Tutorial Sheet 7, Topology 2013

1. Let X be the set of all points in the plane which have at least one rational coordinate. Show that X, with the subspace topology, is a connected space.

2. Prove that $\mathbb{Q}$, with the subspace topology inherited from $\mathbb{R}$, is totally disconnected, but not discrete.

3. (a) If X has a finite number of components, show that each component is both open and closed.

   (b) Find a space for which none of its components are open.

4. Prove that the unit ball $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ is path connected.

5. (a) If $f : X \to Y$ is continuous and γ is a path in X, prove that $f \circ \gamma$ is a path in Y.

   (b) Conclude that path-connectedness is a topological invariant.

6. Explain how to construct the torus as an identification space.

7. Consider the set $A = [0,1] \times [0,1]$ with the usual product topology. Define an equivalence relation on A as follows. Let $(x_1, y_1) \sim (x_2, y_2)$ if either (I) $(x_1, y_1) = (x_2, y_2)$ or if (II) $x_1 = x_2 = 0$ or if (III) $x_1 = x_2 = 1$ or if (IV) $x_1 = x_2 \in (0,1)$ and $y_1 = 0$ and $y_2 = 1$.

   (a) The space $A/\sim$ is homeomorphic to a familiar surface. What surface is it?

   (b) Pick any open set $U \subset A/\sim$ that contains the equivalence class defined by (II). Draw a picture of the inverse image of U in A under the associated identification map.