Theorem: (Bolzano-Weierstrass) An $\omega$ subset of a compact space has a limit point.

(Consider: quit run all to $\infty$ !)

Proof: let $X$ be compact and let $S \subset X$ have no limit point. We'll show $S$ is finite. Given any $x \in X$, we can find an open set containing $x$ and:

$$O_x \cap S = \begin{cases} \emptyset & \text{if } x \notin S \\ \{x\} & \text{if } x \in S \end{cases}$$

$bcl x$ is not a limit pt of $S$. The collection $\{O_x\}_x$ is an open cover of $X$, so has a finite subcover. $X = \bigcup_{i=1}^n O_{x_i}$. But each $O_{x_i}$

Looking at most one point of $S$. Hence

$$S \subset X \subset \bigcup_{i=1}^n O_{x_i}$$

$$S \cap O_{x_i} = \{x_i\} \forall i$$

$$\therefore S \subset \{x_1, \ldots, x_n\}$$

$$\therefore S \text{ is finite.}$$

Now we're going to discuss how to build new dipolged spaces out of known ones!
*Example: Consider the cylinder

\[ \mathbb{C} = \{(x, y, z) : x^2 + y^2 = 1, 0 \leq z \leq 1\} \]

We can give \( \mathbb{C} \) the subspace topology inherited from \( \mathbb{R}^3 \). Alternatively, we can think of it as a product space:

\[ \mathbb{C} = I \times S^1 \]

*Definition:* Given any two sets \( A \) and \( B \), the product

\[ \mathbb{X} = A \times B \]

is the set

\[ \mathbb{X} = \{ (a, b) : a \in A, b \in B \} \]

If \( A \) and \( B \) are topological spaces, with topologies \( \tau_A \) and \( \tau_B \), we can define a topology on \( \mathbb{X} \), called the product topology, as follows.

Denote

\[ \mathcal{B} = \{ \mathbb{U} \times \mathbb{V} : \mathbb{U} \in \tau_A, \mathbb{V} \in \tau_B \} \]

We say a set \( \mathbb{O} \) is open in the product topology if \( \mathbb{O} \) can be written as a union of elements in \( \mathcal{B} \), i.e., if \( \mathbb{O} \) is a union of \( \{ \mathbb{U} \times \mathbb{V} \} \), for \( \mathbb{U} \in \tau_A \), \( \mathbb{V} \in \tau_B \).
**Remark:**

1. \( \beta \) itself is not a topology, because the union of sets in \( \beta \) is not necessarily in \( \beta \):

\[
\begin{array}{c}
V_1 \\
V_2
\end{array}
\]

\[(U \times V_1) \cup (U \times V_2) \notin \beta\]

2. \( \beta \) is the basis for the product topology.

3. We can similarly define a product topology for any finite product of topological spaces:

\[
\prod X = X_1 \times X_2 \times \cdots \times X_n
\]

**Example:**

\[
\mathbb{B} = \mathbb{I} = [0,1] \quad \mathbb{A} = S^1 \quad \text{unit circle}
\]

\[
I = I \cup Y
\]

**Example:**

\( \mathbb{R}^2 \) with the usual metric topology is homeomorphic to \( \mathbb{R}^1 \times \mathbb{R}^1 \) with the product topology.

\( \mathbb{R}^n \) is homeomorphic to \( \mathbb{R}^{n-1} \times \mathbb{R} \).
* For any n metric spaces \((X_i, d_i)\) \(i=1,\ldots,n\), the space \(X_1 \times \cdots \times X_n\) with the metric topology defined by the metric

\[
d(x,y) = \sqrt{d_1(x_1, y_1)^2 + \cdots + d_n(x_n, y_n)^2}
\]

is homeomorphic to the space \(X_1 \times \cdots \times X_n\) with the product topology.

* The torus is \(S^1 \times S^1\).

We'd like some results about when a product space inherits the properties of its components. E.g., if \(X\) and \(Y\) are compact, is \(XXY\) compact? We'll prove some such facts using:

* **Def.** Given a product space \(X \times Y\), the projection maps are:

  \[
  p_1: X \times Y \to X
  \]
  \[
  p_1(x,y) = x
  \]
  \[
  p_2: X \times Y \to Y
  \]
  \[
  p_2(x,y) = y
  \]

* Thus: if \(X \times Y\) has the product topology, then

  1) \(p_1\) and \(p_2\) are continuous
  2) \(p_1\) and \(p_2\) map open sets to open sets
  3) the product topology is the smallest topology (it contains the finest open sets) s.t. both \(p_1\) and \(p_2\) are continuous.
1) Let \( U \) be open in \( X \). Then \( p_1^{-1}(U) = U \times Y \), which is in \( \beta \) so open in \( X \times Y \). So \( p_1 \) is continuous. (Same as proof in \( p_2 \).)

