Tutorial Sheet 7, Topology 2013

1. Let X be the set of all points in the plane which have at least one rational coordinate. Show that X, with the subspace topology, is a connected space.

   Solution: If we visualize this as a union of horizontal and vertical lines with rational intersection with the axes, we can see how to get from one point to another using a path, so intuitively the space is path connected, which implies it is connected. To make this rigorous, one could use the theorem from lecture about a family of connected sets that cover the space, no two of which are separated. (Use $Z(p_1/q_1, p_2/q_2) = (\{p_1/q_1\} \times Y) \cup (X \times \{p_2/q_2\})$.)

2. Prove that $\mathbb{Q}$, with the subspace topology inherited from $\mathbb{R}$, is totally disconnected, but not discrete.

   Solution: It is not discrete because $\{p/q\}$ is not open – if it was $\{p/q\} = U \cap \mathbb{Q}$ for some open set $U \subset \mathbb{R}$. But this isn't possible - the rational numbers are dense, so any open ball contains infinitely many of them.

   To see that it is totally disconnected, let C be a component containing two points, $x_{1,2}$ (wlog $x_1 < x_2$). But there is an irrational number $y \in (x_1, x_2)$, and so $(-\infty, y) \cap \mathbb{Q}$ and $(y, \infty) \cap \mathbb{Q}$ can be used to disconnect C.

3. (a) If X has a finite number of components, show that each component is both open and closed.

   Solution: Let $A_1, \dots, A_n$ be the components of X. By the theorem from lecture, each component is closed. Also, $X = \cup_{i=1}^{n} A_i$, so $X \setminus A_i = \cup_{j \neq i} A_j$, which is a finite union of closed sets, hence closed. Thus each component is open.

   (b) Find a space for which none of its components are open.

   Solution: An example of space that has no open components is in the previous question: the rational numbers.

4. Prove that the unit ball $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ is path connected.

   Solution: Use a straight-line path: if $x, y \in B^n$, then $\gamma(t) = tx + (1-t)y$ is a path in B^n, since $|\gamma(t)| \leq |t||x| + |1-t||y| \leq t + 1 - t = 1$.

5. (a) If $f : X \to Y$ is continuous and γ is a path in X, prove that $f \circ \gamma$ is a path in Y.

   Solution: $f \circ \gamma$ is continuous because it is the composition of continuous functions, and $f \circ \gamma(t) \in Y$ for all t because f maps into Y. Hence, it is a path in Y.

   (b) Conclude that path-connectedness is a topological invariant.

   Solution: Let $f : X \to Y$ be a homeomorphism where X is path connected, and let $y_1, y_2 \in Y$. There exist points $x_1, x_2 \in X$ such that $f(x_i) = y_i$, $i = 1, 2$, and a path γ in X between them. By part a), $f \circ \gamma$ is then a path in Y between y_1 and y_2, so Y is path connected. The same argument can be used with f^{-1}, to show that Y being path-connect implies that X is.

6. Explain how to construct the torus as an identification space.

   Solution: Let $X = [0,1] \times [0,1]$ (or any rectangle). Define $(x_1, y_1) \sim (x_2, y_2)$ if

   (a) $(x_1, y_1) = (x_2, y_2)$

(b) $(x_1, y_1) = (0, y)$ and $(x_2, y_2) = (1, y)$, with $0 < y < 1$.

(c) $(x_1, y_1) = (x, 0)$ and $(x_2, y_2) = (x, 1)$, with $0 < x < 1$.

(d) $(x_1, y_1), (x_2, y_2) \in \{(0, 0), (1, 0), (0, 1), (1, 1)\}$

The resulting identification space is the torus.

7. Consider the set $A = [0, 1] \times [0, 1]$ with the usual product topology. Define an equivalence relation on A as follows. Let $(x_1, y_1) \sim (x_2, y_2)$ if either (I) $(x_1, y_1) = (x_2, y_2)$ or if (II) $x_1 = x_2 = 0$ or if (III) $x_1 = x_2 = 1$ or if (IV) $x_1 = x_2 \in (0, 1)$ and $y_1 = 0$ and $y_2 = 1$.

   (a) The space $A/\sim$ is homeomorphic to a familiar surface. What surface is it?

   Solution: This is just a sphere. For any fixed $0 < x < 1$, the cross section in y is just a circle, because the two points on the boundary have been identified. Also, all points in the cross section $\{x = 1\}$ and $\{x = 0\}$ are identified to a point, so these are two poles of the sphere.

   (b) Pick any open set $U \subset A/\sim$ that contains the equivalence class defined by (II). Draw a picture of the inverse image of U in A under the associated identification map.

   Solution: The equivalence class (II) is just one of the poles of the sphere. An example of an open set containing it is what we get if we slice the sphere near the pole with a cut that has a circular cross section. In A, this corresponds to a set of the form $\{(x, y) \in A : 0 \leq x < \epsilon\}$ for some small $\epsilon > 0$.