→ A neighborhood need not be open itself (but it could be).

- **Lemma:** A set U in a topological space (X, τ) is open iff it contains a neighborhood of each of its points.

Proof:

$\Rightarrow$ Let U be open. Then U is itself a nbhd of x, for any $x \in U$.

$\Leftarrow$ Suppose $U \subset X$ contains a nbhd of each of its pts. So for each $x \in X$, $\exists$ N_x nbhd of x and an open O_x s.t. $x \in O_x \subset N_x \subset U$. Then one can check if we set

$$V = \bigcup_{x \in U} O_x$$

open b/c union of open sets.

That $V = U$. ($V \subseteq U$ and $U \subseteq V$)

$\boxed{\text{Lecture 4}}$ →

More or less by definition, open sets are the most important types of sets in topology. A close second is

- **Definition:** A set $C \subseteq X$ is called closed if $X \setminus C$ is open.

(*) **Key Point:** "Open" and "Closed" are not opposites! A set can be open + closed, or neither open nor closed.

Ex: $X = \mathbb{R}$ w/ usual metric topology.

i) $[a,b]$ is closed b/c $X \setminus [a,b] = (-\infty, a) \cup (b, \infty)$ is open.

ii) $[a,b)$ is neither open ~~and~~ nor closed.

Ex: Given any topological space, X and $\emptyset$ are always both open and closed.

Ex: Given X, $p \in X$, and let τ be the particular point topology. Any set $U \neq \emptyset$ s.t. $p \notin U$ is closed.

→ Open sets are characterized by the fact that they contain nbhds of each of their points.

→ Closed sets can be characterized by limit points...

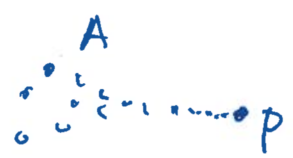
• **Def:** A point $p \in X$ is called a limit point of a set $A \subseteq X$ if $\forall\, \mathcal{O} \in \tau$ s.t. $p \in \mathcal{O}$,

$$\mathcal{O} \cap (A \setminus \{p\}) \neq \emptyset$$

[Note: p need not be in A. It could be that $A \setminus \{p\} = A$.]

· Pictures to keep in mind:

p is on the boundary of A



points of A accumulating at p
(sometimes called accumulation pt.)

• **Example:** $X = \mathbb{R}$ w/ usual topology

i) $A = (a, b)$. a, b are the limit points of A

ii) $A = \{1/n : n = 1, 2, 3, \ldots\}$

Claim: The only limit point of A is 0.

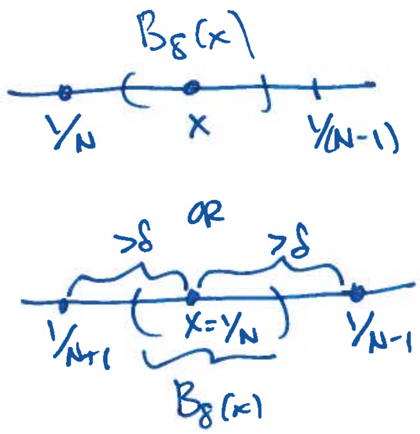
Proof:

• First show 0 is a limit pt. Let $U \in \tau$, $0 \in U$. Then $\exists\ \varepsilon$ s.t. $B_\varepsilon(0) \subset U$. Also, $\exists\ n$ suff large s.t. $1/n < \varepsilon$. Hence

$$\frac{1}{n} \in U \cap \{A \setminus \{0\}\}$$

• Next show that, if x is any other pt in $\mathbb{R}$ then it can't be a limit pt. Let

$$\delta = \tfrac{1}{2} \inf \{ d(x, 1/n) : 1/n \in A,\ \tfrac{1}{n} \neq x \}$$



Then $B_\delta(x)$ is open and

$$B_\delta(x) \cap \{A \setminus \{0\}\} = \phi.$$

$\therefore$ x can't be a limit pt.

iii) $A = \mathbb{Q}$ (rational numbers). All $x \in \mathbb{R}$ are limit points. (Exercise.)

talk about ε's close w/ notion of distance! Stretch/Shrink.

iv) $A = \mathbb{Z}$. There are no limit points. (Exercise: doesn't accumulate anywhere.)

• **Example:** Let X be a set w/ the particular point topology for some $p \in X$. Let $A \subseteq X$ s.t. $p \in A$. Prove that all pts. of X (except p!) are limit points of A.

