- Recall equivalence relations:
  - \( x \equiv x \) (reflexive)
  - \( x \equiv y \rightarrow y \equiv x \) (symmetric)
  - \( x \equiv y \text{ and } y \equiv z \rightarrow x \equiv z \) (transitive)

- Use to define equivalence classes

\[
[x] = \{ y \mid y \equiv x \}
\]

- Def: Given a topological space \( X \), define an equivalence relation \( \equiv \) by setting \( x \equiv y \) if \( x \) and \( y \) are connected subspaces (in subspace topology) \( A \subseteq X \) s.t. \( x, y \in A \). The equivalence classes of \( X \) under \( \equiv \) are the components of \( X \).

Example:

\[
X = \{1, 2\} \cup \{3, 4\}
\]

There are two components:

\[
\{1, 2\} \quad \text{and} \quad \{3, 4\}
\]

There are 3 components.
The components of $\overline{X}$ satisfy

1) The components are disjoint and their union is $\overline{X}$.
2) Each nonempty connected subspace in $\overline{X}$ intersects exactly 1 component.
3) Each component is connected.
4) Each component is closed.

Proof: 1, 2 follow from the definition.

3) Let $x_0 \in C$ and $C$ be a component. $\forall x \in C$, find a connected subspace $A_x \subseteq \overline{X}$ s.t. $x_0 \in A_x$. Then $A_x \subseteq C$ by 2) and $C = \bigcup A_x$. Now apply the theorem on a cover of connected sets ($\overline{X}$).

4) We even $C$ connected implies $\overline{C}$ is connected. By 2), $\overline{C}$ intersects 1 component of $C$, so $C = \overline{C}$.

Ex: 

- If $X$ is discrete, each $\{x\}$ is a component.
- If $X$ is connected it has 1 component itself.
- $\mathbb{R}^2 \setminus S^1$ has 2 components.

Def: A space whose components consist of single points is said to be totally disconnected.
Note: A discrete space is totally disconnected, had a totally space need not be discrete.
(See HW Sheet 7.)

A more useful notion of connectedness is path-connected.

**Definition:** Given \( x, y \in X \), a path in \( X \) from \( x \) to \( y \) is a continuous function \( f: [0, 1] \to X \) s.t. 
\( f(0) = x \), \( f(1) = y \). A space \( X \) is path-connected if every pair of points can be joined by a path in \( X \).

\[
\begin{align*}
x &= f(0) \\
y &= f(1)
\end{align*}
\]

Notation: \( f(1-t) = f(1-t) \) is a path from \( y \) to \( x \).

**Naive Question:** How are connectedness and path-connectedness related?

**True:** A path-connected space is connected.

**Proof:** WLOG let \( \Delta X \) be path-connected but not connected and let \( A \subset X \), \( A \neq \emptyset \) or \( X \), be both open and closed. Pick any \( x \in A \), \( y \in X \setminus A \). Let \( \gamma \) be a path between them.
But then $f^{-1}(A)$, $f^{-1}(X \setminus A)$ are both open in $[0,1]$, nonempty, and disjoint + form a disconnection of $[0,1]$. $
abla$

**Key Fact:** The curve is not one!

**Ex:** Tupdepot's Sine Curve

\[ S = \{(x, \sin(\pi x)) \in \mathbb{R}^2 : x \in [0,1] \} \]

\[ \Rightarrow S \text{ is connected b/c its \text{ preimage} at } (0,0). \]

\[ \Rightarrow \therefore S \text{ is connected, and} \]

\[ \overline{S} = S \cup \{(0,y) : y \in [-1,1] \} \]

\[ \therefore \text{One can check that } \overline{S} \text{ is not path connected.} \]

*Lecture 19*

*Del:** We can define an equivalence relation $\sim_x$ if

\[ \exists \text{ a path } x \text{ from } x \text{ to } y. \]

The resulting equivalence classes are called path components.
• **Def.** Path connected sets satisfy
  1) They’re disjoint and their union is X.
  2) Any path-connected, nonempty subspace intersects exactly one path-connected.
  3) They’re path connected.

