Lecture 13

- **Theorem:** (Bolzano - Weierstrass) An ∞ subset of a compact space has a limit point.

(Intuition: can't run off to ∞!)

Proof: Let X be compact and let $S \subset X$ have no limit point. We'll show S is finite. Given any $x \in X$, we can find an open set containing x s.t.

$$\mathcal{O}_x \cap S = \begin{cases} \emptyset & \text{if } x \notin S \\ \{x\} & \text{if } x \in S \end{cases}$$

b/c x is not a limit pt of S. The collection $\{\mathcal{O}_x\}$ is an open cover of X, so has a finite subcover. $X \subset \bigcup_{i=1}^{n} \mathcal{O}_{x_i}$. But each $\mathcal{O}_{x_i}$ contains at most one point of S. ~~[illegible]~~ IE

$$S \subset X \subset \bigcup_{i=1}^{n} \mathcal{O}_{x_i}$$

$$S \cap \mathcal{O}_{x_i} \subset \{x_i\} \quad \forall i$$

$$\therefore S \subset \{x_1, \dots, x_n\}$$

$\therefore S$ is finite.

$\boxtimes$

Now we're going to discuss how to build new topological spaces out of known ones!

• **Ex:** Consider the cylinder

$$X = \{(x,y,z) : x^2+y^2=1, \ 0 \le z \le 1\}$$

We can give X the subspace topology inherited from $\mathbb{R}^3$. Alternatively we can think of it as a product space:

$$X = I \times S^1$$

• **Def:** Given any two sets A and B, the product $X = A \times B$ is the set

$$X = \{(a,b) : a \in A, b \in B\}$$

(just a set. No topology yet.)

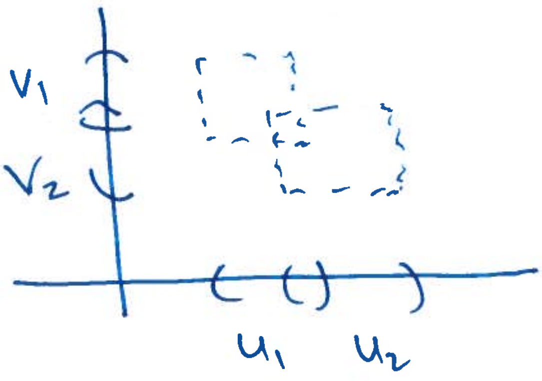
If A and B are topological spaces, with topologies τ_A and τ_B, we can define a topology on X, called the <u>product topology</u>, as follows. Define

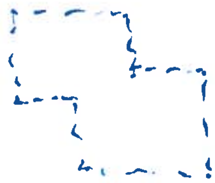
$$\beta = \{U \times V : U \in \tau_A, \ V \in \tau_B\}$$

We say a set Θ is open in the product topology for X if Θ can be written as a union of elements in β, ie if $\exists$ a collection $\{U_\alpha \times V_\alpha\}_{\alpha \in A} \subset B$ s.t.

$$\Theta = \bigcup_\alpha U_\alpha \times V_\alpha$$

• **Remark:** ① β itself is not a topology, b/c the union of sets in β is not necessary in β:





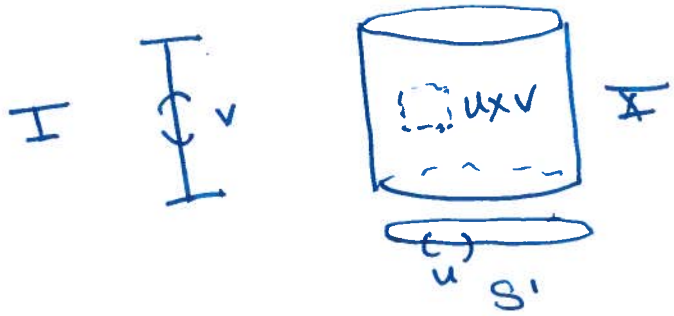
$(U_1 \times V_1) \cup (U_2 \times V_2) \notin \beta$

② β is the basis for the product topology.

③ We can similarly define a product topology for any finite product of topological spaces

$$X = X_1 \times X_2 \times \cdots \times X_n$$

• **Example:** $B = I = [0,1]$ $\quad A = S^1$ unit circle



• **Example:** • $\mathbb{R}^2$ w/ the usual metric topology is homeomorphic to $\mathbb{R}^1 \times \mathbb{R}^1$ w/ the product topology.

• $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^{n-1} \times \mathbb{R}$

- For any n metric spaces (X_i, d_i) $i=1,\dots,n$ the space $X_1\times\cdots\times X_n$ w/ the metric topology defined by the metric

$$d(x,y)=\sqrt{(d_1(x_1,y_1))^2+\cdots+(d_n(x_n,y_n))^2}$$

is homeomorphic to the space $X_1\times\cdots\times X_n$ w/ the product topology.

