Lecture 25

→ We've been discussing the fundamental group. So far we have

- $\pi_1(X, x_0)$: group consisting of all path-homotopic equivalence classes of loops based at x_0, w/ the operation

$$\langle f \rangle \cdot \langle g \rangle = \langle f \cdot g \rangle$$

$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s-1) & 1/2 \le s \le 1 \end{cases}$$

- $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic iff X is path connected.

- for any $x_0 \in \mathbb{R}^n$, $\pi_1(\mathbb{R}^n, x_0)$ is trivial.
  (straight-line homotopy shrinks everything to a pt.)
  $\therefore$ $\mathbb{R}^n$ is <u>simply connected</u> (path connected w/ trivial fund gp.)

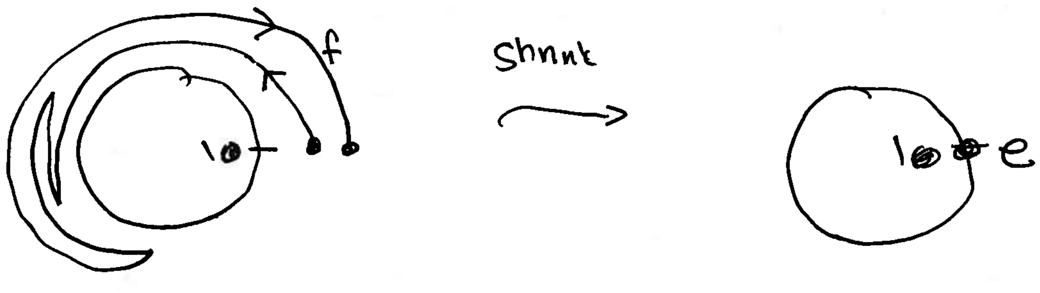
Another example...

- <u>Example</u>: What is $\pi_1(S^1, x_0)$?

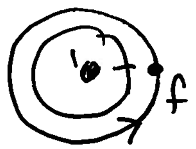
→ Note that S^1 is path-connected, so WLOG take $x_0 = 1$

→ <u>Intuitively</u>:

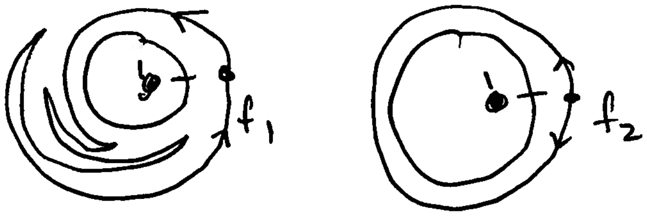
- If f doesn't wind around S^1, then we can shrink it to e:



• If f does wind around S^1, then we can't shrink it to e w/o breaking it.



can't shrink to e.



$f_1 \simeq f_2$ b/c they go around the same # of times.

• $f \simeq_p g$ is they wind around S^1 the same # of times.

• Expect $\pi_1(S^1, 1)$ to be isomorphic to $(\mathbb{Z}, +)$

$\langle f \rangle \longleftrightarrow n$ if f winds around S^1 n times.

• Theorem: For any $x_0 \in S^1$, $\pi_1(S^1, x_0)$ is isomorphic to $\mathbb{Z}$ w/ the group operation of addition.

• Sketch of proof: WLOG $x_0 = 1$. Define

$$\left.\begin{array}{l} p : \mathbb{R} \to S^1 \\ p(x) = e^{2\pi i x} \end{array}\right\} \quad \text{note } p(n) = 1 \ \forall\, n \in \mathbb{Z}$$

For any $n \in \mathbb{Z}$ define

$$\left.\begin{array}{l} \gamma_n : [0,1] \to \mathbb{R} \\ \gamma_n(s) = ns \end{array}\right\} \quad \text{path in } \mathbb{R} \text{ from } 0 \text{ to } n$$

Note that $(p \circ \gamma_n)(s) = e^{2\pi i n s}$ is a loop in S^1 that winds around S^1 n times, clockwise if $n > 0$, counterclockwise if $n < 0$. One can show that

$$\phi: \mathbb{Z} \to \pi_1(S^1, 1)$$
$$n \mapsto \langle p \circ \gamma_n \rangle$$

is an isomorphism. (Won't go through the details.) $\boxed{\equiv}$

→ We'd like to show that homeomorphic spaces necessarily have isomorphic fundamental groups.

• <u>Def</u>: Let $h: X \to Y$ be continuous with $h(x_0) = y_0$ for some $x_0 \in X$. Define $h_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$, via

$$h_*(\langle f \rangle) = \langle h \circ f \rangle$$

↑ composition as functions

h_* is called the <u>homomorphism</u> <u>induced by</u> h.

