Lecture 7

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- **Last time:** $f: (X, \tau_1) \to (Y, \tau_2)$ continuous if $\forall\, U \in \tau_2$, $f^{-1}(U) \in \tau_1$.

- **Ex:** Suppose (X, d_1), (Y, d_2) are metric spaces w/ τ_1, τ_2 metric topologies. Show $f: X \to Y$ is cont. iff $\forall\, x \in X$ and $\varepsilon > 0$ $\exists\, \delta > 0$ s.t. $\forall\, \tilde{x}$ s.t. $d_1(x, \tilde{x}) < \delta$ we have $d_2(f(x), f(\tilde{x})) < \varepsilon$. [Note: δ can depend on x and ε.]

Proof: ($\Rightarrow$) Let f be continuous. Take any $x \in X$, $\varepsilon > 0$. Since $B_\varepsilon(f(x))$ is open in Y, $f^{-1}(B_\varepsilon(f(x)))$ is open in X. Since $x \in f^{-1}(B_\varepsilon(f(x)))$, $\exists\, \delta > 0$ s.t. $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x)))$. But then $\forall\, \tilde{x}$ s.t. $d_1(x, \tilde{x}) < \delta$, $d(f(x), f(\tilde{x})) < \varepsilon$. ✓

($\Leftarrow$) Suppose the converse. Take any open $U \in \tau_2$. Take any $x \in f^{-1}(U)$. Since $f(x) \in U$, $\exists\, \varepsilon$ s.t. $B_\varepsilon(x) \subset U$. Take the corresponding δ, ~~so~~ So we know $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x))) \subset f^{-1}(U)$. But then $B_\delta(x)$ is a nbhd of x in $f^{-1}(U)$, so $f^{-1}(U)$ is open.

- **Prop:** Let (X, τ_1), (Y, τ_2), (Z, τ_3) be topological spaces.

  1) If $f: X \to Y$ and $g: Y \to Z$ are continuous, then so is $g \circ f: X \to Z$.

2) Let $A \subset X$ be any subset w/ the subspace topology. Let $f: X \to Y$ be continuous. Then $f|_A: A \to Y$ is continuous.

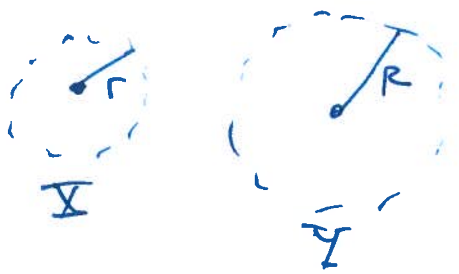
3) The identity map $id: X \to X$, $id(x) = x$, is continuous.

→ We can now define what we mean for 2 topological spaces to be "the same."

• <u>Def:</u> A <u>homeomorphism</u> between X and Y is a continuous function $f: X \to Y$ that is a bijection (1-1 and onto) with a continuous inverse. If such a function exists, we say X and Y are homeomorphic.

<u>Ex:</u> $X = B_r(0)$ in $\mathbb{R}^2$ w/ usual topology.
$Y = B_R(0)$ "

with $R > r > 0$



Stretch X into Y to see they're the same.

• <u>Prove they're homeomorphic:</u> just need to construct an $f: X \to Y$ and show it's a homeo.

<u>Claim:</u> $f(x,y) = \left(\frac{R}{r}x, \frac{R}{r}y\right)$ works!

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Proof:

1) f is cont. Can use ε, δ proof. Given $\varepsilon > 0$, $f(x,y) \in B_R(0)$, we need to find a $\delta > 0$ s.t.

$\forall$ $(\tilde{x},\tilde{y})$ s.t. $d((x,y),(\tilde{x},\tilde{y})) < \delta$,

$d(f(\tilde{x},\tilde{y}), f(x,y)) < \varepsilon$.

$$d(f(\tilde{x},\tilde{y}), f(x,y)) = d\left(\frac{R}{r}(\tilde{x},\tilde{y}), \frac{R}{r}(x,y)\right)$$

$$= \sqrt{\frac{R^2}{r^2}(\tilde{x}-x)^2 + \frac{R^2}{r^2}(x-y)^2} = \frac{R}{r} d((x,y),(\tilde{x},\tilde{y}))$$

$$< \frac{R}{r}\delta \qquad \therefore \text{ take } \delta = \frac{r}{R}\varepsilon.$$

2) f 1-1. Suppose $f((x,y)) = f((\tilde{x},\tilde{y}))$.

