Lecture 18

- Recall equivalence relationship:
  - $x \sim x$ (reflexive)
  - $x \sim y \rightarrow y \sim x$ (symmetric)
  - $x \sim y$ and $y \sim z \rightarrow x \sim z$ (transitive)

- Use to define equivalence classes

$$[x] = \{ y \in X : x \sim y \}$$

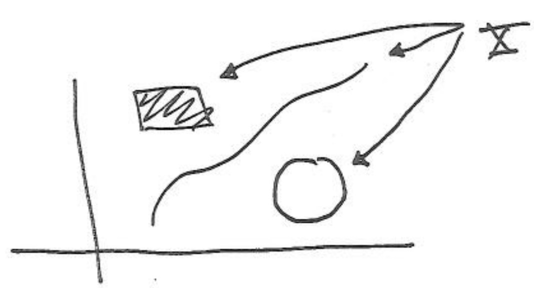
- **Def:** Given a topological space X, define an equivalence relation $\sim$ by setting $x \sim y$ if $\exists$ a connected subspace (in subspace topology) $A \subseteq X$ s.t. $x, y \in A$. The equivalence classes of X under $\sim$ are the components of X.

Ex: • $X = [1,2] \cup [3,4]$

There are two components

$[1,2]$ and $[3,4]$

•



There are 3 components.

- **Thm:** The components of X satisfy
  1) The components are disjoint and their union is X.
  2) Each nonempty connected subspace in X intersects exactly 1 component.
  3) Each component is connected.
  4) Each component is closed.

Proof: 1, 2 follow from the definition.

3) Let $x_0 \in C$ and C be a component. $\forall\, x \in C$, find a connected subspace $A_x \subseteq X$ s.t. $x_0, x \in A_x$. Then $A_x \subseteq C$ by 2) and $C = \bigcup_x A_x$. Now apply the theorem on a cover of connected sets (Z & $\tilde{Z}$.).

4) We know C connected implies $\overline{C}$ is connected. By 2), $\overline{C}$ intersects 1 component only, so $C = \overline{C}$.

∎

Ex:

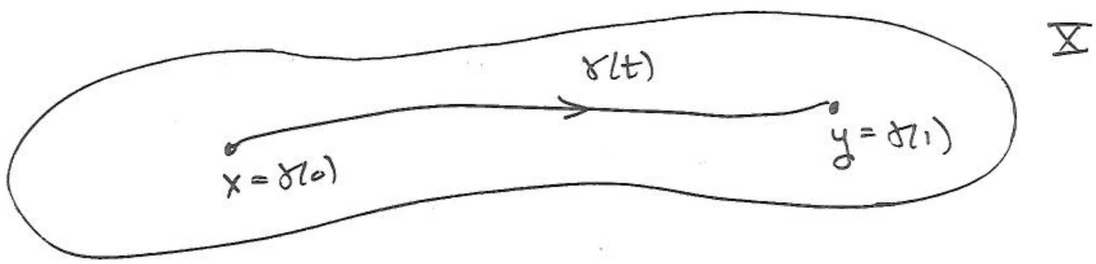
- If X is discrete, each $\{x\}$ is a component.
- If X is connected it has 1 component — itself.
- $\mathbb{R}^2 \setminus S^1$ has 2 components

- **Def:** A space whose components consist of single points is said to be *totally disconnected*.

Note: A discrete space is totally disconnected, but a tot. dis. space need not be discrete. (See tut. sheet 7.)

→ Another useful notion of connectedness is path-connected

- Def: Given $x, y \in X$, a path in X from x to y is a continuous function $\gamma : [0,1] \to X$ s.t. $\gamma(0) = x$, $\gamma(1) = y$. A space X is path-connected if every pair of points can be joined by a path in X.



Notation: $\gamma^{-1}(t) = \gamma(1-t)$ is a path from y to x.

- Natural Question: How are connectedness and path-connectedness related?

- Thm: A path-connected space is connected.

Proof: BWOC let X be path connected but not connected and let $A \subset X$, $A \neq \emptyset$ or X, be both open and closed. Pick any $x \in A$, $y \in X \setminus A$. Let γ be a path between them.

But then $\gamma^{-1}(A)$, $\gamma^{-1}(X \setminus A)$ are both open in $[0,1]$, nonempty, and disjoint + form a disconnection of $[0,1]$. $\lightning$. $\blacksquare$

(*) Key fact: The converse is not true!