**Key Fact:** They satisfy properties analogous to connected components, except they need not be closed.

**Ex:** \( S = \{ (x, y) : x \in (0, 1), y = \frac{1}{2}, y = \frac{1}{2x} \} \).

Path connected \( S \) are \( S \) and \( Y \), \( S \) is not closed.
(\( S \) has 1 component, connected, and not itself.)

> Later for the discussion of fundamental groups, we’ll need a notion of...

• **Def.** \( X \) is locally connected at \( y \in X \) if \( A \) contains \( N \) at \( y \).
  - A neighborhood \( N \) at \( y \).
  - \( U \in N \) s.t. \( U \subseteq N \) and \( U \) is connected. If \( U \) includes \( y \), then \( x \) is locally connected. \( X \) is locally
  - path connected at \( y \) in \( X \) if \( A \) contains \( N \) at \( y \).
  - \( U \) of \( y \), \( U \subseteq N \), s.t. \( U \) is path-connected.
  - If \( U \) includes \( y \), then \( X \) is locally path connected.
Ex: \( X = Q \cup [0,1] \)

\( \rightarrow X \) is locally connected at any \( y \in (0,1) \).

\( X = [1,2] \cup [3,4] \)

is locally connected but not connected (also p.p.m.)

Time for identification of quotient spaces ...

Recall: The Möbius Strip is

\[
\begin{align*}
R \xrightarrow{\text{half twist}} & \xrightarrow{\text{glue}} M \\
\end{align*}
\]

? How do we turn \( M \) into a topological space?
How do we "glue" the ends of \( R 
\)
dogether after twisting?

Ex: \( R = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 3, 0 \leq y \leq 1\} \)

Divide up \( R \) using the following equivalence relation: 
\[ (x_1, y_1) \sim (x_2, y_2) \] if

\[ (x_1, y_1) = (x_2, y_2) \]
2) \( x_1 = 0, \ x_2 = 3, \ y_2 = 1 - y_1 \) for any \( y_1 \in [0, 1] \).

The equivalence class 2) is the mathematically rigorous way to glue the ends together with a half twist.

**Question:** How do we define a topology on this space?

→ Define a map \( \pi : R \to M \), where \( M \) is the set of equivalence classes defined by \( \sim \), so that \( \pi(x, y) = [x, y] \). i.e. \( \pi \) sends a point to its equivalence class.

\[ [\pi/0, y] = \pi(3, 1 - y) \]

→ Define a set \( \Theta \subset M \) to be open if

\( \pi^{-1}(\Theta) \) is open in \( R \).

Examples of open sets in \( M \):

\( \pi^{-1}(\Theta) \)
• **Definition:** Let \( \pi: X \rightarrow Y \) be a onto, continuous function between hyperreal spaces. It is called an **ideleically map** or **quochent map** if \( U \) is open in \( Y \) if and only if \( \pi^{-1}(U) \) is open in \( X \).

**Key Point:** This is stronger than continuity.

• **Definition:** Let \( X \) be a hyperreal space and let \( A \) be any set. If \( \pi: X \rightarrow A \) is onto, then \( A \) topology for \( A \) st. It is an **ideleically/quochent map**. This topology is called the **quochent topology** or **ideleically topology**.

Web: \( T_A = \{ U \subset A : \pi^{-1}(U) \text{ is open in } X \} \)

→ In our example, \( H \) is an ideleically space, and \( \pi \) is an ideleically map.
Lecture 20

Note: I must rectify the delusion of locally canceled!

Def: $T$ is locally canceled at $y \in T$ if
A word $N$ of $y$, $T$ a canceled word not $y$
s.t. $U \subseteq N$. (Same for $p_0$ locally proper canceled.)

[Different is a bit subtle... don't worry about it too much.]

---

We studied destroy identity-like spaces

* $\Pi: T \to A$, auto, $T$ typical-like space
  $\tau_A = \{U \cup A : \Pi^{-1}(U) \text{ is open in } T\}$
  is the identity-like topology on $A$, and $\Pi$
  makes $\Pi$ an identity-like map.