Then

$$\frac{R}{r}(\tilde{x},\tilde{y}) = \frac{R}{r}(x,y)$$

$$\therefore (\tilde{x},\tilde{y}) = (x,y) \quad \checkmark$$

3) f onto: given any $(x,y) \in B_R(0)$,

$$f\left(\frac{r}{R}(x,y)\right) = (x,y) \quad \checkmark$$

4) $f^{-1}(x,y) = \frac{r}{R}(x,y)$. Check cont. similar to 1.

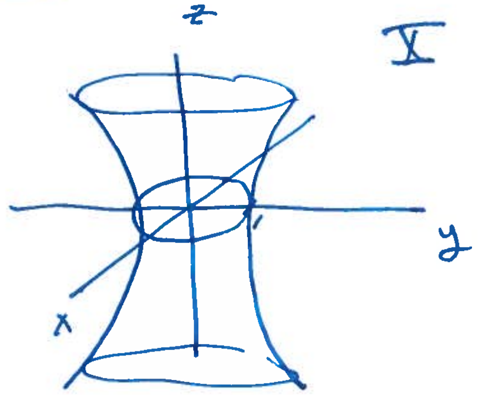
<u>Stretch / shrink disks and see they're the same!</u>

Lecture 8

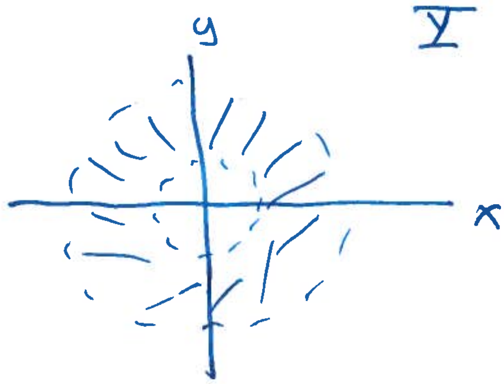
- **Ex:** $X = \mathbb{R}$ and $Y = (-1,1)$ are homeomorphic.
- **Proof:** $f(x) = \tanh x$
  $f^{-1}(y) = \frac{1}{2}\log\left(\frac{1+y}{1-y}\right)$

- **Ex:** The hyperboloid $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2-z^2 = 1\}$ is homeomorphic to the annulus $\{(x,y) \in \mathbb{R}^2 : 1 < x^2+y^2 < 3\}$.

Proof:



2D surface of ∞ extent



2-d bounded (open) region.

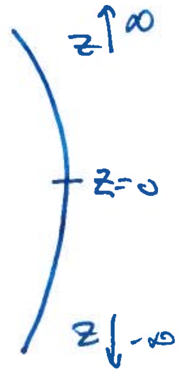
→ hyperboloid in cylindrical coordinates:

$$\{(r,\theta,z) : r^2 - z^2 = 1\}$$

$$g(r,\theta,z) = \left(2 + \frac{z}{1+|z|},\ \theta\right)$$

$$g : X \to Y$$

eg: $\theta = 0$:



$$\Longrightarrow \left(2 + \frac{z}{1+|z|},\ 0\right)$$

↑ part of hyperboloid when $y=0$ so $\theta = 0$, $x>0$

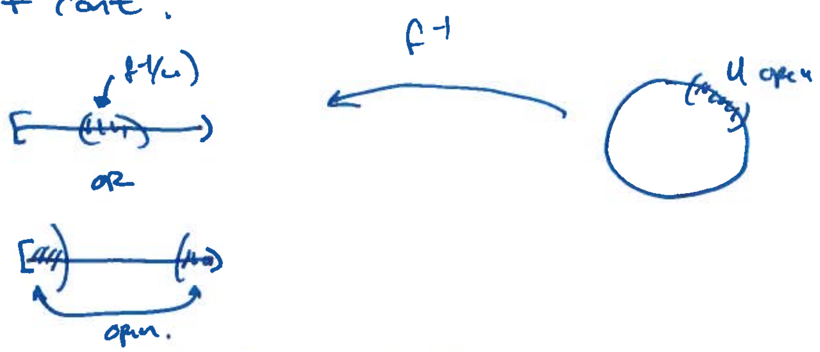
∞ stretching/shrinking!

