Tutorial Sheet 8, Topology 2013

1. Consider the identification space $\mathbb{R}^2/\sim$ under the following equivalence relations. What familiar spaces are they homeomorphic to?

   (a) $(x_1, y_1) \sim (x_2, y_2)$ if (I) $(x_1, y_1) = (x_2, y_2)$ or if (II) $x_1 + y_1^2 = x_2 + y_2^2$

   Solution: This is just the real line. To see this, think about a parabola of the form $x + y^2 = c$ for each $c \in \mathbb{R}$. If I consider the collection of all such parabolas, I've covered the entire plane. Each equivalence class, defined above, is just one particular parabola. Since this family of parabolas is indexed by the number c, and $c \in \mathbb{R}$, the collection of equivalence classes is the same as the real numbers. (If you wanted to prove $\mathbb{R}^2/\sim$ was homeomorphic to $\mathbb{R}$, you could use the function $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = x + y^2$ and check that g is an identification map such that $(x_1, y_1), (x_2, y_2) \in g^{-1}(c)$ if and only if $x_1 + y_1^2 = x_2 + y_2^2$. Then apply the theorem from lecture.)

   (b) $(x_1, y_1) \sim (x_2, y_2)$ if (I) $(x_1, y_1) = (x_2, y_2)$ or if (II) $x_1^2 + y_1^2 = x_2^2 + y_2^2$

   Solution: Using an argument similar to above, this space is just $[0, \infty)$. The index here is the radius of the circle $x^2 + y^2 = r^2$, with $r \in [0, \infty)$. (To prove this, consider $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = x^2 + y^2$. Check that g is an identification map such that $(x_1, y_1), (x_2, y_2) \in g^{-1}(c)$ if and only if $x_1^2 + y_1^2 = x_2^2 + y_2^2$.)

2. Let $f : X \to Y$ be an onto continuous map. Prove that, if f maps open sets to open sets, or if f maps closed sets to closed sets, then f is an identification map.

   Solution: We must show that U is open in Y if and only if $f^{-1}(U)$ is open in X. If U is open in Y, then by continuity $f^{-1}(U)$ is open in X. If $f^{-1}(U)$ is open in X and f maps open sets to open sets, then since f is onto, $f(f^{-1}(U)) = U$, so U is open in Y. The argument for closed sets is similar.

3. Prove that any two continuous functions $f, g : X \to A$, where A is a convex subset of $\mathbb{R}^n$ and X is an arbitrary topological space, are homotopic.

   Solution: We can use a straight-line homotopy $F(x, t) = (1-t)f(x) + tg(x)$, which maps $X \times I \to A$ because A is convex.

4. (a) Let $f, g : X \to S^n$ be continuous functions such that $f(x)$ and $g(x)$ are never antipodal (ie $f(x) \neq -g(x)$ for any $x \in X$). Prove that

   $$F(x, t) = \frac{(1-t)f(x) + tg(x)}{|(1-t)f(x) + tg(x)|}$$

   is a homotopy between f and g.

   Solution: This function is continuous because the denominator is never zero. This is guaranteed by the assumption that the functions are never antipodal. If $(1-t)f(x) + tg(x) = 0$, then $f(x) = -(tg(x))/(1-t)$ But since $|f(x)| = |g(x)| = 1$, this implies $t = 1/2$ and $f(x) = -g(x)$, which can't happen.

   (b) Suppose that $f : S^1 \to S^1$ is continuous and not homotopic to the identity. Prove that $f(x) = -x$ for some $x \in S^1$.

Solution: BWOC, if not, then by part a) f would be homotopic to the function $g(x) = x$, which is the identity – hence, a contradiction.