Proof: Take any $x \in X$, $O \in \tau$ s.t. $x \in O$. Then $p \in O$.

$\therefore$ $O \cap A \setminus \{x\} \ni p$ so $\neq \phi$. ∎

- As promised, the characterisation of a closed set in terms of limit points.

- **Theorem:** A set $C \subseteq X$ is closed iff it contains all its limit points.

Proof:

($\Rightarrow$) Let C be closed and BWOC suppose $p \notin C$ is a limit point. Then $p \in X \setminus C$, which is open, so $\exists$ a nbhd N of p and an open set $O \in \tau$ s.t.

$$p \in O \subset N \subset X \setminus C$$

But this implies

$$O \cap C \setminus \{p\} = \emptyset \quad \lightning$$

$X \setminus C$ is an open set containing p s.t. $(X \setminus C) \cap C = \emptyset$ $\lightning$

($\Leftarrow$) Suppose C contains all its limit points. We must show $X \setminus C$ is open. Let $q \in X \setminus C$, so q is not a limit point of C. Hence, $\exists$ $O \in \tau$ s.t. $q \in O$ and

$$O \cap C \setminus \{q\} = \emptyset$$

$\therefore$ $q \in O \subset X \setminus C$

$\therefore$ O is a nbhd of q

$\therefore$ $X \setminus C$ contains a nbhd of each of its pts.

$\therefore$ $X \setminus C$ is open

$\boxtimes$

→ Recall that, for a topology $\mathcal{T}$, finite intersections of open sets are open and arbitrary unions of open sets are open. For closed sets ...

- Properties of closed sets:

  1) ϕ, X are closed
  2) If C, D are closed, then $C \cup D$ is closed.
  3) If C_α is closed $\forall\, \alpha \in A$, then $\bigcap_\alpha C_\alpha$ is closed.

  Exercise: check these.

$\boxed{\text{Lecture 5}}$

- Def: The closure of a set $A \subseteq X$, denoted by $\bar{A}$ or $cl(A)$, is the union of A with the set of its limit points.

Ex: $X = \mathbb{R}$ with the usual topology.

1) $\overline{(0,1)} = \overline{(0,1]} = \overline{[0,1)} = \overline{[0,1]} = [0,1]$
2) $\overline{\mathbb{Q}} = \mathbb{R}$
3) $\overline{\mathbb{Z}} = \mathbb{Z}$

- Lemma: If C is closed, $\bar{C} = C$.

- Proposition: $\bar{A}$ is closed and it is contained in any closed set that contains A.

15/

Proof:

1) $\bar{A}$ clearly contains all limit pts. of A, but we need to check we haven't created any new ones that aren't in $\bar{A}$. Need to show that $X \setminus \bar{A}$ is open. Let $x \in X \setminus \bar{A}$, so x is not a limit pt. of A. Then $\exists$ open U s.t. $x \in U$ and $U \cap (A \setminus \{x\}) = \phi$. Hence $U \subset X \setminus A$. Also, U can't contain any limit pts of A b/c if p was such a pt, then $p \in U$, $U \cap A = \phi$, which is a contradiction. Thus, $U \subset X \setminus \bar{A}$ and so $X \setminus \bar{A}$ contains a nbhd of each of its pts. Hence $X \setminus \bar{A}$ is open, so $\bar{A}$ is closed.

2) Let C be closed, $A \subset C$. One can check that any limit pt of A is also a limit pt. of C (exercise), and is thus in C. Hence $\bar{A} \subseteq C$.

[p limit pt of A. Let $p \in O$, O open. Then $O \cap A \setminus \{p\} \neq \phi$, and so $O \cap C \setminus \{p\} \neq 0$. Hence p is a limit pt. of C and so in C b/c C is closed.]

- One can check that this proposition implies
  1) $\bar{A}$ is the smallest closed set containing A.
  2) If $\{C_\alpha\}_{\alpha \in A}$ is the collection of all closed sets containing A, then $\bar{A} = \bigcap_\alpha C_\alpha$.
  3) A is closed if and only if $\bar{A} = A$.

- 3 ideas related to closures of sets:

- __Definition:__ A set $A \subseteq X$ s.t. $\bar{A} = X$ is said to be __dense__ (in X).

