What is topology?

- Study of shapes, surfaces, and space.
- Goal is to do this abstractly, without explicit measure of distance or size.

Often described as "rubber sheet geometry": two objects that can be stretched/squeezed/ twisted/ etc. w/o breaking or tearing them are topologically "the same".

Ex: are all the same, but distinct from

Ex: via paper and scissors...

1. Make a cylinder. How many sides does it have? How many edges? What happens if you cut it in half?

(2, 2, get 2 cylinders)
(2) Make a Möbius strip: cylinder w/ a half-twist.
- Sides: 1 (!)
- Edges: 1 (!)
- Cut in half: get one “cylinder”
  (full twist: 2 sides, 2 edges)
- Cut in thirds (use a new Möbius strip)
  get one Möbius strip linked w/ a “cylinder” (2 full twists)

(3) Make a “cylinder” with a full twist.
- Sides: 2
- Edges: 2
- Cut in half: 2 linked “cylinders”

One goal of topology is to define precisely what we mean when we say the standard cylinder is the same as the cylinder w/ the full twist. Let’s start with some formal definitions…

- Definition: A topological space is a set $X$ together with a collection $\mathcal{T}$ of subsets, $\tau$, called the topology on $X$. Members of $\tau$ are called open sets, and $\tau$ must satisfy:

- Lecture 2
1) $\emptyset, X \in T$

2) If $U, V \in T$, then $U \cap V \in T$

3) If $U_{\alpha} \in T$ for all $\alpha \in A$ (some index set), then $\bigcup_{\alpha \in A} U_{\alpha} \in T$

In words: a topology $T$ must contain the empty set, the entire space $X$, finite intersections, and arbitrary (finite, countable, uncountable...) unions.

**Example:**

$X = \{1, 2, 3\}$  
$T = \{\emptyset, 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3\}$

**Proof:** $T$ is a topology.

**Solution:** Just check the 3 axioms.

1) $\emptyset, \{1, 2, 3\} = X$ are both in $T$. ✓

2) $\emptyset \cup U = \emptyset \in T \forall U \in T$ ✓

   $\emptyset \cup \emptyset = \emptyset \in T$ ✓

   $\emptyset \cup \{1\} = \{1\} \in T$ ✓

   $\emptyset \cup \{2\} = \{2\} \in T$ ✓

   $\emptyset \cup \{3\} = \{3\} \in T$ ✓

3) Similarly check all unions. ✓

**Example:**

$X = \{1, 2, 3\}$  
$T = \{\emptyset, \{1, 2, 3\}, \{1, 3\}, \{1, 2, 1\}\}$

**Question:** Is $T$ a topology?

**Solution:** No. $\{1, 2\} \cup \{1, 3\} = \{1, 2, 3\} \notin T$.

... 2) is violated.
Example: For any set $X$, $\tau = \emptyset, X \cong$ is a topology called the trivial topology. ($\emptyset$ is not very interesting.)

Example: For any set $X$, $\tau = P(X)$, where $P(X)$ is the collection of all subsets of $X$, is a topology called the discrete topology. (For reasons we'll discuss later.)

Example: Given any set $X$ and any point $p \in X$, let $\tau$ consist of all subsets that contain $p$, plus $\emptyset$. This is a topology called the particular point topology.

\[ \text{Prove this is a topology:} \]

\[ \text{Sol'n (by Proof):} \]

1) For $\tau$ by definition, $p \in X$ so $X \in \tau$.
2) If $U, V \in \tau$, then $p \in U$, $p \in V$. So $p \in U \cap V$ and $\therefore U \cap V \in \tau$.
3) Consider $U \in \tau$ and $A \subseteq A$. Then $p \in U \cap A$ and so $p \in U \cap A = U \in \tau$.

Example: Given any set $X$, define $\tau$ to be the collection of subsets whose complements are finite (i.e., contain finitely many elements), plus $\emptyset$. Then $\tau$ is a topology called the finite complement topology.
Example: (Metric Spaces) Recall that a metric space is a set \( X \) together with a distance function \( d : X \times X \rightarrow [0, \infty) \)

1) \( d(x, y) = d(y, x) \)
2) \( d(x, y) = 0 \text{ iff } x = y \)
3) \( d(x, z) \leq d(x, y) + d(y, z) \) (\( \Delta \) inequality)

The metric topology on \( X \), or the topology induced by the metric \( d \), is defined as follows. Let

\[ B_\varepsilon(x) = \{ y \in X : d(x, y) < \varepsilon \} \]

This is called the \( \varepsilon \)-ball, ball of radius \( \varepsilon \), or \( \varepsilon \)-neighborhood of \( x \). We visualize it typically as

\[ \begin{array}{c}
\varepsilon \\
\hline
x \\
\varepsilon \\
y
\end{array} \]

But see Manual (sheet #1)

A set \( U \) is open, i.e. \( U \in T \) where \( T \) is the metric topology, iff it contains a neighborhood of every point. In other words, \( U \in T \) iff \( \forall y \in U, \exists \varepsilon = \varepsilon(y) > 0 \text{ s.t. } B_\varepsilon(y) \subseteq U \).
For example, if $X = \mathbb{R}^n$ and
\[
d(x,y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}
\]
\[x = (x_1, \ldots, x_n)
\]
\[y = (y_1, \ldots, y_n)
\]

The topology induced by this metric is the usual topology on $\mathbb{R}^n$.

