Tutorial Sheet 9, Topology 2013

1. Describe the homomorphism h_* that is induced for each of the following continuous functions on the circle S^1:

   (a) $h(e^{i\theta}) = e^{i(\theta+\pi)}$, $\theta \in [0, 2\pi]$

   (b)
   $$h(e^{i\theta}) = \begin{cases} e^{i\theta} & \text{if} \quad \theta \in [0, \pi] \\ e^{i(2\pi-\theta)} & \text{if} \quad \theta \in [\pi, 2\pi] \end{cases}$$

   Solution:

   (a) This function just rotates the entire circle counterclockwise by π degrees. This doesn't affect the number of times a loop goes around the circle, or the direction in which it goes around. Also, $h : S^1 \to S^1$ is in fact a homeomorphism. Thus, h_* is the identity from $\pi_1(S^1, 1)$ to $\pi_1(S^1, -1)$.

   (b) This function is the identity on the top half of the circle, and maps the bottom half of the circle onto the top half of the circle. Thus, if a loop approaches and attempts to pass through -1 counterclockwise, it is forced by h to turn around a go back up. A similar thing happens if the loop tries to pass through 1. Thus, $h \circ \alpha$ will never go around the circle, and so it will be homotopic to the constant loop. Thus, $h_*(\langle\alpha\rangle) = \langle e\rangle$ for all α. In other words, h_* sends everything to the identity, and maps $\pi_1(S^1, 1)$ to the subgroup of $\pi_1(S^1, 1)$ given by the trivial group.

2. Compute the fundamental groups of the following spaces:

   (a) A cylinder: $\pi_1(S^1 \times J, x_0)$ where J is any interval in $\mathbb{R}$ and $x_0 = (y_0, z_0) \in S^1 \times J$ is any point.
   Solution: By the theorem from class on product spaces, this fundamental group is isomorphic to $\pi_1(S^1, y_0) \times \pi_1(J, z_0) = \mathbb{Z} \times \{e\}$, which is isomorphic to $\mathbb{Z}$. These are all independent of the base point, because the spaces are path connected.

   (b) The punctured plane: $\pi_1(\mathbb{R}^2 \setminus \{(0, 0)\}, x_0)$, for any point $x_0 \in \mathbb{R}^2 \setminus \{0\}$.
   Solution: Using polar coordinates (r, θ), one can see that the punctured plane $\mathbb{R}^2 \setminus \{(0, 0)\}$ is homeomorphic to $(0, \infty) \times S^1$, and so by part a), the fundamental group is isomorphic to $\mathbb{Z}$.

   (c) A punctured disk: $\pi_1(B^2 \setminus \{0\}, x_0)$, for any point $x_0 \in B^2 \setminus \{0\}$.
   Solution: As in b), the space is homeomorphic to $(0, 1] \times S^1$, so its fundamental group is the integers.