  __Ex:__ $\mathbb{Q}$ is dense in $\mathbb{R}$ (w/ the usual topology).

- __Definition:__ The interior of a set A, denoted by $\mathring{A}$ or $\mathrm{Int}(A)$, is

$$\mathring{A} = \bigcup_\alpha O_\alpha$$

where $\{O_\alpha\}$ is the collection of all open sets contained in A.

~~Remark: You can check this is equivalent to~~

$$\cancel{\mathring{A} = X \setminus \mathrm{cl}(X \setminus A)}$$

- __Definition:__ The frontier of A, denoted by $\mathrm{Front}(A)$ or ∂A, is

$$\partial A = \bar{A} \setminus \mathring{A}.$$

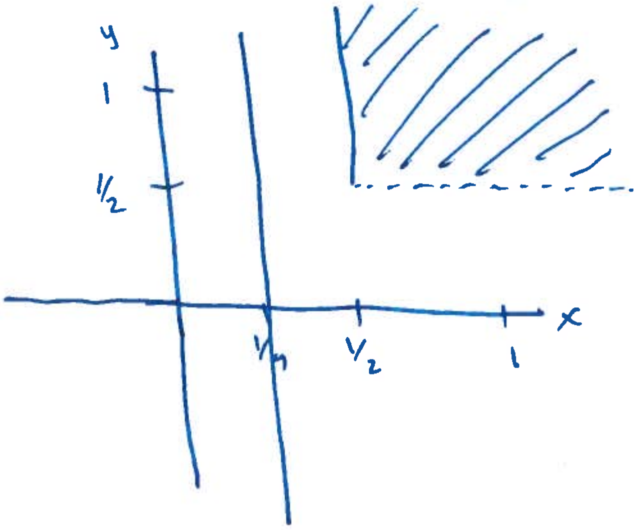
• Examples:

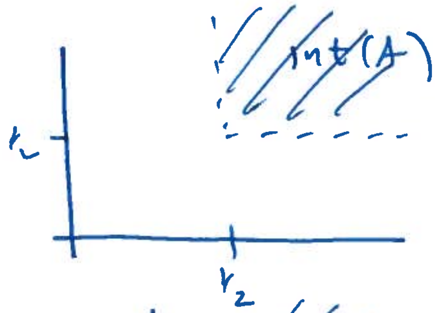
① $X = \mathbb{R}$ w/ usual topology. $A = (a,b]$.

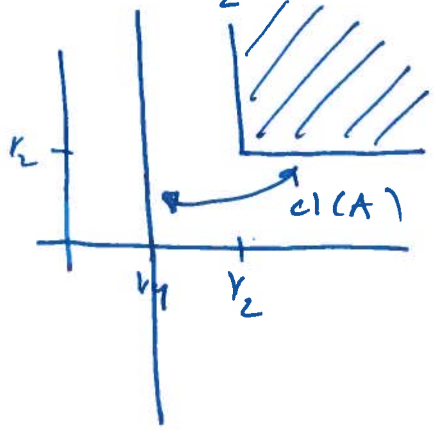
$$\mathring{A} = (a,b) \qquad \bar{A} = [a,b] \qquad \partial A = \{a,b\}$$