**Def:** Let $T$ be a typical-like space and let $X^\ast$ be a collection of closed sets of $T$
where union is $T$; let $\Pi: X \to X^\ast$ send
each $X$ to $\{U \subseteq \{x\} : \Pi^{-1}(U) \text{ is closed in } T\}$ the subset
containing $x$. Then $X^\ast$, together with the
identity-like topology induced by $\Pi$, is
an identity-like (or quotient) space.
Remark: The disjoint subsets partition \( \mathbb{I} \), and one can view \( \mathbb{I} \) as an equivalence relation: \( x \equiv y \) if they're in the same partition element. This is why people often write \( \mathbb{I}/\sim \) instead of \( \mathbb{I}^\ast \) for the associated quotient space.

Note: Sometimes, it will be convenient to define a \( \mathbb{R}^n \) Euclidean space directly via an equivalence relation, and other times via a partition.

Example: Klein Bottle

\[
\begin{array}{c}
\text{\includegraphics[width=\textwidth]{klein_bottle.png}} \\
\text{(Hard to draw - need 4 dimensions!)} \\
\text{(Note: it doesn't really exist itself!)}
\end{array}
\]

Consider \( \mathbb{R} = [0,1] \times [0,1] \) and define equivalence classes via

\[
(I) \ (x_1, y_1) = (x_2, y_2)
\]
(II) \((0, y) \sim (1, y)\)
(III) \((x, 0) \sim (1-x, 1)\)

Then \(\mathbb{R}/\sim\) w/ the associated identically
hyper is the Klein bottle.

Remarks

1) If \(\pi: X \to Y\) and \(\tilde{\pi}: Y \to Z\) are idemrlearly
maps, so is \(\tilde{\pi} \circ \pi: X \to Z\).

2) If \(A\) is a subspace of \(X\) and we set
\(\pi: \mathbb{R} \times A \to X\), \(\pi(x)=x\), Then \(\pi\) is
not necessarily an idemrlear map.

3) If \(X\) is Hausdorff, \(Y\) need not be.

\(\text{Identically maps don't always behave as nicely as you might hope.}\)

\[\text{Ex:}\] \(B^2 = \{ x \in \mathbb{R}^2 : |x| \leq 1 \}, \ S^1 = \{ x \in \mathbb{R}^2 : |x|=1 \}\)

Define an equivalence relatin on \(B^2\) via

(I) \(x_1 \sim x_2\) if \(x_1 = x_2\)
(II) \(x_1 \sim x_2\) if \(x_1, x_2 \in S^1\)

So the curve crossing \(B^2\) is been
identified to a point. \(B^2/\sim\) is a
familiar surface... what is it?
How can we show $B^2/v$ is homeomorphic to $S^2$?

We'll do this via a sequence of theorems...

**Theorem:** Let $\pi : X \rightarrow Y$ be an idenditical map.
Let $g : X \rightarrow Z$ be a continuous function that is constant on each subset of $X$ of the form $\pi^{-1}(y)$, for $y \in Y$. Consider the induced map $f$ defined via $f \circ \pi = g$.

1) $f$ is continuous iff $g$ is continuous.
2) $f$ is an identity map iff $g$ is.

**Proof:** First note that $f(y) = g(\pi^{-1}(y))$

1) $\rightarrow$ Let $f$ be cont. Since $g = f \circ \pi$ is the
   composition of cont. functions, $f$ is continuous.
(⇒) Let $g$ be cont. Let $U \subset Z$ be open, so $g^{-1}(U)$ is open in $X$. One can check
\[ g^{-1}(U) = \pi^{-1}(f^{-1}(U)) \]
Since $\pi$ is an identity map, $f^{-1}(U)$ is open in $Y$.

(⇐) Let $f$ be an identity map. Then so is $g$ because it is the composition of identity maps.

(⇒) Let $g$ be an identity map. So $g$ is cont. and $U \subset Z$ is open iff $g^{-1}(U)$ is open in $X$. By (1)
f is cont. Let $f^{-1}(U)$ be open in $Y$. We must show $U$ is open in $Z$. By (⇒) and (⇒) identity, $g^{-1}(U)$ is open in $X$. Be $g$ ident.

Also, $f$ is cont. by given $x \in Z \Rightarrow$
\[ \pi(\pi^{-1}(f(x))) = f(x) = z. \]