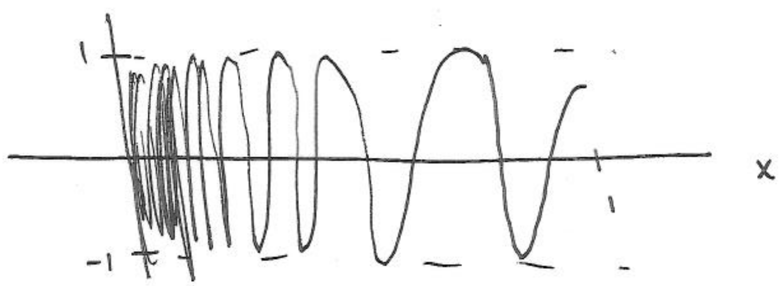
EX: Topologist's Sine Curve

$$S = \{ (x, \sin \tfrac{1}{x}) \in \mathbb{R}^2 : x \in (0,1] \}$$

→ S is connected b/c it's the cont. image of $(0,1]$.

→ $\therefore \overline{S}$ is connected, and

$$\overline{S} = S \cup \{ (0,y) : y \in [-1,1] \}$$



→ One can check that $\overline{S}$ is not path connected.

$\boxed{\text{Lecture 19}}$

- Def: We can define an equivalence relation $x \sim y$ if $\exists$ a path in X from x to y. The resulting equivalence classes are called path-components.

- <u>Thm:</u> Path components satisfy
  1) They're disjoint and their union is X.
  2) Any path-connected, nonempty subspace intersects exactly one path-component.
  3) They're path connected.

(*) <u>Key fact:</u> They satisfy properties analogous to connected components, except they need not be closed!

<u>EX:</u> $S = \{(x, \sin\frac{1}{x}) : x \in (0,1]\}$. $\overline{S} = S \cup Y$, $Y = \{(0,y) : y \in [-1,1]\}$

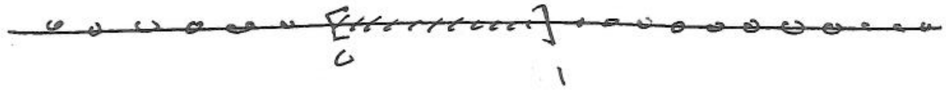
Path components of $\overline{S}$ are S and Y, S is <u>not</u> closed. ($\overline{S}$ has 1 component, connected, and that is itself.)

→ Later, for the discussion of fundamental groups, we'll need a notion of ...

- <u>Def:</u> X is <u>locally connected</u> at $y \in X$ if $\forall$ nbhds N of y, $\exists$ a nbhd U of y s.t. $U \subseteq N$ and U is connected. If this holds $\forall$ $y \in X$ then X is <u>locally connected</u>. X is locally <u>path connected</u> at $y \in X$ if $\forall$ nbhds N of y $\exists$ a nbhd U of y, $U \subseteq N$, s.t. U is path-connected. If this holds $\forall y \in X$ then X is <u>locally path connected</u>.

EX: • $X = \mathbb{Q} \cup [0,1]$

→ X is localy connected at any $y \in (0,1)$.



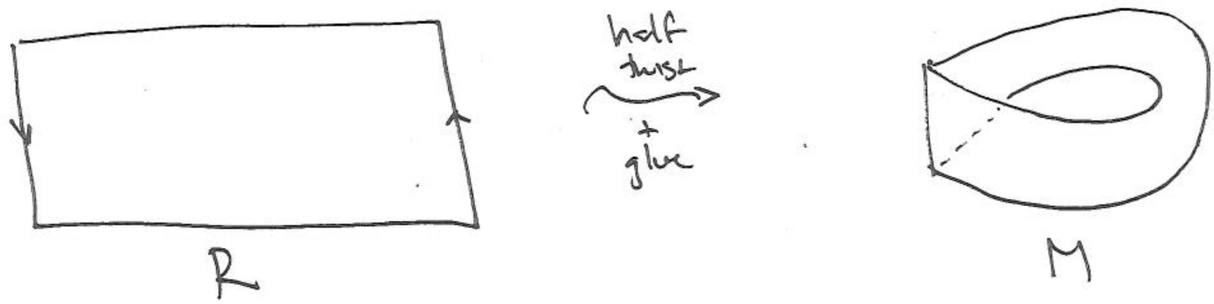
• $X = [1,2] \cup [3,4]$

is localy connected but not connected

(also path.)

Time for Identification, of Quotient, spaces ...

• Recall: The Möbius Strip is



$\boxed{?}$ How do we turn M into a topological space?

How do we "glue" the ends of R

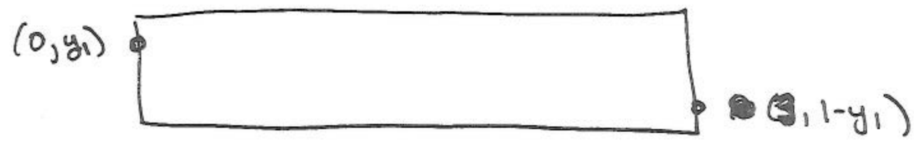
together after twisting?