2) Let \( \Theta \) be open in \( X \times Y \), so \( \Theta = \bigcup x U_x \), where each \( U_x \) \((V_x)\) is open in \( X \) (Y).

By \( p_1 \) \( p_1^{-1}(\Theta) = \bigcup_x U_x \) and \( p_1^{-1}(V_x) = X \times V_x \) are open for \( \Theta \).

We have \( p_1(\Theta) = \bigcup_x U_x \), which is open in \( X \).

Similarly for \( p_2 \). (Note: \( p_1, p_2 \) are cont. w.r.t. \( \tau \).)

3) Let \( \tau \) be any other topology on \( X \times Y \). Let \( \Theta = \bigcup x U_x \) be open in the product topology. Since \( p_1 \) and \( p_2 \) are cont. w.r.t. \( \tau \),

\( p_1^{-1}(\Theta) = U \times Y \) and \( p_2^{-1}(V_x) = X \times V_x \) are open in \( \tau \). This implies

\[ (U \times Y) \cap (\bigcup x X \times V_x) = U \times V_x \]

is open in \( \tau \), and hence \( \Theta \) is open in \( \tau \).

Since \( \tau \) contains all sets open in the product topology, i.e., \( \tau \) contains at least as many sets as the product topology.

Since the topology on \( X \times Y \) is defined on both of its bases, it would be wise to know that we can prove things just by wary of beingolean.
Lemma: Let $f: X \to Y$ be a function between topological spaces and let $\beta$ be a base in the topology on $Y$. $f$ is continuous iff $A \cup \beta f^{-1}(\beta)$ is open in $X$.

Proof: ($\Rightarrow$) $V$ open in $Y$, so $V = \bigcup U_x$, $U_x \in \beta$. Since $f^{-1}(U_x)$ is open $A \cup \alpha$ and $f^{-1}(V) = \bigcup f^{-1}(U_x)$, $f^{-1}(V)$ is open.

($\Leftarrow$) Let $f$ be cont. each $U_x \in \beta$ is open.

Theorem: A function $f: Z \to X \times Y$ is continuous iff $p_1 f: Z \to X$ and $p_2 f: Z \to Y$ are continuous.

Proof: ($\Rightarrow$) Composition of continuous is continuous.

($\Leftarrow$) Consider $A \cup \beta \in \mathcal{B}$. Since $(p_1 f)^{-1}(A)$ and $(p_2 f)^{-1}(\beta)$ are open, so is $(p_1 f)^{-1}(A) \cup (p_2 f)^{-1}(\beta) = f^{-1}(A \cup \beta)$

Now use the lemma.

Lemma: Let $\beta$ be a base for $X$. $X$ is compact iff every open cover by $\beta$ has a finite subcover.

Proof: ($\Rightarrow$) Clear.

($\Leftarrow$) Let $\mathcal{U}$ be an open cover. We know $A \subset \bigcup U_x$
Let $A = \bigcup_{\alpha} A_\alpha$ and define

$$\hat{X} = \bigcup_{\alpha \in A} B_\alpha = \bigcup_{\alpha} U_\alpha$$

is an open cover of $\hat{X}$ of basic elements. Since $\hat{X}$ has a finite subcover,

$$\hat{X} \subset \bigcup_{i=1}^n B_{\alpha_i}$$

For each $i$, pick an $\alpha_i$ s.t. $B_{\alpha_i} \subset U_{\alpha_i}$. Then

$$\hat{X} \subset \bigcup_{i=1}^n U_{\alpha_i}$$

Thus, $X \times Y$ is compact iff both $X$ and $Y$ are compact.

Proof: 

($\Rightarrow$) $\hat{X} = p_1(X \times Y)$, $Y = p_2(X \times Y)$,

Compact space + compact space is compact.

($\Leftarrow$) Let $\hat{X} \times \hat{Y}$ be an open cover of basic elements. Fix $x \in \hat{X}$. One can check that

$p_2: \hat{X} \times \hat{Y} \rightarrow \hat{Y}$ is a homeomorphism,

so $\hat{X} \times \hat{Y}$ is compact. Thus, $\exists$ a finite subcover $\{U_{\alpha_i} \times V_{\alpha_i} \}_{i=1}^m$.

