- **Proof:**

($\Rightarrow$) Let A be compact as a subset of X, and let $\{U_\alpha\}$ be any open cover of A in the subspace topology. Then $\forall\, U_\alpha$, $\exists$ a V_α, open in X, s.t. $U_\alpha = A \cap V_\alpha$. Since $A \subset \bigcup_\alpha V_\alpha$, $\{V_\alpha\}$ is an open cover of A in X, so $\exists$ a finite subcover $\{V_{\alpha_i}\}_{i=1}^n$. But then $\{U_{\alpha_i}\}_{i=1}^n$ is also a finite subcover.

($\Leftarrow$) Exercise

Recall: Defined compact sets: every open cover has a finite subcover. Studied Heine-Borel: closed & bdd in $\mathbb{R}^n$ is compact. Preliminary to prove it: $A \subset X$ compact subset/subspace equivalent.

Lecture 10

- **Proposition:** Let $f: X \to Y$ be continuous and let $C \subset X$ be compact. Then $f(C)$ is compact in Y.

Proof: Let $\{U_\alpha\}$ be an open cover of $f(C)$ in Y. Then $f^{-1}(U_\alpha)$ is open for each α, so $\{f^{-1}(U_\alpha)\}$ is an open cover of C in X. Hence, $\exists$ a finite subcover

$$C \subset \bigcup_{i=1}^n f^{-1}(U_{\alpha_i})$$

But this implies $f(C) \subset \bigcup_{i=1}^n U_{\alpha_i}$. $\blacksquare$

- **Key Fact:** Compactness is a topological invariant, meaning if X and Y are homeomorphic, then they are either both compact or neither is compact. This is implied by the above proposition.

$f: X \to Y$ homeo

$\therefore$ f is cont and onto, so X compact implies $f(X) = Y$ is compact.

$\therefore$ f^{-1} is cont and onto, so Y compact implies $f^{-1}(Y) = X$ is compact.

• **Ex:** Let X be any set with the ~~[illegible]~~ discrete topology. Then X is compact iff it is finite.

Proof:

($\Rightarrow$) Let X be compact. Consider the open cover $U_x = \{x\}$ $\forall$ $x \in X$. Since $\exists$ a finite subcover, $X \subset \bigcup_{i=1}^{n} U_{x_i} = \{x_1, \dots, x_n\}$ so it's finite.

($\Leftarrow$) Let X be finite. Let $\{U_\alpha\}$ be any open cover. For each $x \in X$, find an α_x s.t. $x \in U_{\alpha_x}$. Then $\{U_{\alpha_x}\}_{x \in X}$ is a finite subcover.

$\blacksquare$

• **Thm:** A closed subset of a compact space is compact.

Proof: Let $\{U_\alpha\}$ be an open cover of a closed subset $C \subset X$, where X is compact. Notice that

$$X \subset \left(\bigcup_{\alpha} U_\alpha\right) \cup (X \setminus C)$$

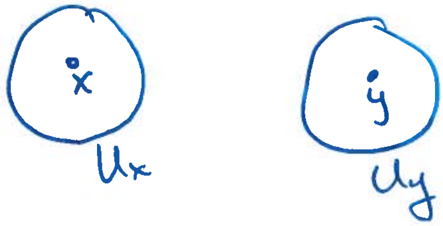
$\uparrow$ open b/c C is closed.

Hence, $\exists$ a finite subcover $\{U_{\alpha_i}\}_{i=1}^{n}$ plus $(X \setminus C)$.

But then $C \subset \bigcup_{i=1}^{n} U_{\alpha_i}$.

$\blacksquare$

- **Def:** A space X is called Hausdorff if, given any $x, y \in X$, $\exists$ disjoint open sets U_x and U_y such that $x \in U_x$, $y \in U_y$.



$\therefore$ Distinct points in a Hausdorff space are separated in this sense.

- **Examples of non-Hausdorff spaces:**

1) A set containing at least two elements w/ the trivial topology.

$x, y \in X$, $x \neq y$. Only open sets are $\emptyset$, X!

2) A set X with the particular point topology.

No two open sets are disjoint (unless one is $\emptyset$).

$\boxed{\text{Lecture 11}}$

- **Ex:** If X is Hausdorff and has finitely many elements, then its topology must be the discrete one.

Proof: Take any $x \in X$. $\forall\, y \neq x$, $\exists$ $U_{y(x)}$ s.t. $x \in U_{y(x)}$ but $y \notin U_{x(y)}$. Consider

$$V = \bigcap_{y \in X} U_{x(y)}$$ ← finite intersection, so open.

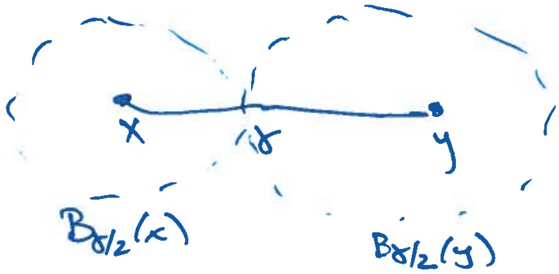
$$= \{x\}$$ ← $\forall\, y \in X$, $y \notin$ at least one set in the intersection $\therefore y \notin V$.

This implies the topology is discrete.

$\blacksquare$

EX: Any metric space w/ the metric topology is Hausdorff.

Proof: $x \neq y \to d(x,y) = \delta > 0$. $B_{\delta/2}(x) \cap B_{\delta/2}(y) = \emptyset$



• **Thm:** A compact subset of a Hausdorff space is closed.

(**Note:** This is not true w/o the Hausdorff assumption. See tut. sheet 4.)

Proof: Let $C \subset X$ be compact and X Hausdorff.

