Lecture 16

Def: An interval is a subset $I \subset \mathbb{R}$ s.t. if $a, b \in I$ then $\forall\, x \in (a,b)$, $x \in I$.

- **Thm:** A subset of $\mathbb{R}$ is connected iff A is an interval.
  (in subspace topology.)

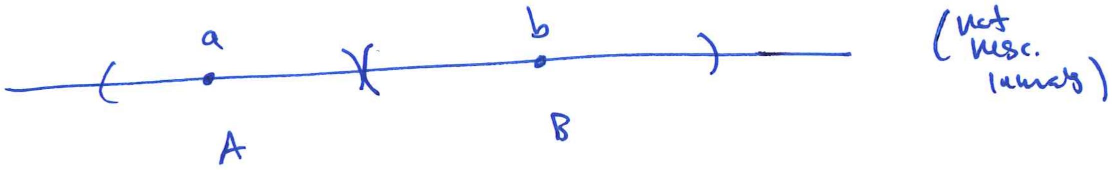
Proof:

($\Longrightarrow$) BWOC suppose $\cancel{A}\ I \subset \mathbb{R}$ is connected but not an interval.
Then $\exists\ a, b \in A$ and $x \in (a,b)$ s.t. $x \notin \cancel{A}\ I$.
But then $(-\infty, x) \cap \cancel{A}\ I$ and $(x, \infty) \cap \cancel{A}\ I$ disconnect A.

($\Longleftarrow$) Let $\cancel{A}\ I$ be an interval and BWOC suppose A and B disconnect A in the sense of the previous theorem. Take $a \in A$, $b \in B$, WLOG $a < b$.

Define

$$S = \sup\{\tilde{a} \in A : \tilde{a} < b\}$$



i) If $s \notin A$, then $s \in B$ and $s \in \bar{A}$ ↯

ii) If $s \in A$ then $s < b$ and $\forall\, y \in (s, b)$, $y \in b$.
But then $s \in \bar{B}$ ↯.

- **Theorem:** The continuous image of a connected set is connected.

Proof: Exercise. (On wk. sheet 6.)

- **Corollary:** Connectedness is a topological invariant!

- **Thm:** Let $Z \subset X$. If Z is connected and dense in X, then X is connected.

Proof: Let $A \subset X$ be both open and closed, $A \neq \phi$ or X. Since it is open and Z is dense, $Z \cap A \neq \phi$. ~~This~~ is ~~b/c for any $z \in Z$, z is a limit pt of Z, so~~ (This is b/c, for any $a \in A$, if $a \in Z$ then $Z \cap A \neq \phi$. If $a \notin Z$, we know a is a limit pt of Z, and A is open and contains a, so $A \cap Z \neq \phi$)

Claim: $Z \cap A$ is both open and closed in Z.

Since Z is connected, this implies $Z \cap A = \phi$ (can't be true) and so $Z \cap A = Z$. Hence, $Z \subseteq A$ and so

$$X = \overline{Z} \subseteq \overline{A} = A$$

$$\therefore X = A \quad \text{↯}$$

Thus X is connected.

Proof of claim: ~~Z is open and closed in itself (at the subspace topology) and~~ A is open in X, so $A \cap Z$ is open in the subspace topology on Z. Also, $X \setminus A$ is open so $(X \setminus A) \cap Z$ is open.

$$(X \setminus A) \cap Z = Z \setminus (A \cap Z)$$

$\boxtimes$

• Corollary: Z being connected implies $\overline{Z}$ is connected.

We'd like to show that the product of connected spaces is connected. We'll need:

• **Thm:** Let $\mathcal{Z}$ be a collection of subsets of X that cover X (not nec. open). If each $Z \in \mathcal{Z}$ is connected and no two sets $Z_1, Z_2 \in \mathcal{Z}$ satisfy $\overline{Z_1} \cap \overline{Z_2} = \emptyset$, then X is connected.

Proof: Let $A \subset X$, $A \neq \emptyset$, be both open and closed. By the argument in the proof of the preceding thm, $A \cap Z$ is both open and closed, so $A \cap Z = \emptyset$ or Z, b/c Z is connected. (True for all $Z \in \mathcal{Z}$.)

i) If $A \cap Z = \emptyset$ $\forall Z \in \mathcal{Z}$, then $A = \emptyset$, since $\mathcal{Z}$ covers X.

ii) Otherwise, $\exists Z \in \mathcal{Z}$ s.t. $A \cap Z = Z$, so $Z \subset A$. Pick another $W \in \mathcal{Z}$. If $W \cap A = \emptyset$, then $W \subset X \setminus A$ and so $\overline{W} \subset \overline{X \setminus A} = X \setminus A$ b/c

exercise, check this

A is open so $X \setminus A$ is closed. Also, $\overline{Z} \subset \overline{A} = A$, so $\overline{Z} \cap \overline{W} = \emptyset$ ↯.

$$\therefore A \cap Z = Z \quad \forall Z \in \mathcal{Z}$$

$$\therefore A = X. \qquad \blacksquare$$

Remark: If two sets Z_1 and Z_2 satisfy $\overline{Z_1} \cap \overline{Z_2} = \emptyset$ we say Z_1 and Z_2 are *separated*.

