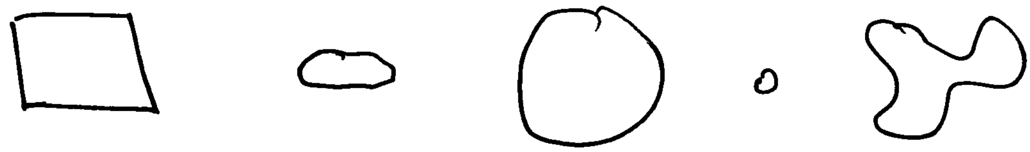
Lecture 1

- **What is topology?**
  - → Study of shapes, surfaces, and space.
  - → Goal is to do this abstractly, without explicit measure of distance or size
  - → Often described as "rubber sheet geometry": two objects that can be stretched/squeezed/twisted/etc w/o breaking or tearing them are topologically "the same".

Ex:



are all the same

but distinct from

Ex: via paper + scissors...

① Make a cylinder. How many sides does it have? How many edges? What happens if you cut it in half?

(2, 2, get 2 cylinders)

② Make a Möbius strip: cylinder w/ a half twist.

- Sides: 1 (!)
- Edges: 1 (!)
- Cut in half: get one "cylinder" (full twist; 2 sides, 2 edges)
- Cut in thirds (use a new Möbius strip) get one Möbius strip linked w/ a "cylinder" (2 full twists)

③ Make a "cylinder" with a full twist.

- Sides: 2
- edges: 2
- cut in half: 2 linked "cylinders"

- One goal of topology is to define precisely what we mean when we say the standard cylinder is the same as the cylinder with the full twist. Let's start with some formal definitions...

Lecture 2

- Definition: A topological space is a set X together with a collection of subsets, τ, called the topology on X. Members of τ are called open sets, and τ must satisfy:

1) $\phi, X \in \tau$

2) If $U, V \in \tau$, then $U \cap V \in \tau$

3) If $U_\alpha \in \tau$ for all $\alpha \in A$ (some index set)

then $\bigcup_{\alpha \in A} U_\alpha \in \tau$

In words: a topology τ must contain the empty set, the entire space X, finite intersections, and arbitrary (finite, countable, uncountable...) unions.

- Example: $X = \{1,2,3\}$ $\tau = \{\phi, \{1\}, \{1,2\}, \{1,2,3\}\}$

Prove: τ is a topology.

Solution: Just check the 3 conditions.

1) $\phi, \{1,2,3\} = X$ are both in τ. ✓

2) $\phi \cap U = \phi \in \tau \quad \forall\ U \in \tau$ ✓

$\{1\} \cap \{1,2\} = \{1\} \in \tau$ ✓

$\{1\} \cap \{1,2,3\} = \{1\} \in \tau$ ✓

$\{1,2\} \cap \{1,2,3\} = \{1,2\} \in \tau$ ✓

3) Similarly check all unions. ✓

- Example: $X = \{1,2,3\}$ $\tau = \{\phi, \{1,2\}, \{1,3\}, \{1,2,3\}\}$

Question: Is τ a topology?

Sol'n: No. $\{1,2\} \cap \{1,3\} = \{1\} \notin \tau$.

$\therefore$ 2) is violated.

- Example: For any set $\mathbb{X}$, $\tau = \{\phi, \mathbb{X}\}$ is a topology called the trivial topology. (B/c its not very interesting.)

- Example: For any set $\mathbb{X}$, $\tau = \mathcal{P}(\mathbb{X})$, where $\mathcal{P}(\mathbb{X})$ is the collection of all subsets of $\mathbb{X}$, is a topology called the discrete topology. (For reasons we'll discuss later.)

- Example: Given any set $\mathbb{X}$ and any point $p \in \mathbb{X}$, let τ consist of all subsets that contain p, plus ϕ. This is a topology called the particular point topology.

→ Prove this is a topology:

→ Sol'n: (ie Proof):

1) $\phi \in \tau$ by definition. $p \in \mathbb{X}$ so $\mathbb{X} \in \tau$.

