Tutorial Sheet 6, Topology 2013

1. Prove that a space X is connected if and only if there do not exist nonempty disjoint sets A and B (not necessarily open) such that $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ and whose union is X.

   Solution: Suppose there do not exists such sets A and B, and bwoc let X be disconnected. Then there exists disjoint nonempty open sets U and V that separate X. Suppose $\bar{U} \cap V \neq \emptyset$. Then there is a limit point p of U that is also in V. But then V would be an open set containing p, and so the definition of limit point implies that $V \cap U \neq \emptyset$, which isn't true. Therefore, U and V are sets of the form A and B, described above, which is a contradiction. Hence, X is connected.

   Suppose now that X is connected, and bwoc suppose there exist sets as described above. Consider $\text{int}(A)$ and $\text{int}(B)$, which are disjoint because $\text{int}(A) \subset A$ and $\text{int}(B) \subset B$. We will show they are nonempty and their union is the entire space, thus contradicting the fact that X is connected. Suppose there exists an $x \in X$, $x \notin \text{int}(A) \cup \text{int}(B)$. Then wlog $x \in A \setminus \text{int}(A)$. But then for any open set U containing x, there is a point $y \in U$ that is not in A – ie $y \in B$. But then x is a limit point of B, and so $A \cap \bar{B} \neq \emptyset$, which isn't true. Thus, $\text{int}(A) \cup \text{int}(B) = X$. To see that both $\text{int}(A)$ and $\text{int}(B)$ are nonempty, suppose instead that $\text{int}(A) = \emptyset$. Since A is nonempty, there exists an $x \in A \setminus \text{int}(A)$. But again the above argument shows this can't happen, because it would imply that $\bar{A} \cap B \neq \emptyset$.

2. Are the following sets connected?

   (a) The rational numbers, considered as a subset of the real numbers.

   (b) The subset of $\mathbb{R}^2$ defined by

   $$X = \{(x, y) : y = 0\} \cup \{(x, y) : x > 0 \text{ and } y = 1/x\}$$

   (c) Any set with the discrete topology.

   Solution: The first is not connected because the sets $(-\infty, \pi) \cap \mathbb{Q}$ and $(\pi, \infty) \cap \mathbb{Q}$ are both open in $\mathbb{Q}$, in the subspace topology, and thus they form a disconnection of the rationals. (The above argument works if π is replaced by any irrational number.) The second set is also not connected; just apply the result in question 1, with $A = \{(x, y) : y = 0\}$ and $B = \{(x, y) : x > 0 \text{ and } y = 1/x\}$. Finally, the third set is also not connected (unless it only consists of a single point), since $\{x\}$ and $X \setminus \{x\}$ are disjoint nonempty open sets that disconnect the space, for any $x \in X$.

3. Let $f : [a, b] \to \mathbb{R}$ be continuous with $f(a) < 0 < f(b)$. Use the connectedness of $[a, b]$ to prove the intermediate value theorem: there must be a $c \in (a, b)$ such that $f(c) = 0$.

   Solution: BWOC. Since $(-\infty, 0)$ and $(0, \infty)$ are open and $0 \notin f([a, b])$, $f^{-1}(-\infty, 0)$ and $f^{-1}(0, \infty)$ are open, and $[a, b] \subset f^{-1}(-\infty, 0) \cup f^{-1}(0, \infty)$. But since $[a, b]$ is connected, it cannot be written as the union of two open, disjoint sets. Hence this is a contradiction, which proves the result.

4. Prove that the continuous image of a connected set is connected.

   Solution: Let $f : X \to f(X)$ be continuous with X connected, and let $A \subset f(X)$ be a subset that is both open and closed. But then $f^{-1}(A)$ is open, and $f^{-1}(f(X) \setminus A)$ is also open. But this implies $f^{-1}(A)$ is closed. Thus, $f^{-1}(A) = \emptyset$ or X. Hence, $A = \emptyset$ or $f(X)$.