Tutorial Sheet 8, Topology 2013

1. Consider the identification space $\mathbb{R}^2/\sim$ under the following equivalence relations. What familiar spaces are they homeomorphic to?

   (a) $(x_1, y_1) \sim (x_2, y_2)$ if (I) $(x_1, y_1) = (x_2, y_2)$ or if (II) $x_1 + y_1^2 = x_2 + y_2^2$

   (b) $(x_1, y_1) \sim (x_2, y_2)$ if (I) $(x_1, y_1) = (x_2, y_2)$ or if (II) $x_1^2 + y_1^2 = x_2^2 + y_2^2$

2. Let $f : X \to Y$ be an onto continuous map. Prove that, if f maps open sets to open sets, or if f maps closed sets to closed sets, then f is an identification map.

3. Prove that any two continuous functions $f, g : X \to A$, where A is a convex subset of $\mathbb{R}^n$ and X is an arbitrary topological space, are homotopic.

4. (a) Let $f, g : X \to S^n$ be continuous functions such that $f(x)$ and $g(x)$ are never antipodal (ie $f(x) \neq -g(x)$ for any $x \in X$). Prove that

   $$F(x,t) = \frac{(1-t)f(x) + tg(x)}{|(1-t)f(x) + tg(x)|}$$

   is a homotopy between f and g.

   (b) Suppose that $f : S^1 \to S^1$ is continuous and not homotopic to the identity. Prove that $f(x) = -x$ for some $x \in S^1$.