2) If $U, V \in \tau$, then $p \in U$, $p \in V$. So $p \in U \cap V$ and $\therefore$ $U \cap V \in \mathbb{X}$

3) Consider $U_\alpha \in \tau$ $\forall$ $\alpha \in A$. Then $p \in U_\alpha$ $\forall \alpha$ and so $p \in \bigcup_\alpha U_\alpha$ $\therefore$ $\bigcup_\alpha U_\alpha \in \mathbb{X}$

∎

- Example: Given any set $\mathbb{X}$, define τ to be the collection of subsets whose complements are finite (ie contain finitely many elements), plus ϕ. Then τ is a topology called the finite complement topology.

<u>Example:</u> (Metric Spaces) Recall that a metric space is a set X together with a distance function satisfying

$$d: X \times X \to [0,\infty)$$

i) $d(x,y)=d(y,x)$

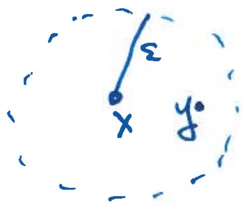
ii) $d(x,y)=0$ iff $x=y$

iii) $d(x,z) \leq d(x,y)+d(y,z)$ (Δ inequality)

The <u>metric topology</u> on X, or the topology induced by the metric d, is defined as follows. Let

$$B_\varepsilon(x)=\{y\in X : d(x,y)<\varepsilon\}$$

This is called the <u>ε-ball</u>, <u>ball of radius ε</u>, or <u>ε-neighborhood</u> of x. We visualize it typically as



(But see tutorial sheet !!)

A set U is open, ie $U \in \tau$ where τ is the metric topology, iff it contains a neighborhood of this type at each of its points. In other words, $U \in \tau$ iff $\forall\, y \in U$, $\exists\, \varepsilon=\varepsilon(y)$ s.t. $B_\varepsilon(y) \subset U$.

For example, if $X = \mathbb{R}^n$ and

$$d(x,y) = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}$$

$x = (x_1, \dots, x_n)$
$y = (y_1, \dots, y_n)$

(usual metric)

The topology induced by this metric is the usual topology on $\mathbb{R}^n$.

Visualize in 1D:

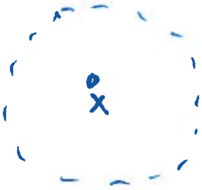
open interval = open set

fails @ 1 pt.

fails @ 2 pts.

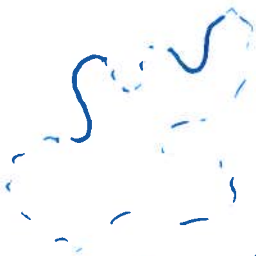
not open! $B_\varepsilon(a) \not\subset U \;\; \forall \varepsilon > 0$

Visualize in 2D:



an open ball is itself an open set

Region w/o boundary is open



not open if it includes any boundary!

Warning: Drawing open sets as open balls/intervals only makes sense in the metric topology! It can be misleading otherwise!

Example: Let (X, d) be a metric space with finitely many elements. Prove that the metric topology is the discrete topology.

Proof: Recall the discrete topology τ is defined to be the collection of all subsets of X. So, we must show that any subset of X is open in the metric topology.

Let U be any subset of X and pick $y \in U$.

We must find an $\varepsilon = \varepsilon(y)$ s.t. $B_\varepsilon(y) \subset U$.

$$X = \{x_1, \dots, x_n, y\}$$

Define $\delta = \min_{i=1\dots n} d(x_i, y)$ and $\varepsilon = \delta/2$.

Then $B_\varepsilon(y) = \{y\} \subset U$ $\blacksquare$

Lecture 3

(recall def. of topology...)

- **Example:** Let (X, τ) be any topological space and let $Y \subseteq X$ be any subset of X. Define

$$\tau_Y = \{U : U = \mathcal{O} \cap Y \text{ for some } \mathcal{O} \in \tau\}$$

Then τ_Y is a topology for Y called the <u>subspace topology</u>.

- **Key Fact:** For the subspace topology, a set that is open in Y is <u>not</u> necessarily open in X.

Ex: $X = \mathbb{R}$ w/ the usual metric topology

ie $d(x,y) = |x-y|$

$Y = [0,1]$

- $[0, 1/2) = (-1/2, 1/2) \cap Y$ so it is <u>open</u> in Y!
- $[0, 1/2)$ is <u>not</u> open in X!

When discussing a topological space, we'd like to have a way to ~~discuss~~ describe the simplest, or most basic, open sets. This is done via a basis.