• <u>Remarks</u>

1) h_* is a homomorphism b/c

$$h_*(\langle f \rangle \cdot \langle g \rangle) = h_*(\langle f \cdot g \rangle)$$

$$= \langle h \circ (f \cdot g) \rangle = \langle (h \circ f) \cdot (h \circ g) \rangle$$

can check these func. are equal.

$$= \langle h \circ f \rangle \cdot \langle h \circ g \rangle$$

$$= h_*(\langle f \rangle) \cdot h_*(\langle g \rangle)$$

2) h_* depends not just on h but also on x_0.

• <u>Theorem</u>: If $h: X \to Y$ and $k: Y \to Z$ are continuous with $h(x_0) = y_0$ and $k(y_0) = z_0$, then $(k \circ h)_* = k_* \circ h_*$. Also, if $i: X \to X$ is the identity, then i_* is also the identity.

(Won't prove – just directly uses the definition)

- Corollary: If $h: X \to Y$ is a homeomorphism, then h_* is an isomorphism.

Proof: Let $k: Y \to X$ be defined via $k = h^{-1}$. Then by the previous theorem, $k_* \circ h_* = h_* \circ k_* = i_*$, so $h_*^{-1} = k_*$. This shows h_* is a bijection, and hence an isomorphism.

$\therefore$ The fundamental group is a topological invariant!

→ Our next example will be to compute the fundamental group of the sphere. To do so, we'll need a couple preliminary results.

Lecture 26

- Lemma (Lebesgue's lemma): Let Z be a compact metric space and $\{O_\alpha\}$ any open cover. Then $\exists$ a $\delta > 0$ s.t. any subset of Z with diameter less than δ is contained entirely in some O_α.

Recall: $\mathrm{diam}(A) = \sup\{d(x,y) : x, y \in A\}$

Proof: BWOC. Then $\exists$ a sequence of subsets $\{A_n\}_{n=1}^{\infty}$ s.t. $\mathrm{diam}(A_n) \to 0$ and none of the A_n's are contained in a single element of the cover. For each n, pick any $x_n \in A_n$. Either

i) $\{x_n\}$ contains finitely many distinct points and $x_n = p$ for ∞ many n's, for some p.

ii) $\{x_n\}$ has a limit point in X by Bolzano Weierstrass. Call it p.

Find an O_α s.t. $p \in O_\alpha$ and an $\varepsilon > 0$ s.t. $B_\varepsilon(p) \subset O_\alpha$.
Then take N suff large s.t.

a) $\operatorname{diam} A_n < \varepsilon/2 \quad \forall\, n \geq N$

b) $x_n \in B_{\varepsilon/2}(p)$ (ok b/c limit pt.) $\forall\, n \geq N$

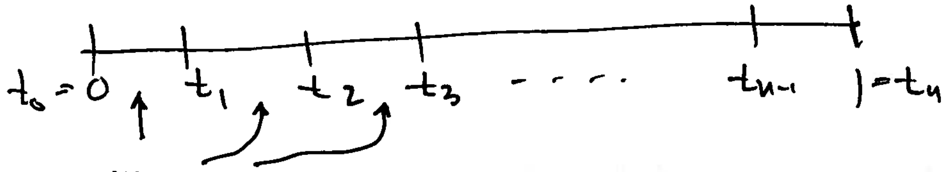
Then $\forall\, x \in A_N$,

$$d(x,p) \leq d(x, x_N) + d(x_N, p)$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

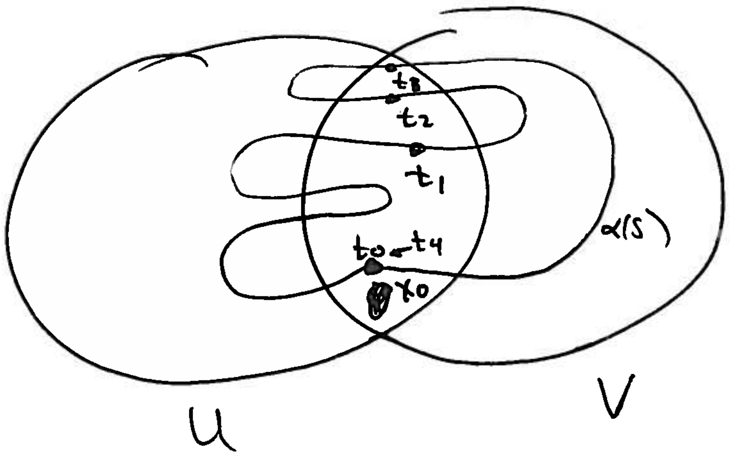
This implies $A_N \subset B_\varepsilon(p) \subset O_\alpha$. ↯ ∎

• **Theorem:** Let X be a space that can be written as the union of two open, simply connected sets U and V s.t. $U \cap V$ is nonempty and path connected. Then X is simply connected.