- The torus is $S^1\times S^1$

→ We'd like some results about when a product space inherits the properties of its components. Eg, if X and Y are compact, is $X\times Y$ compact? We'll prove such things using:

- Def: Given a product space $X\times Y$, the projection maps are

$$p_1: X\times Y\to X \qquad p_1(x,y)=x$$
$$p_2: X\times Y\to Y \qquad p_2(x,y)=y$$

- Thm: If $X\times Y$ has the product topology, then
  1) p_1 and p_2 are continuous
  2) p_1 and p_2 map open sets to open sets
  3) the product topology is the smallest topology (ie contains the fewest open sets) s.t. both p_1 and p_2 are continuous.

Proof:

1) Let U be open in X. Then $p_1^{-1}(U) = U \times Y$, which is in β so open in $X \times Y$. So p_1 is continuous. (Similar proof for p_2.)

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2) Let O be open in $X \times Y$, so $O = \bigcup_\alpha U_\alpha \times V_\alpha$, where each U_α (V_α) is open in X (Y).

~~By (1), $p_1^{-1}(U_\alpha) = U_\alpha \times Y$ and $p_2^{-1}(V_\alpha) = X \times V_\alpha$ are open, so U~~

We have $p_1(O) = \bigcup_\alpha U_\alpha$, which is open in X.

Similar for p_2.

3) Let τ be any other topology on $X \times Y$ for which p_1, p_2 are cont. Let $O = \bigcup_\alpha U_\alpha \times V_\alpha$ be open in the product topology. Since p_1 and p_2 are cont. w.r.t. τ, $p_1^{-1}(U_\alpha) = U_\alpha \times Y$ and $p_2^{-1}(V_\alpha) = X \times V_\alpha$ are open in τ. This implies

$$(U_\alpha \times Y) \cap (X \times V_\alpha) = U_\alpha \times V_\alpha$$

is open in τ, and hence O is open in τ. Hence τ contains all sets open in the product topology. ie τ contains at least as many sets as the product topology.

exercise.

Since the topology on $X \times Y$ is defined in terms of its basis, it would be nice to know that we can prove things just by working w/ basis elements.

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- **Lemma:** Let $f: X \to Y$ be a function between topological spaces and let β be a basis for the topology on Y. f is continuous iff $\forall\ U \in \beta$ $f^{-1}(U)$ is open in X.

Proof: ($\Leftarrow$) V open in Y, so $V = \bigcup_\alpha U_\alpha$, $U_\alpha \in \beta$. Since $f^{-1}(U_\alpha)$ is open $\forall\ \alpha$ and $f^{-1}(V) = \bigcup f^{-1}(U_\alpha)$, $f^{-1}(V)$ is open.
$\therefore$ f is cont.

($\Rightarrow$) Let f be cont. each $U \in \beta$ is open. ✓

☒

- **Thm:** A function $f: Z \to X \times Y$ is continuous iff $p_1 \circ f: Z \to X$ and $p_2 \circ f: Z \to Y$ are continuous.

Proof: ($\Rightarrow$) Composition of cont. functions are continuous.

($\Leftarrow$) Consider $U \times V \in \beta$. Since $(p_1 \circ f)^{-1}(U)$ and $(p_2 \circ f)^{-1}(V)$ are open, so is

$$(p_1 \circ f)^{-1}(U) \cap (p_2 \circ f)^{-1}(V) = f^{-1}(U \times V)$$

Now use the lemma.

$f(z) \in U \times V$
$\therefore$ $p_1 \circ f(z) \in U$
$p_2 \circ f(z) \in V$

- **Lemma:** Let β be a basis for X. X is compact iff every open cover using members of β has a finite subcover.

Proof: ($\Rightarrow$) clear.

($\Leftarrow$) Let $\{U_\alpha\}$ be an open cover. We know $\forall\ \alpha$

$$U_\alpha = \bigcup B_\gamma$$

Let $\tilde{A} = \bigcup_\alpha A(\alpha)$ and notice

$$\mathbb{X} \subset \bigcup_{\gamma\in\tilde{A}} B_\gamma = \bigcup_\alpha U_\alpha$$

is an open cover for $\mathbb{X}$ of basis elements. Hence, $\exists$ a finite subcover.

$$\mathbb{X} \subset \bigcup_{i=1}^{n} B_{\gamma_i}$$

For each i, pick an α_i s.t. $B_{\gamma_i} \subset U_{\alpha_i}$. Then

$$\mathbb{X} \subset \bigcup_{i=1}^{n} U_{\alpha_i}.$$

∎

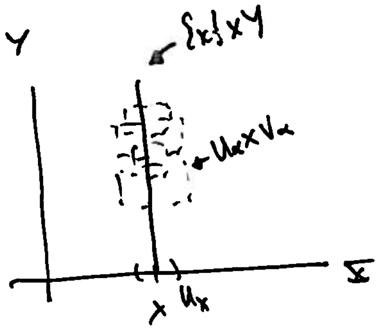
• **Thm:** $X\times Y$ is compact iff both X and Y are compact.

Proof: $(\Rightarrow)$ $\mathbb{X} = p_1(X\times Y)$ $\quad Y = p_2(X\times Y)$

Cont. image of compact spaces are compact.

put in lecture 15

$(\Leftarrow)$ Let $\{U_\alpha \times V_\alpha\}$ be an open cover of basis elements. Fix $x\in\mathbb{X}$. One can check that $p_2 : \{x\}\times Y \to Y$ is a homeomorphism, so $\{x\}\times Y$ is compact. Thus, $\exists$ a finite subcover $\{U^x_{\alpha_i} \times V^x_{\alpha_i}\}_{i=1}^{n_x}$.