• **Def:** A basis, or base, for a topology τ on X is a collection of open sets β s.t. for any open $O \in \tau$, $\exists$ a family of sets $B_\alpha \in \beta$, for $\alpha \in A$, such that

$$O = \bigcup_\alpha B_\alpha$$

• **Example:** Let (X, d) be a metric space with the associated metric topology. Prove that

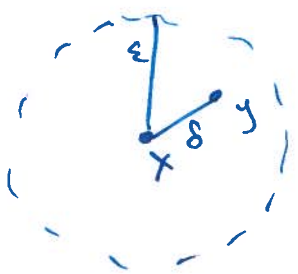
$$\beta = \{ B_\varepsilon(x) : x \in X \text{ and } \varepsilon > 0 \}$$

is a basis.

Proof: ~~Let $O \in \tau$. For each $y \in O$, $\exists$ $\varepsilon = \varepsilon(y)$ s.t. $B_{\varepsilon(y)}(y) \subset O$.~~

1) We must first show that each $B \in \beta$ is itself open. Given $B = B_\varepsilon(x)$ and $y \in B$, let

$$\delta = d(x,y)$$



Consider $B_{\varepsilon-\delta}(y)$. To show $B_{\varepsilon-\delta}(y) \subset B_\varepsilon(x)$, let $z \in B_{\varepsilon-\delta}(y)$. Then

$$d(x,z) \le d(x,y) + d(y,z)$$
$$< \delta + \varepsilon - \delta = \varepsilon$$

$$\therefore \; z \in B_\varepsilon(x)$$

$$\therefore \; B_{\varepsilon-\delta}(y) \subset B \quad \checkmark$$

2) Now, given any $\mathcal{O} \in \tau$, we must find a collection $B_\alpha \in \beta$ s.t. $\mathcal{O} = \bigcup_\alpha B_\alpha$.

Given any $y \in \mathcal{O}$ $\exists$ $\varepsilon(y)$ s.t. $B_{\varepsilon(y)}(y) \subset \mathcal{O}$.

$\therefore$ let this collection B_α be given by

$$B_{\varepsilon(y)}(y) \text{ s.t. } y \in \mathcal{O}$$

One can check that

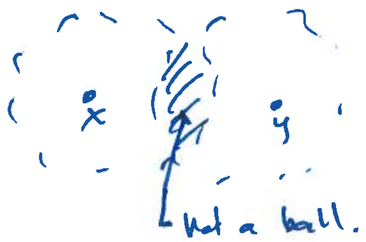
$$\bigcup_y B_{\varepsilon(y)}(y) = \mathcal{O}$$

[ie set $V = \bigcup_y B_{\varepsilon(y)}(y)$ and show $V \subseteq \mathcal{O}$ and $\mathcal{O} \subseteq V$]

$\left(x \in V \Rightarrow x \in B_{\varepsilon}(y) \subset \mathcal{O} \quad \therefore x \in \mathcal{O} \right)$ $\left(x \in \mathcal{O} \Rightarrow x \in B_{\varepsilon}(x) \quad \therefore x \in V \right.$

• <u>Remark:</u> The collection of all open balls in a metric space does not itself form a topology.

$$B_\varepsilon(x) \cap B_\delta(y) \neq B_\delta(z)$$



Another way to think of open sets is in terms of neighborhoods.

• <u>Definition:</u> A neighborhood of $x \in X$ is a subset $N \subseteq X$ s.t. $x \in N$ and $\exists$ $\mathcal{O} \in \tau$ s.t. $x \in \mathcal{O} \subset N$.

<u>Ex:</u> $X = \mathbb{R}$ w/ usual metric topology.

$N = (1/2, 3/2]$ is a nbhd of 1 b/c $1 \in N$

and $(1/2, 3/2) \subset N$ and $(1/2, 3/2)$ is open.

→ A neighborhood need not be open itself (but it could be).

- <u>Lemma:</u> A set U in a topological space (X, τ) is open iff it contains a neighborhood of each of its points.

<u>Proof:</u>

⇒ Let U be open. Then U is itself a nbhd of x, for any $x \in U$.

⇐ Suppose $U \subset X$ contains a nbhd of each of its pts. So for each $x \in X$, $\exists$ N_x nbhd of x and an open O_x s.t. $x \in O_x \subset N_x \subset U$. Then one can check if we set

$$V = \bigcup_{x \in U} O_x$$ open b/c union of open sets.

That $V = U$. $(V \subseteq U$ and $U \subseteq V)$