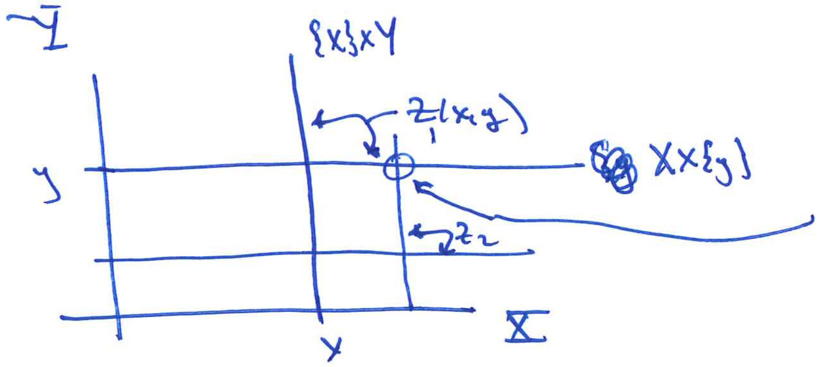
• **Thm:** $X \times Y$ is connected iff X and Y are connected.

Proof:

($\Rightarrow$) Suppose $X \times Y$ is connected. Since $p_1(X \times Y) = X$ and the cont. image of a connected set is connected, X is connected. Similar for Y.

($\Leftarrow$) Let X, Y be connected. Since $\{x\} \times Y$ is homeomorphic to Y, it is connected. Same for $X \times \{y\}$. Define

$$Z(x,y) = (X \times \{y\}) \cup (\{x\} \times Y)$$ ← can check this is connected.



note: $Z_0(x_1,y_1) \cap Z(x_2,y_2) \neq \emptyset$.

Take the family to be $\mathcal{F} = \{Z(x,y) : x \in X, y \in Y\}$.

Now apply the preceding theorem. ∎

- Corollary:
  - $I_1 \times \cdots \times I_n$, a product of intervals is connected.
  - $\mathbb{R}^n$ is connected.

$\boxed{?}$ Is $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ connected?

Should be, but how to prove...?

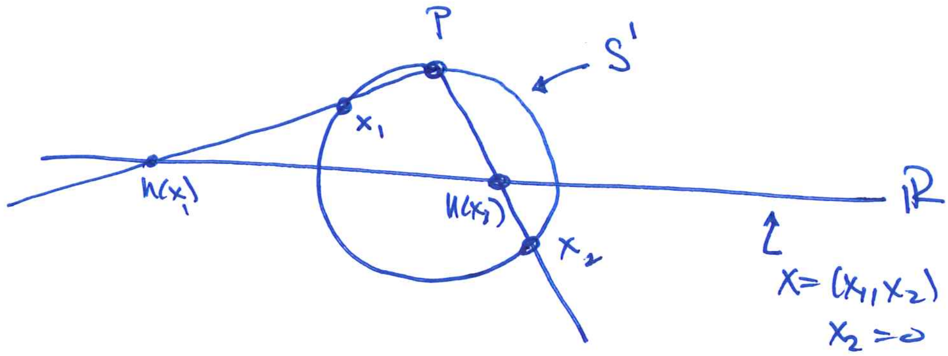
- Proposition: Given any $p \in S^n$, $n \geq 1$, $S^n \setminus \{p\}$ is homeomorphic to $\mathbb{R}^n$.

49

Proof: WLOG $p = (0,0,\dots,0,1)$ (otherwise rotate sphere).

Define $h: S^n \setminus p \to \mathbb{R}^n$ (homeo; can check easy in $\mathbb{R}^2$.)
as follows. Given any $x \in S^n \setminus p$, draw the line between x and p in $\mathbb{R}^{n+1}$. Define $h(x)$ to be the point where this line intersects the set $\{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$

In $\mathbb{R}^2$:



This function h is called the stereographic projection of the sphere. Exercise: check it's a homeo.

[$\ell(t) = tx + (1-t)p \quad t \in \mathbb{R}$ line from x to p

Set t^* s.t. $(1-t^*) = -t^* x_{n+1}$

$$t^* = \frac{1}{1-x_{n+1}}$$

$h(x) = \ell(t^*)$.]

■

- Corollary: $S^n \setminus p$ is connected.
- Corollary: S^n is connected. (Any disconnection of S^n disconnects $S^n \setminus p$.)

→ If a space isn't connected, we can divide it up into connected components.