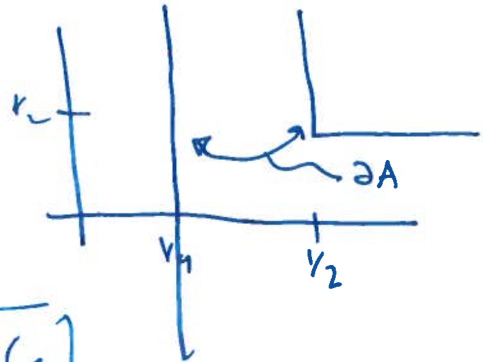
② $X = \mathbb{R}^2$ w/ usual topology,

$$A = \{(x,y) : x = 1/4\} \cup \{(x,y) : x \geq 1/2 \text{ and } y > 1/2\}$$





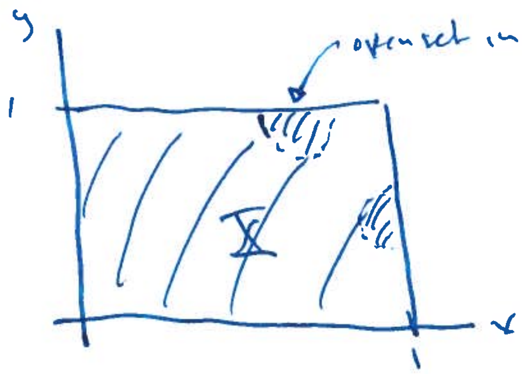




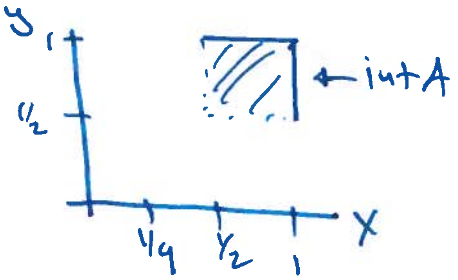
$\boxed{\text{Lecture 6}}$

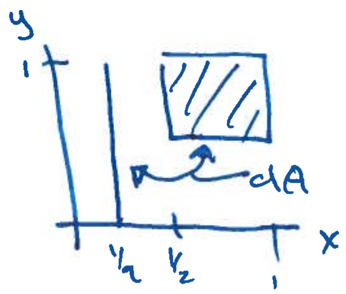
③ $X = [0,1] \times [0,1] \subset \mathbb{R}^2$ w/ subspace topology

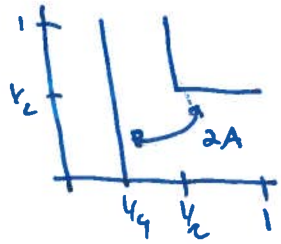
open set in X can include the edges of X!



Same set A







Ex: Let $X = \mathbb{R}$ w/ the finite complement topology.
Let $A = [0,1]$. What is $\mathring{A}$?

Claim: $\mathring{A} = \phi$.

Proof: Let O be any open set s.t. $O \neq \phi$.
Since $X \setminus O$ has finitely many elements, $O \not\subset A$.
(otherwise, $(-\infty, 0) \cup (1, \infty) \subset X \setminus O$.)
$\therefore$ The any open set in A is ϕ.

$$\therefore \mathring{A} = \bigcup_{\alpha} O_\alpha \quad \text{s.t. } O_\alpha \subset A$$
$$= \phi.$$

Ex: Is $\text{int}(\bar{A}) = A$?

No! $A = (a,b]$. $\bar{A} = [a,b]$. $\text{int}(\bar{A}) = (a,b)$.

Ex: If U is open, is $\text{int}(\bar{U}) = U$?

No. X be an ∞ set w/ the finite comp topology.
Take any $x \in X$ and set $U = X \setminus \{x\}$,
which is open. But $\bar{U} = X$, since if
V is any open set contg x, $V \cap U \neq \phi$.
Also, X itself is open, so $\text{int}(\bar{U}) = \text{int}(X) = X$.

$$\therefore \text{int}(\bar{U}) \neq U$$

Ex: $X = \{1,2,3\}$ $\quad \tau = \{\emptyset, \{1\}, \{1,2\}, \{1,3\}, X\}$

a) List all closed subsets.

$\emptyset, X, \{2,3\}, \{3\}, \{2\}$

b) Find $\overline{\{1\}}$. Using a), it is smallest closed subset containing 1, so it is X

b) $\overline{\{2\}} = \{2\}$.

c) $\text{int}(\{2,3\}) = \emptyset$ (contains no open sets)

d) $\partial\{2,3\} = \overline{\{2,3\}} \setminus \text{int}\{2,3\} = \{2,3\} \setminus \emptyset = \{2,3\}$.

- Definition: Let (X, τ_1) and (Y, τ_2) be 2 topological spaces. A function $f: X \to Y$ is continuous if for each $U \in \tau_2$, $f^{-1}(U) \in \tau_1$.

- In words: "A function is continuous if inverse images of open sets are open."

- Ex: Let $\tau_1 = P(X)$ (discrete topology) and τ_2 be arbitrary. Then any function is continuous.

$f^{-1}(U) \in \tau_1$ b/c all subsets are in τ_1

$\boxed{\text{Lecture 7}}$

- Ex: Suppose X and Y are both metric spaces with metrics d_1 and d_2, respectively. Show f is continuous iff given any $x \in X$ and $\varepsilon > 0$, $\exists$ $\delta > 0$ s.t. $d_2(f(x), f(\tilde{x})) < \varepsilon$ $\forall \tilde{x}$ s.t. $d_1(x, \tilde{x}) < \delta$.

↑ depends on ε, x.