hausdorff if and only if both X and Y are Hausdorff.

   Solution: Suppose X and Y are Hausdorff and $(x_1,y_1) \neq (x_2,y_2)$. Then wlog $x_1 \neq x_2$, so there are disjoint open sets U_1 and U_2 in X that contain $x_{1,2}$. But then $U_1 \times Y$ and $U_2 \times Y$ are disjoint open sets in $X \times Y$ that contain (x_1,y_1) and (x_2,y_2).

Next, suppose the product is Hausdorff, and let $x_1 \neq x_2$. Pick any $y \in Y$ and two disjoint open sets O_1 and O_2 in the product space that contain (x_1, y) and (x_2, y). This implies there exist basis elements $U_1 \times V_1 \subset O_1$ and $U_2 \times V_2 \subset O_2$ that contain (x_1, y) and (x_2, y). But since $y \in V_1$ and $y \in V_2$, U_1 and U_2 must be disjoint. Thus, they are the desired open sets.