**Theorem:** Let \( \pi : X \to Y \) be an idematically map. Let \( g : X \to Z \) be a continuous function that is constant on each subset of \( X \) of the form \( \pi^{-1}(y) \) for \( y \in Y \). Consider the induced map \( f \) defined by \( f = g \).

1) \( f \) is continuous iff \( g \) is continuous.
2) \( f \) is an idematically map iff \( g \) is an idematically map.

\[ \begin{array}{ccc}
X & \xrightarrow{\pi} & Y & \xrightarrow{f} & Z \\
\downarrow g & & \downarrow f & & \downarrow
g(y) = g(\pi^{-1}(y))
\end{array} \]

**Proof:**

1) \( \implies \) \( f \) is cont. \( \iff \) \( g \) is the comp. of cont. functions. So continuous.

2) \( \iff \) \( g \) is continuous. Let \( U \subset Z \) be open, and note that

\[ g^{-1}(U) = \pi^{-1}(f^{-1}(U)) \] (\( \ast \))

Since \( g^{-1}(U) \) is open and \( \pi \) is an idematically map, \( f^{-1}(U) \) is open in \( Y \).
2) \( \Rightarrow \) Again, comp. of idnt. maps are idnt. maps.

(\(\Leftarrow\)) Let \( f \) be an idnt. map. \( f \) is an idnt. map.

We must show \( f \) is cont. and \( U \subset \mathbb{Z} \) is open \( \Rightarrow f^{-1}(U) \subset \mathbb{Y} \) is open.

**Onto:**

Certainly \( z \in \mathbb{Z} \), \( \exists \ x \in X \) s.t. \( f(x) = z \).

Also, \( \pi(x) = y \) for \( s \in \mathbb{Y} \). Hence

\[
 f(y) = f(\pi(x)) = f(x) = z.
\]

**Open:** \( U \subset \mathbb{Z} \) open \( \Rightarrow f^{-1}(U) \subset \mathbb{Y} \) open by cont.

Let \( f^{-1}(U) \subset \mathbb{Y} \) be open. Since \( \pi \) is idnt.,

\( U \) must be open. Then \( f \) idnt.

\( U \) must be \( U \subset \mathbb{Z} \) open.

A related result:

- **Prop:** Let \( g : X \to \mathbb{Z} \) be onto and continuous. Let \( X^* \) be the collection of partitions elements defined by

\[
 X^* = \{ g^{-1}(z) : z \in \mathbb{Z} \}.
\]

Cover \( X^* \) the idnt. map \( g \).

1) \( g \) induces a continuous bijection \( f : X^* \to \mathbb{Z} \) that

is a homeomorphism if \( g \) is an idnt. map.
b) If \( Z \) is Hausdorff, then so is \( X^\times \).

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & X^\times \\
\downarrow & & \downarrow \phi \\
\pi \circ f & \rightarrow & Z
\end{array}
\]

\[f \circ \pi = g\]
\[f(y) = g(\pi^{-1}(y))\]

**Proof:** Note that we have a natural identification map \( \pi: X \rightarrow X^\times \) that sends a point to its partition element and \( f \) is continuous on subsets of the form \( \pi^{-1}(y) \). Hence the previous theorem applies.

a) 1st Show \( f \) is a cont. bijection: \( f \) is cont. by the proved theorem. It is cont. by the proof of the previous theorem. To see it 1-1, suppose \( y_1 \neq y_2, y_1, y_2 \in X^\times \). Then they are distinct partition elements and so

\[f(y_1) = g(\pi^{-1}(y_1)) \neq g(\pi^{-1}(y_2)) = f(y_2)\]
If $g$ is a homeo. of $X$, then $g$ is cont. WLC $f$ is.

$\leq\Rightarrow$ Let $g : X \to X$ be cont. Need to show $U \subseteq X$ is open.

Consider $f^{-1}(U) = \{ y \in X : f(y) \in U \} = \bigcap_{x \in U} f^{-1}(x)$.

Note that $f^{-1}(U) = \bigcap_{x \in U} f^{-1}(x)$, which is open in $X$.

Since $f \circ g$ is cont., $V \subseteq X$ is open. Since $f^{-1}$ is cont., $(f^{-1})^{-1}(V) = U$ is open in $Z$.

$\Rightarrow\leq$ Let $g : X \to X$ be cont. Need to show $f^{-1}$ is cont. Let $U \subseteq X$ be open. $f^{-1}(U) \subseteq X$ is open by $f$ (idnt. mtp.) and

$$f^{-1}(U) = \bigcap_{x \in U} f^{-1}(x)$$

Hence, $f^{-1}(U)$ is open by $f$ is an idnt. mtp.

---

b) Let $Z$ be Hausdorff and take $y_1, y_2 \in X^*$, $y_1 \neq y_2$.

Then $g(f^{-1}(y_1)) \neq g(f^{-1}(y_2))$ and so $Z$ open.

For $U_1, U_2 \subseteq Z$, disjoint, $g(f^{-1}(y_2)) \subseteq U_1$.

Since $f$ is cont., $f^{-1}(U_1) = V_1$, $f^{-1}(U_2) = V_2$ are open in $X^*$. One can also check they're disjoint and $y_j \in V_j$. 

$\square$