→ We just started discussing:

∘ **Thm:** Let $\pi: X \to Y$ be an identification map. Let $g: X \to Z$ be a continuous function that is constant on each subset of X of the form $\pi^{-1}(y)$ for $y \in Y$. Consider the induced map f defined by $f \circ \pi = g$.

1) f is continuous iff g is continuous
2) f is an identification map iff g is an identification map.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "g"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

$$f(y) = g(\pi^{-1}(y))$$

Proof:

1) $(\Rightarrow)$ If f is cont., then g is the comp. of cont. functions so continuous.

$(\Leftarrow)$ Let g be continuous. Let $U \subset Z$ be open, and note that

$$g^{-1}(U) = \pi^{-1}(f^{-1}(U)) \quad (*)$$

Since $g^{-1}(U)$ is open and π is an identification map, $f^{-1}(U)$ is open in Y. //

63

2) ($\Rightarrow$) Again, comp. of ident. maps are ident. maps.

($\Leftarrow$) Let g be an ident map. f is cont by 1). We must show f is onto and $U \subset Z$ is open iff $f^{-1}(U) \subset Y$ is open.

<u>Onto:</u> Given $z \in Z$, $\exists$ $x \in X$ s.t. $g(x) = z$. Also, $\pi(x) = y$ for some y. Hence

$$f(y) = f \circ \pi(x) = g(x) = z.$$

<u>Open:</u> $U \subset Z$ open $\Rightarrow$ $f^{-1}(U) \subset Y$ open by cont. Let $f^{-1}(U) \subset Y$ be open. Since π is ident., (*) implies $g^{-1}(U)$ is open. Then g ident implies $U \subset Z$ is open.

$\boxed{Z}$

A related result:

• <u>Prop:</u> Let $g: X \to Z$ be onto and continuous. Let X^* be the collection of partition elements defined by

$$X^* = \{ g^{-1}(z) : z \in Z \}$$

Given X^* the identification topology.

a) g induces a continuous bijection $f: X^* \to Z$ that is a homeomorphism iff g is an identification map.

b) If Z is Hausdorff then so is X^*.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "g"] & \\ X^* \arrow[r, "f"] & Z \end{tikzcd}$$

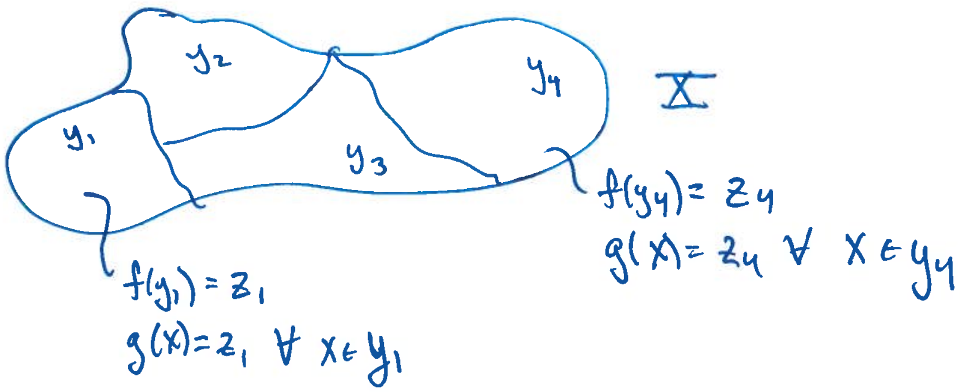
$f \circ \pi = g$

$f(y) = g(\pi^{-1}(y))$

<u>Proof:</u> Note that we have a natural identification map $\pi: X \to X^*$ that sends a point to its partition element and g is constant on subsets of the form $\pi^{-1}(y)$. Hence the previous theorem applies.

a) <u>1st show f is a cont. bijection:</u> f is cont. by the previous theorem. It is onto by the proof of the previous theorem. To see it's 1-1, suppose $y_1 \neq y_2$, $y_1, y_2 \in X^*$. Then they are different partition elements and so

$$f(y_1) = g(\pi^{-1}(y_1)) \neq g(\pi^{-1}(y_2)) = f(y_2)$$



f is a homeo iff g is ident.:

(⇒) Let f be a homeo. g is cont. b/c f is.
Let $g^{-1}(U) \subset X$ be open. Need to show $U \subset Z$ is open.
Consider $f^{-1}(U) = \{y \in X^* : f(y) \in U\} =: V \subset X^*$.
Note that $\pi^{-1}(V) = g^{-1}(U)$, which is open in X. Since π is an ident. map, $V \subset X^*$ is open. Since f^{-1} is cont, $(f^{-1})^{-1}(V) = U$ is open in Z.

(⇐) Let g be an ident map. We need to show f^{-1} is cont. Let $U \subset X^*$ be open. $\pi^{-1}(U) \subset X$ is open b/c π ident. map, and

$$\pi^{-1}(U) = g^{-1}(f(U)) \qquad \text{(see } *)$$

Hence, $f(U)$ is open b/c π is an ident. map.

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Lecture ~~23~~ 22

b) Let Z be Hausdorff and take $y_1, y_2 \in X^*$, $y_1 \neq y_2$.
Then $g(\pi^{-1}(y_1)) \neq g(\pi^{-1}(y_2))$ and so $\exists$ open sets $U_1, U_2 \subset Z$, disjoint, $g(\pi^{-1}(y_i)) \in U_i$.
Since f is cont, $f^{-1}(U_1) = V_1$, $f^{-1}(U_2) = V_2$ are open in X^*. One can also check they're disjoint and $y_j \in V_j$.

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