Define $U_x = \bigcap_{i} U_{\alpha_i}$ which is open, $x \in U_x$.

Do this for each $x$ to obtain the collection $\{U_x \times V_x \}$ which covers $X$. Let $\{U_{x_j} \times V_{x_j} \}_{j=1}^m$ be a finite subcover. But then

$$\{U_{x_j} \times V_{x_j} \}_{j=1}^m$$

is a finite subcover of $\hat{X} \times Y$. 

\[ \square \]
Examples:

1) \( S' \times S' \) is compact (equiv.) \{ previously known \}
2) \( S' \times S' \) is compact (tens.) \{ Baire-Dedek \}
3) \( H \)-code \( C_H \) : product of \( H \) invar. of length \( \infty \).
   alt. proof \{ Baire-Dedek \}.
4) Let \( X \) = \( \{0,1,2\} \) with open basis. Then \( S' \times X \) is compact. (See p. 4, sheet 4.)

\[ \rightarrow \] We've already seen that compactness is a topological invariant.
Another topological invariant is "connectedness."

**Def.** A space \( X \) is **disconnected** if \( X \) nonempty, disjoint open sets \( U, V \) s.t. \( U \cup V = X \). If \( X \)

**Note:** The sets \( U \) and \( V \) in the above def. are called a
disconnection of \( X \).

**Ex:** \( X = \{0,1,2\} \) with subspace topology is not connected.

\[ \rightarrow \] Below giving some examples, we'll give some equivalent
cardings. Basically, a space is connected if
it is "all in one piece."

**Thm:** The following statements are equivalent.

1) \( X \) is connected
2) The only subset of \( X \) that are both open and
closed are \( X \) and \( \emptyset \).
3) There does not exist a continuous function from \( X \) to a discrete space \( Y \) unless more than one element.

**Proof:**

1 \( \rightarrow \) 2: BWOC. Let \( A \subset X \) be open and closed, \( A \neq \emptyset \) and \( A \neq X \). Then both \( A \) and \( X \setminus A \) are open and form a disconnection of the space. \( \ast \)

2 \( \rightarrow \) 3: BWOC. Suppose \( 3 \) a space \( Y \) with a discrete topology, connected with more than one element and a continuous function \( f: X \rightarrow Y \) with \( X \) connected. Since \( Y \) has the discrete topology, take \( y_1, y_2 \in Y \) and \( U = y_1, V = Y \setminus y_2 \). These are both open and nonempty, and disjoint. Also, \( f^{-1}(U), f^{-1}(V) \) are open. But \( f^{-1}(V) = X \setminus f^{-1}(U) \), so \( f^{-1}(U) \) is both open and closed. \( \ast \)

3 \( \rightarrow \) 1: We'll prove the contrapositive: \( f \) is not connected, then such a function exists. Let \( U, V \) be a disconnection of \( X \) and let \( Y = \{0, 1\} \) with the discrete topology. Then

\[
f(x) = \begin{cases} 0 & \text{if } x \in U \\ 1 & \text{if } x \in V \end{cases}
\]

is continuous.

**Remark:** \( \#3 \) says you can't tear a connected space into 2 pieces w/ a continuous function.
Examples:

1) Any set \( A \in \mathbb{R}^2 \) is not connected if there are gaps between its connected components.

2) Let \( \overline{X} \) have the particular point topology. Then \( \overline{X} \) is connected.

   \[ \Rightarrow \text{There do not exist sets } A, B \text{ both open and closed (except } \varnothing, \overline{X} \text{) } \overline{X} \text{ belongs to } \text{one of them.} \]

   \[ \Rightarrow \overline{X} = (0,1) \cup (1,2) \text{ } \text{w/ } \text{point } p \text{- heap is connected.} \]

* Theorem: A space \( \overline{X} \) is connected iff there do not exist nonempty \( \varnothing \)-open subsets \( A, B \) s.t. \( A \cup B = \overline{X} \)

   \[ \text{and } \overline{A \cap B} = \overline{A} \cap \overline{B} = \varnothing. \]

* Proof: Exercise.

   \[ \Rightarrow \text{Intuitively, something like an interval should always be connected: let's prove it.} \]

* Definition: An interval is a subset \( \mathbb{R} \) s.t. if \( a \leq b \), then \( x \in (a, b), \ x \in [a, b] \).