Proof: Given an arbitrary loop α in X, we must show that $\alpha \simeq_p e$. (X is clearly path connected: U and V are and $U \cap V$ is nonempty.) If $\alpha(s) \in V$ or $\alpha(s) \in U$ $\forall\, s$, then this is true b/c U and V are simply connected. So assume $\exists\, s_0$ s.t. $\alpha(s_0) = U \cap V$. Call $\alpha(s_0) = x_0$ and wlog assume this is the vast part of the loop. Since $\alpha^{-1}(u)$ and $\alpha^{-1}(v)$ are open and cover $[0,1]$, we can apply lebesgues lemma. A set in $[0,1]$ w/ diameter $< \delta$ is just a subinterval in $[0,1]$ of length $< \delta$.



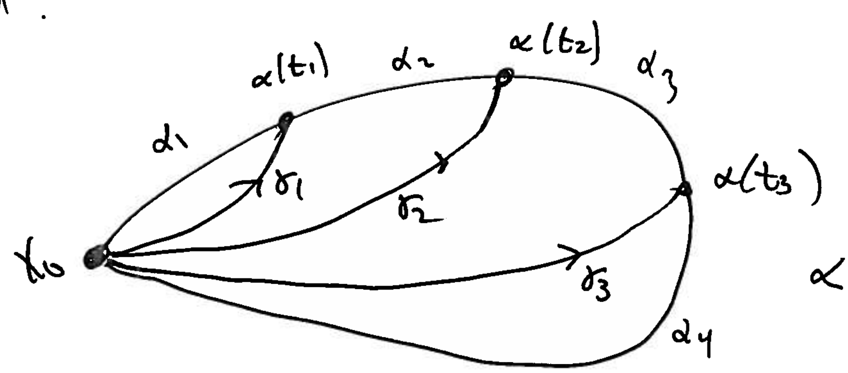
in $\alpha^{-1}(u)$ or $\alpha^{-1}(v)$ $\Rightarrow$ $[t_k, t_{k+1}] \subset \alpha^{-1}(u)$ or $\alpha^{-1}(V)$



For each k, let $\gamma_k(s)$ be a path from x_0 to $\alpha(t_k)$ s.t.

1) If $\alpha(t_k) \in U$ then $\gamma_k(s) \in U \ \forall s$
2) If $\alpha(t_k) \in V$ then $\gamma_k(s) \in V \ \forall s$
3) If $\alpha(t_k) \in U \cap V$ then $\gamma_k(s) \in U \cap V \ \forall s$

which is possible b/c U and V are path connected.



Define $\alpha_k(s) = \alpha((t_k - t_{k-1})s + t_{k-1})$ $\quad s \in [0,1]$
so that α_k is a path from $\alpha(t_{k-1})$ to $\alpha(t_k)$.
Since both α_k and γ_k are contained entirely in U, V or $U \cap V$, we have

$$\alpha_1 \cdot \gamma_1^{-1} \simeq_p e \qquad \text{(constant loop at } p\text{)}$$

$$\gamma_k \cdot \alpha_{k+1} \cdot \gamma_{k+1}^{-1} \simeq e$$

$$\gamma_{n-1} \cdot \alpha_n \simeq_p e$$

This implies:

$$\alpha \simeq_p [(\alpha_1 \cdot \gamma_1^{-1}) \cdot (\gamma_1 \cdot \alpha_2 \cdot \gamma_2^{-1}) \cdots (\gamma_k \cdot \alpha_{k+1} \cdot \gamma_{k+1}^{-1}) \cdots (\gamma_{n-1} \cdot \alpha_n)]$$

$$\simeq_p e$$

$\blacksquare$

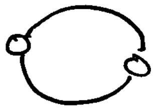
$\boxed{\text{Lecture 27}}$

- Corollary: S^n, for $n \geq 2$, is simply connected.

Proof: Take any $x, y \in S^n$, $x \neq y$, and set $U = S^n \setminus x$, $V = S^n \setminus y$. Both U and V are open and homeomorphic to $\mathbb{R}^n$, so simply connected. Also, $U \cap V = S^n \setminus \{x, y\}$ is nonempty and path connected, since $n \geq 2$, so the theorem applies.