Define $U_x = \bigcap_{i=1}^{n_x} U^x_{\alpha_i}$ which is open, $x\in U_x$.

Do this for each x to obtain the collection $\{U_x\}$ which covers $\mathbb{X}$. Let $\{U_{x_j}\}_{j=1}^{m}$ be a finite subcover. But then

$$\left\{ \{U^{x_j}_{\alpha_i} \times V^{x_j}_{\alpha_i}\}_{i=1}^{n_{x_j}} \right\}_{j=1}^{m}$$

is a finite subcover of $\mathbb{X}\times\mathbb{Y}$.

∎

- any compact, $p_1(x \times y) = x$, $p_2(x \times y) = y$ cont. image of compact is compact.

Examples:

1) $S^1 \times I$ is compact. (cylinder)
2) $S^1 \times S^1$ is compact (torus)

knew already from Heine-Borel

3) M-cube C_M: product of n intervals of length N. alt. proof of Heine-Borel.

4) Let $X = [0,1]$ w/ finite comp topology. Then $S^1 \times X$ is compact. (See HW sheet 4.)

→ We've already seen that compactness is a topological invariant. Another topological invariant is "connectedness"

- **Def.** A space X is <u>disconnected</u> if $\exists$ nonempty, disjoint open sets U, V s.t. $U \cup V = X$. If X is not disconnected, it is said to be <u>connected</u>.

Note: The sets U and V in the above def are called a disconnection of X.

EX: • $X = (0,1) \cup (1,2)$ w/ subspace topology is not connected.
• $(0,2)$ is connected.

→ Before giving some more examples, we'll give some equivalent conditions. Basically, a space is connected if it is "all in one piece".

- **Thm:** The following statements are equivalent.
  1) X is connected
  2) The only subsets of X that are both open and closed are X and ϕ

3) There does not exist a continuous function from X to a discrete space consisting of more than one element.

Proof:

1 ⇒ 2: BWOC let $A \subset X$ be open and closed, $A \neq \phi$ and $A \neq X$. Then both A and $X \setminus A$ are open and form a disconnection of the space. ↯

2 ⇒ 3: BWOC. Suppose ∃ a space Y with the discrete topology containing more than one element and a continuous function $f: X \to Y$, with X connected. Since Y has the discrete topology, take $y_1 \in Y$, and $U = \{y_1\}$, $V = Y \setminus \{y_1\}$. These are both open and nonempty, and disjoint. Also $f^{-1}(U)$, $f^{-1}(V)$ are open. But $f^{-1}(V) = X \setminus f^{-1}(U)$, so $f^{-1}(U)$ is both open and closed. ↯

3 ⇒ 1: We'll prove the contrapositive: If X isn't connected, then such a function exists. Let U, V be a disconnection of X and let $Y = \{0, 1\}$ w/ the discrete topology. Then

$$f(x) = \begin{cases} 0 & x \in U \\ 1 & x \in V \end{cases}$$

is continuous.

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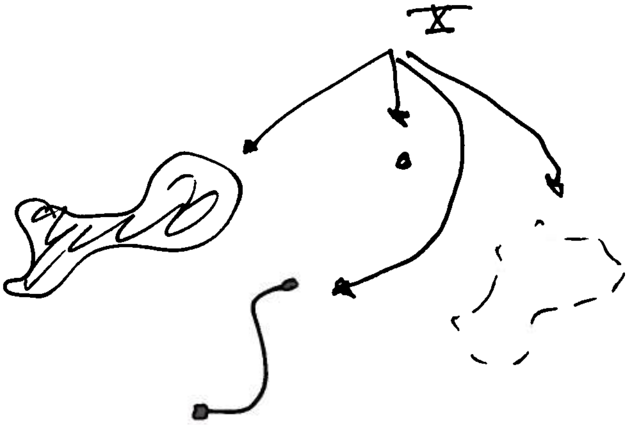
→ Remark: #3 says you can't tear a ~~space~~ connected space into 2 pieces w/ a continuous function.

Examples:

1) Anything like



as a subset in $\mathbb{R}^2$ is not connected b/c there are gaps between things.

2) Let X have the particular point topology. Then X is connected.

→ There do not exist sets that are both open and closed (except $\emptyset$, X) b/c both a set and its complement can't contain p.

→ $X = (0,1) \cup (1,2)$ w/ part. pt. top is connected.

• Thm: A space X is connected iff there do not exist nonempty subsets (not nesc. open) A, B s.t. $A \cup B = X$ and $\bar{A} \cap B = A \cap \bar{B} = \emptyset$.

Proof: exercise.

→ Intuitively, something like an interval should always be connected. Let's prove it.

• Def: An interval is a subset $I \subset \mathbb{R}$ s.t. if $a, b \in I$ then $\forall\, x \in (a,b)$, $x \in I$.