EX: $R = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 3,\ 0 \le y \le 1\}$

Divide up R using the following equivalence relation: $(x_1,y_1) \sim (x_2,y_2)$ if

I) $(x_1,y_1) = (x_2,y_2)$

11) $x_1 = 0$, $x_2 = 3$, $y_2 = 1 - y_1$ for any $y_1 \in [0,1]$.



The equivalence class 11) is the mathematically rigorous way to glue the ends together with a half twist.

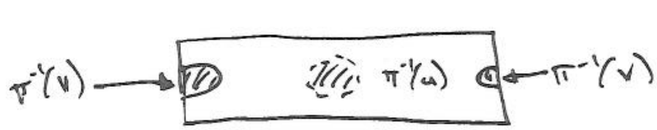
$\boxed{?}$ How do we define a topology on this space?

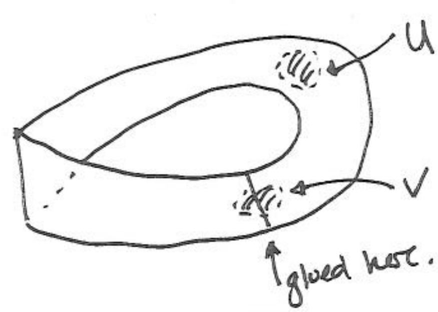
→ Define a function $\pi: R \to M$, where M is the set of equivalence classes defined by $\sim$, so that $\pi(x,y) = [(x,y)]$. ie π sends a point to its equivalence class.

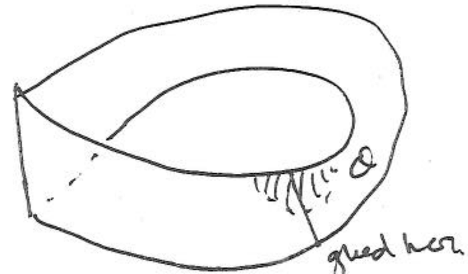
$$[\pi(0,y) = \pi(3,1-y)]$$

→ Define a set $\Theta \subset M$ to be open if $\pi^{-1}(\Theta)$ is open in R.

Examples of open sets in M







- <u>Definition</u>: Let $\pi: X \to Y$ be an onto, continuous function between topological spaces. π is called an <u>identification map</u> or <u>quotient map</u>, if U is open in Y <u>iff</u> $\pi^{-1}(U)$ is open in X.

(*) <u>Key Point</u>: This is stronger than continuity.

- <u>Definition</u>: Let X be a topological space and let A be any set. If $\pi: X \to A$ is onto, then $\exists!$ topology for A s.t. π is an identification/quotient map. This topology is called the <u>quotient topology</u> or <u>identification topology</u>.

~~The space A with the topology is called an identification or quotient space~~

<u>Note</u>: $\tau_A = \{U \subset A : \pi^{-1}(U) \text{ is open in } X\}$

→ In our example, M is an identification space, and π is an identification map.

Lecture 20

Note: I misstated the definition of locally connected!

Def: X is locally connected at $y \in X$ if $\forall$ nbhds N of y, $\exists$ a connected nbhd U of y s.t. $U \subseteq N$. (Same for ~~[illegible]~~ locally path connected.)

[Difference is a bit subtle... don't worry about it too much.]

→ We started discussing identification spaces

- $\pi : X \to A$ onto, X topological space
  $$\tau_A = \{ U \subset A : \pi^{-1}(U) \text{ is open in } X \}$$
  is the identification topology on A, and it makes π an identification map.

- Def: Let X be a topological space and let X^* be a collection of disjoint subsets of X whose union is X. Let $\pi : X \to X^*$ send each pt. to ~~[illegible]~~ the subset containing it. Then X^*, together with the identification topology induced by π, is an identification (or quotient) space.