**Visualize in 1D:**

(→)

open interval → open set

falls @ 1 pt.

falls @ 2 pts.

\[a \rightarrow b\]

\[\not\text{open!} \quad B_e(a) \notin U \forall \varepsilon > 0\]

**Visualize in 2D:**

an open ball is not an open set

Borel w/o boundary is open

not open if it includes an boundary!

**Warning:** Drawing open sets as open balls/intervals only makes sense in the metric topology! It can be misleading otherwise!

**Example:** Let $(X,d)$ be a metric space with finitely many elements. Prove that the metric topology is the discrete topology.
Proof: Recall that a topology $\tau$ is defined as the collection of all subsets of $X$. So, we must show that any subset of $X$ is open in the usual topology.

Let $U$ be any subset of $X$, and pick $y \in U$.
We must find an $\varepsilon = \varepsilon(y)$ s.t. $B_\varepsilon(y) \subseteq U$.

Define $\delta = \min_{i=1,\ldots,n} d(x_i, y)$ and $\varepsilon = \delta/2$.

Then $B_\varepsilon(y) = \{x \in X \mid d(x, y) < \varepsilon\} \subseteq U$.

**Lecture 3**
(Recall def. of topology...)

**Example:** Let $(\mathcal{X}, \tau)$ be any topological space and let $Y \subseteq \mathcal{X}$ be any subset of $\mathcal{X}$. Define $\tau_Y = \{U \subseteq Y \mid U \in \tau Y\}$. Then $\tau_Y$ is a topology on $Y$ called the **subspace topology**.

**Key Fact:** For the subspace topology, a set that is open in $Y$ is not necessarily open in $\mathcal{X}$.

**Example:** $\mathcal{X} = \mathbb{R}$ with the usual metric topology

$\tau = \{(-x, x) \mid x \in \mathbb{R}\}$

$\tau_Y = \{[0,1] \}$

$\{[0,1/2]\} = (-\frac{1}{2}, \frac{1}{2}) \cap Y$ so it is open in $Y$!

$\{[0,1/2]\}$ is not open in $\mathcal{X}$!
When discussing a topological space, we'd like to have a way to describe the simplest, or most basic, open sets. This is done via a basis.

**Def:** A basis, or base, for a topology $\tau$ on $X$ is a collection of open sets $\beta$ s.t. for any open $O \in \tau$, if a family of sets $B_\alpha \in \beta$, $\alpha \in \mathcal{A}$, such that

$$O = \bigcup \beta \alpha$$

**Example:** Let $(X, d)$ be a metric space with the associated metric topology. Prove that

$$\beta = \{ B_\varepsilon(x) : x \in X \text{ and } \varepsilon > 0 \}$$

is a basis.

**Proof:**

1) We must first show that each $B_\varepsilon \beta$ is itself open. Given $B = B_\varepsilon(x)$ and $y \in B$, let $s = d(x, y)$.

Consider $B_{s - \varepsilon}(y)$. To show $B_{s - \varepsilon}(y) \subset B_\varepsilon(x)$, let $z \in B_{s - \varepsilon}(y)$. Then

$$d(x, z) \leq d(x, y) + d(y, z)$$

$$\leq s + s - s = \varepsilon$$

$$\therefore z \in B_\varepsilon(x)$$

$$\therefore B_{s - \varepsilon}(y) \subset B \checkmark$$
2) Now, given any \( O \subset T \), we must find a collection \( \{ B_x \subset O \} \) s.t. \( O = \bigcup B_x \).

Given any \( y \in O \), \( \exists x(y) \subset O \) s.t. \( B_{x(y)}(y) \subset O \).

Let my collection \( \{ B_x \} \) be given by \( B_{x(y)}(y) \) s.t. \( y \in O \).

One can check that
\[
\bigcup B_{x(y)}(y) = O
\]

[le set \( V = \bigcup B_{x(y)}(y) \) and show \( V \subset O \) and \( O \subset V \)]

\[
\begin{align*}
(x \in V & \iff x \subset B_{x(y)}(y)) \subset O \subset V \\
& \iff x \in O \\
& \iff x \in O
\end{align*}
\]

\[ \therefore O \subset V. \]

Remark: The collection of all open balls in a metric space does not itself form a topology.

\[ B_x(z) \cap B_y(z) \neq B_r(z) \]

Another way to think of open sets is in terms of neighborhoods.

* Definition: A neighborhood of \( x \in X \) is a subset \( N \subset X \)
  s.t. \( x \in N \) and \( \exists O \subset T \) s.t. \( x \in O \subset N \).

Ex: \( X = \mathbb{R} \) w/ usual metric topology.
\( N = (\frac{1}{2}, \frac{3}{2}) \) is a nbhd of \( 1 \) w/ \( 1 \in N \)
and \( (\frac{1}{2}, \frac{3}{2}) \subset N \) and \( (\frac{1}{2}, \frac{3}{2}) \) is open.
A neighborhood need not be open itself (but it could be).

**Lemma:** A set $U$ in a topological space $(X,\tau)$ is open iff it contains a neighborhood of each of its points.

**Proof:**

$\Rightarrow$ Let $U$ be open. Then $U$ is itself a nbhd of $x$, in any $x \in U$.

$\Leftarrow$ Suppose $U \subseteq X$ contains a nbhd of each of its pts. So for each $x \in X$, $X \setminus U$ is closed and an open $O_x$ s.t. $x \in O_x \subseteq X \setminus U$. Then we can check if we set

$$V = U \cup_{x \in U} O_x$$

**Proof** $V = U$. ($V \subseteq U$ and $U \subseteq V$)