• **Ex:** Consider $f: [0,1) \to S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$
$x \mapsto e^{2\pi i x}$
subspace topologies

Is f a homeo?

→ Intuition: shouldn't be b/c they're not the same!

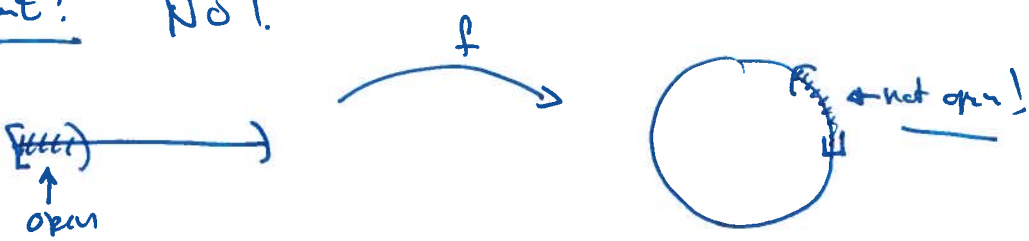
→ Is f cont?



→ f 1-1? $e^{2\pi i x} = e^{2\pi i y}$ Then $x = y + n$, $n \in \mathbb{Z}$.
But $x, y \in [0,1)$ so $x = y$.

→ f onto? Yes; x is like "θ". $e^{2\pi i \theta}$ covers the circle for $\theta \in [0,1)$.

→ Is f^{-1} cont? No!



(Did in lecture 07)

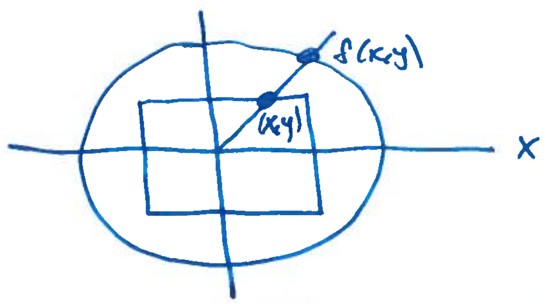
• **Ex:** $X = \mathbb{R}$ w/ discrete topology; $Y = \mathbb{R}$ w/ usual topology.
$f(x) = x$. Is f a homeo?

→ Intuition: Shouldn't be! Different topologies!
$\{x\}$ open in X but $f(\{x\}) = \{x\}$ not open in Y. f^{-1} not

EX: The closed square $S = [-1/2, 1/2] \times [-1/2, 1/2]$ is homeomorphic to the Disk $D = \{(x,y) : 0 \le x^2+y^2 \le 1\}$.

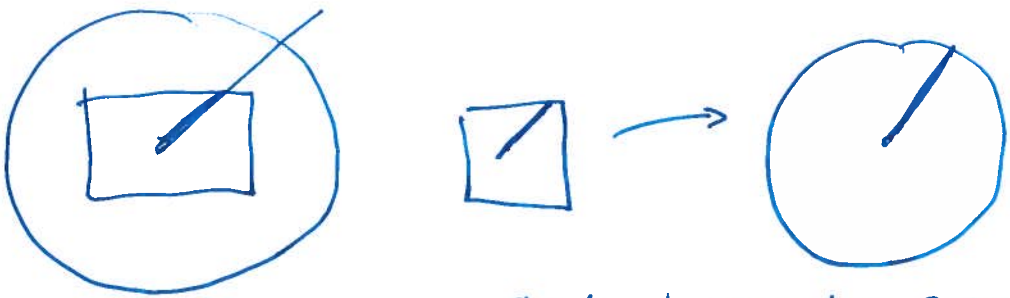
→ We'll construct a homeo in 2 steps.

1) $\forall\ (x,y) \in \partial S$ defined $f(x,y)$ as follows. Draw the line from $(0,0)$ out to (x,y) and continue to ∂D. Set that equal to $f(x,y)$



2) Given any $(x,y) \in \partial S$, $\forall\ \lambda \in [0,1]$ set

$$f(\lambda x, \lambda y) = \lambda f(x,y)$$



Stretch out each line segment to disk.