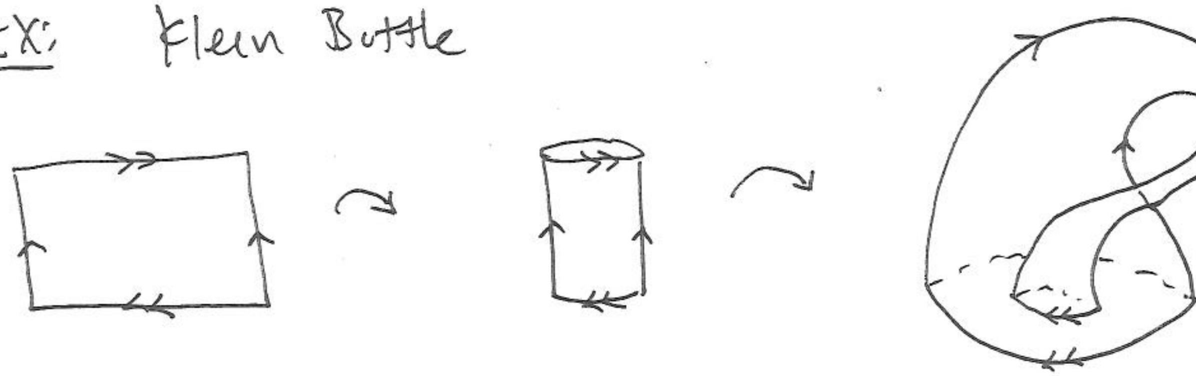
- Remark: The disjoint subsets partition X, and one can view this as an equivalence relation: $x \sim y$ if they're in the same partition element. This is why people often write

$$X/\sim \quad \text{instead of} \quad X^*$$

for the associated identification space.

Note: Sometimes it will be convenient to define an identification space directly via an equivalence relation, and other times via a partition.

Ex: Klein Bottle



(Hard to draw - need 4 dimensions!)
(Note: it doesn't really intersect itself!)

Consider $R = [0,1] \times [0,1]$ and define equivalence classes via

(I) $(x_1, y_1) = (x_2, y_2)$

(II) $(0,y) \sim (1,y)$

(III) $(x,0) \sim (1-x,1)$

Then $R/\sim$ w/ the associated identification topology is the Klein bottle.

- Remarks

1) If $\pi: X \to Y$ and $\tilde{\pi}: Y \to Z$ are identification maps, so is $\tilde{\pi} \circ \pi : X \to Z$.

2) If A is a subspace (ie subspace topology.) of X and we set $\pi: A \subset X \to A$, $\pi(x) = x$, then π is not nescesary an identification map.

3) If X is Hausdorff, Y need not be.

→ Identification maps don't always behave as nicely as you might hope.

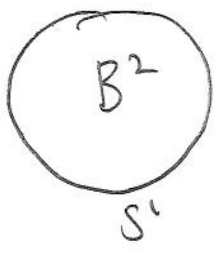
- EX: $B^2 = \{x \in \mathbb{R}^2 : |x| \leq 1\}$, $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$

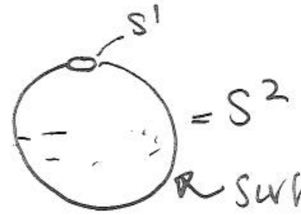
Define an equivalence relation on B^2 via

(I) $x_1 \sim x_2$ if $x_1 = x_2$

(II) $x_1 \sim x_2$ if $x_1, x_2 \in S^1$

So the entire boundary of B^2 as been identified to a point. $B^2/\sim$ is a familiar surface ... what is it?





surface of sphere is B^2 stretched over it!

How can we show $B^2/\sim$ is homeomorphic to S^2?

We'll do this via a sequence of theorems...

- Thm: Let $\pi : X \to Y$ be an identification map. Let $g : X \to Z$ be a continuous function that is constant on each subset of X of the form $\pi^{-1}(y)$, for $y \in Y$. Consider the induced map f defined via $f \circ \pi = g$.

$$
\begin{tikzcd}
X \arrow[d, "\pi"'] \arrow[dr, "g"] & \\
Y \arrow[r, "f"'] & Z
\end{tikzcd}
$$

1) f is continuous iff g is continuous
2) f is an identification map iff g is.

Proof: First note that $f(y) = g(\pi^{-1}(y))$

1) ($\Rightarrow$) Let f be cont. Since $g = f \circ \pi$ is the composite of cont. functions, it is continuous.

($\Leftarrow$) let g be cont. let $U \subset Z$ be open, so $g^{-1}(U)$ is open in X. One can check

$$g^{-1}(U) = \pi^{-1}(f^{-1}(U)) \quad (*)$$

Since π is an identification map, $f^{-1}(U)$ is open in Y. //

2) ($\Rightarrow$) let f be an ident. map. Then so is g b/c it is the composition of ident. maps.

($\Leftarrow$) let g be an ident. map. So g is cont. + onto and $U \subset Z$ is open iff $g^{-1}(U)$ is open in X. By 1) f is cont. Let $f^{-1}(U)$ be open in Y. We must show U is open in Z. By $(*)$ and π identification, $g^{-1}(U)$ is open in X. B/c g ident. U is open in Y. //

$\boxed{Z}$

(Also, f is onto b/c given $z \in Z$ $\exists$ x s.t. $g(x) = z$. But $f(\pi(x)) = g(x) = z$.)