$\blacksquare$

Note: Doesn't work for $n = 1$! $\qquad S^1 \setminus \{x, y\}$



Not path connected!

- So far we know:

$$\pi_1(\mathbb{R}^n) = \{\langle e \rangle\}$$
$$\pi_1(S^n) = \{\langle e \rangle\} \quad n \geq 2$$

$\Big\}$ simply connected.

$$\pi_1(S^1) = (\mathbb{Z}, +)$$

To compute the fundamental group of other spaces, we'll use the product structure.

- Theorem: If X and Y are path-connected, then $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$.

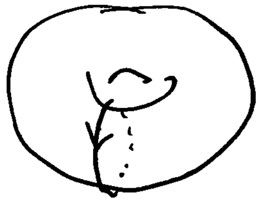
→ Remark: If $(G_1, \circ)$ and $(G_2, \times)$ are groups, then the set $G_1 \times G_2$ is a group when given the operation $(g_1, h_1) \circ (g_2, h_2) = (g_1 \circ g_2, h_1 \times h_2)$.

- Example: What is $\pi_1(S^1 \times S^1)$? (torus)

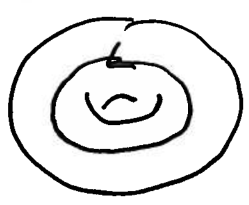
(Note: Can think of the torus as an identification space, but to compute π_1 it is easier to use the product structure.)

$$\pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1)$$
$$= (\mathbb{Z}, +) \times (\mathbb{Z}, +)$$

Intuition: loops can go around in two ways

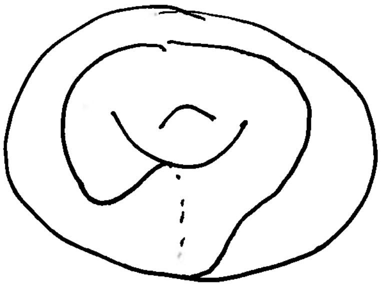


[Slice along loop to make cylinder]



[Slice along loop to make a bagel sandwich]

So $\langle \alpha \rangle \in \pi_1(S^1 \times S^1)$ is just (m,n) if it goes around m times in the "cylinder way" and n times in the "bagel" way.



← example of a $(1,1)$ loop.

- Proof of Theorem: WLOG take $x_0 \in X$ and $y_0 \in Y$ and $(x_0, y_0) \in X \times Y$ as base points. The projections p_1 and p_2 are continuous so they induce homomorphisms

$$p_{1*} : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0)$$
$$p_{1*}(\langle \alpha \rangle) = \langle p_1 \circ \alpha \rangle$$
$$p_{2*} : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(Y, y_0)$$
$$p_{2*}(\langle \alpha \rangle) = \langle p_2 \circ \alpha \rangle$$

Define

$$\psi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

$$\psi(\langle \alpha \rangle) = (p_{1*}(\langle \alpha \rangle), p_{2*}(\langle \alpha \rangle))$$

which is a homomorphism b/c p_{1*} and p_{2*} are. We must show it's a bijection.

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onto: Given any loops α in X and β in Y, consider $\gamma(s) = (\alpha(s), \beta(s))$, which is a loop in $X \times Y$. By construction, $\psi(\langle\gamma\rangle) = (\langle\alpha\rangle, \langle\beta\rangle)$.

1-1: Since ψ is a homomorphism, it suffices to check that the any thing that gets mapped to the identity is the identity.

Suppose $h(g) = e \Rightarrow g = e$. Note that
$$e = h(e) = h(gg^{-1}) = h(g)h(g^{-1}) \Rightarrow (h(g))^{-1} = h(g^{-1})$$
If $h(g_1) = h(g_2)$, then $e = h(g_1)(h(g_2))^{-1} = h(g_1)h(g_2^{-1}) = h(g_1 g_2^{-1})$.
But then $g_1 g_2^{-1} = e$ and so $g_2 = g$.

So suppose $\psi(\langle\gamma\rangle) = (\langle e_{x_0}\rangle, \langle e_{y_0}\rangle)$, and so $\langle p_1 \circ \gamma\rangle = \langle e_{x_0}\rangle$, $\langle p_2 \circ \gamma\rangle = \langle e_{y_0}\rangle$. This implies $\exists$ path homotopies F and G from $p_1 \circ \gamma$ to e_{x_0} and from $p_2 \circ \gamma$ to e_{y_0}, respectively. But then $H(s,t) = (F(s,t), G(s,t))$ is a path homotopy from γ to $e_{(x_0, y_0)}$. ∎