Fix $x \in X \setminus C$ and take any $y \in C$. Since $x \neq y$, $\exists$ disjoint open V_y and U_y containing them. We can find such a U_y $\forall\, y \in C$, so $\{U_y\}$ is an open cover of C. Since C is compact, $\exists$ a finite subcover $\{U_{y_i}\}_{i=1}^n$. Define

$$V = \bigcap_{i=1}^{n} V_{y_i}$$ ← finite intersection of corresponding V_{y_i}'s.

This is open b/c it's a finite intersection. Also, $x \in V$ and $V \cap C = \emptyset$. (This is b/c for any $z \in V$, $z \in V_{y_i}$ $\forall i$, so $z \notin U_{y_i}$ for any i, and $C \subset \bigcup_{i=1}^{n} U_{y_i}$.) Hence, V is a nbhd of $x \in X \setminus C$, and x was arbitrary. Thus $X \setminus C$ is open, so C is closed. ∎

We can now finally prove Heine-Borel!

Thm: (Heine-Borel) A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

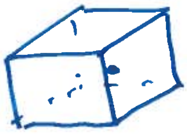
Proof: ($\Rightarrow$) Let $C \subset \mathbb{R}^n$ be compact. Since $\mathbb{R}^n$ has its metric topology, and any metric space w/ more than 1 element in it is Hausdorff, C is closed. Consider the open cover $\{B_n(0) : n = 1, 2, 3, \ldots\}$ of $\mathbb{R}^n$, which is also an open cover of C. $\exists$ a finite subcover, $\{B_{n_i}(0)\}_{i=1}^{m}$, and each ball in this is contained in the largest ball, WLOG $B_N(0)$, $N = n_m$. But then

$$C \subset B_N(0)$$

so it is bounded. ✓

($\Leftarrow$) Let C be closed and bounded. If we can show that C is contained in a compact subset, this will imply C is compact. Consider the n-cube

$$C_M = \{x \in \mathbb{R}^n : -\tfrac{M}{2} \le x_i \le \tfrac{M}{2},\ i = 1, \ldots, n\} \subset \mathbb{R}^n$$

$\mathbb{R}^3$  ← centered at origin w/ sides length M.

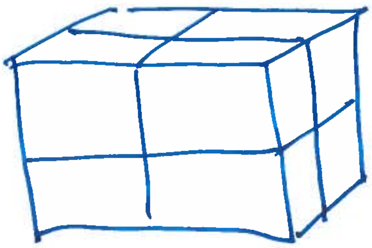
Lecture 12

Need to prove:

Claim: C_M is compact

Since C is bounded, $C \subset C_M$ for M suff large. We'll show C_M is compact.

BWOC: Suppose $\exists$ an open cover $\{U_\alpha\}$ of C_M that doesn't have a finite subcover. Divide C_M into 2^n smaller cubes that have side length $M/2$



$n=3$, get 8 cubes $8=2^3$.

33/

B/c there doesn't exist a finite subcover of C_M, there must exist some subcube of this type (side length $M/2$) that isn't contained in any finite subcover. Call this subcube S_1. Now divide S_1 into subcubes w/ sides length $M/4$ and find a subcube that isn't contained in any finite subcover and call it S_2. Continue to get a sequence $\{S_k\}$ of nested cubes, each having sides of length $M/2^k$.

<u>Claim:</u> $\bigcap_{k=1}^{\infty} S_k = \{x\}$ for some point $x \in C_M$.

Note that, if this claim is true, then since $x \in C_M \subset \bigcup_\alpha U_\alpha$, $\exists$ an α^* s.t. $x \in U_{\alpha^*}$. Since U_{α^*} is open, $\exists$ an $\varepsilon > 0$ s.t. $B_\varepsilon(x) \subset U_{\alpha^*}$. But then if k is suff. large, $x \in S_k \subset B_\varepsilon(x) \subset U_{\alpha^*}$, which is a contradiction.

<u>Proof of Claim:</u> Let I_k be a nested sequence of closed intervals: $I_k = [a_k, b_k]$

$$a_k \leq a_{k+1} < b_{k+1} \leq b_k$$

Such that $\lim_{k\to\infty} |b_k - a_k| = 0$. Since a_k is bdd and increasing, $\exists$ x s.t. $\lim_{k\to\infty} a_k = x$. Also, $x \leq b_k \ \forall k$.

$$\therefore x \in \bigcap_{k=1}^{\infty} I_k$$

$c = \sup_k a_k$ exists (least upper bound)
$\therefore a_k \to c \ldots$

Suppose $y \in \bigcap_{k=1}^{\infty} I_k$ and $x \neq y$. Then $d(x,y) = \varepsilon > 0$ for some ε. But $|a_k - b_k| < \varepsilon/2$ for k suff large. ↯

$$\therefore \{x\} = \bigcap_{k=1}^{\infty} I_k.$$

For a nested sequence of cubes S_k,

$$S_k = \{x \in \mathbb{R}^n : a_k^i \leq x \leq b_k^i\}$$

So we can just apply the above argument for each i to get $x = (x_1, \dots, x_n)$ s.t.

$$\{x\} = \bigcap_{k=1}^{\infty} S_k$$

$$\left(\text{ie } x_i = \bigcap_{k=1}^{\infty} [a_k^i, b_k^i] \text{ for each } i\right).$$

$\boxtimes$

Other facts related to compactness:

• **Thm:** Let $f : X \to Y$ be a continuous bijection, with X compact and Y Hausdorff. Then f is a homeomorphism.

Proof: Need to show f^{-1} is cont. Let Θ be open in X. Then $X \setminus \Theta$ is closed and hence compact. Therefore $f(X \setminus \Theta)$ is compact in Y, and hence closed.
But then

$$f(\Theta) = Y \setminus f(X \setminus \Theta)$$

is open.

$\boxtimes$