Note: A similar proof can be used to show that any closed convex region in $\mathbb{R}^2$ is homeomorphic to a closed disk.

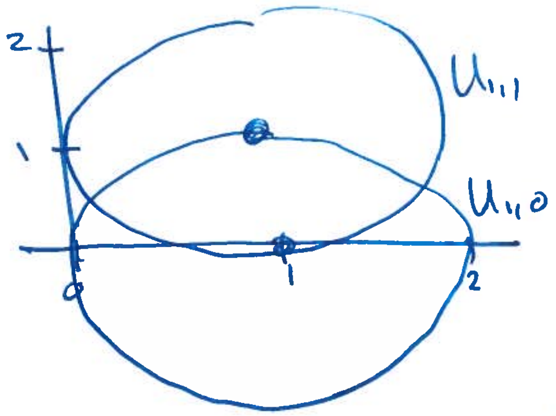
 (any convex region homeo to a disk!)

Theme: Classify objects that are the same, up to homeomorphism. In general this is hard! If we can find some "topological invariants" that differ between the spaces, then they can't be homeomorphic!

- Def: A topological space is compact if, given any family of open sets $\{U_\alpha\}$, $\alpha \in A$, s.t. $X = \bigcup_\alpha U_\alpha$, $\exists$ a finite number of them, $U_{\alpha_1}, \dots U_{\alpha_n}$ s.t. $X = \bigcup_{i=1}^{n} U_{\alpha_i}$. The collection $\{U_\alpha\}$ is called an open cover and $\{U_{\alpha_i}\}_{i=1}^{n}$ is called a finite subcover.

→ X is compact if every open cover has a finite subcover.

Ex: $X = \mathbb{R}^2$. Consider $U_{m,n} = B_1(m,n) = \{(x,y) : d((x,y),(m,n)) < 1\}$, for $m,n \in \mathbb{Z}$.



Then $\{U_{m,n}\}$ is an open cover, but $\nexists$ a finite subcover. If I remove $U_{M,N}$ for some

(M, N), then $(M,N) \notin \bigcup_{M,N} U_{M,N}$

$\therefore$ $\mathbb{R}^2$ w/ the usual topology is not compact.

- Ex: $X = [0,1]$ w/ subspace topology. Consider the open cover

$$[0, 1/10), \; (1/3, 1], \; \left\{ \left(\frac{1}{n+2}, \frac{1}{n} \right) \right\} \quad n \geq 2$$

$$\left(\tfrac{1}{4}, \tfrac{1}{2}\right), \left(\tfrac{1}{5}, \tfrac{1}{3}\right) \ldots$$

An example of a finite subcover is

$$[0, 1/10), \; (1/3, 1], \; \left\{ \left(\frac{1}{n+2}, \frac{1}{n} \right) \right\}_{n=2}^{9}$$

- Remark: This doesn't prove $X = [0,1]$ is compact!

- Theorem: (Heine-Borel Thm) A subset of $\mathbb{R}^n$ (usual topology) is compact iff it is closed and bounded.

This is an impt. theorem that we'll eventually prove. However, we need several intermediate results first.

- Def: A subset ~~of~~ $A \subset X$ if all open covers of A (open in X) have finite subcovers.

- Lemma: A subset $A \subset X$ is compact iff it is compact as a topological space when given the subspace topology.

Proof:

($\Rightarrow$) Let A be compact as a subset of $\mathbb{X}$, and let $\{U_\alpha\}$ be any open cover of A in the subspace topology. Then $\forall\, U_\alpha$, $\exists$ a V_α, open in $\mathbb{X}$, s.t. $U_\alpha = A \cap V_\alpha$. Since $A \subset \bigcup_\alpha V_\alpha$, $\{V_\alpha\}$ is an open cover of A in $\mathbb{X}$, so $\exists$ a finite subcover $\{V_{\alpha_i}\}_{i=1}^n$. But then $\{U_{\alpha_i}\}_{i=1}^n$ is also a finite subcover.

($\Leftarrow$) Exercise