Lecture 22

→ We're in the middle of proving:

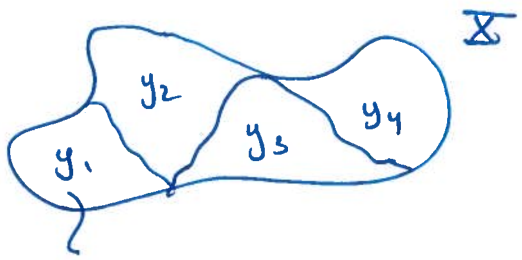
- **Prop:** Let $g: X \to Z$ be onto and continuous. Let X^* be the collection of partition elements defined by
$$X^* = \{ g^{-1}(z) : z \in Z \}$$
Give X^* the identification topology.

a) g induces a continuous bijection $f: X^* \to Z$ that is a homeomorphism iff g is an identification map.

b) If Z is Hausdorff then so is X^*.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "g"] & \\ X^* \arrow[r, "f"] & Z \end{tikzcd}$$

$f \circ \pi = g$



$f(y_1) = z_1$
$g(x) = z_1 \ \forall x \in y_1 \subset X^*$

Proof: a) last lecture.

b) Let Z be Hausdorff and take $y_1, y_2 \in X^*$, $y_1 \neq y_2$. Then $g(\pi^{-1}(y_1)) \neq g(\pi^{-1}(y_2))$ so $\exists$ disjoint open sets $U_1, U_2 \subset Z$ s.t. $g(\pi^{-1}(y_1)) \in U_1$, $g(\pi^{-1}(y_2)) \in U_2$.

Since f is continuous (by a)), $f^{-1}(U_1) = V_1$ and $V_2 = f^{-1}(U_2)$ are open. One can check they're disjoint and $y_1 \in V_1$, $y_2 \in V_2$.

→ Back to our example:

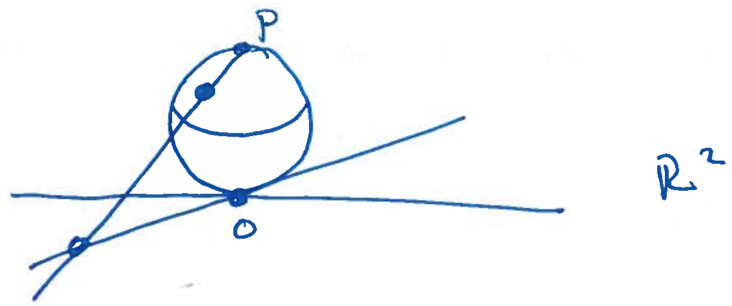
Claim: Consider $B^2/\sim$, $B^2 = \{ x \in \mathbb{R}^2 : |x| \leq 1 \}$, with $x_1 \sim x_2$ if

I) $x_1 = x_2$

II) $x_1, x_2 \in S^1 = \{ x \in \mathbb{R}^2 : |x| = 1$

Then $B^2/\sim$ is homeomorphic to $S^2 = \{ x \in \mathbb{R}^3 : |x| = 1 \}$.

Proof: Recall that for any $p \in S^2$, $S^2 \setminus \{p\}$ is homeomorphic to $\mathbb{R}^3$ via the stereographic projection. Let $h_1 : \mathbb{R}^2 \longrightarrow S^2 \setminus \{p\}$ be such a homeomorphism



Define $h_2 : B^2 \setminus S^1 \to \mathbb{R}^2$ via

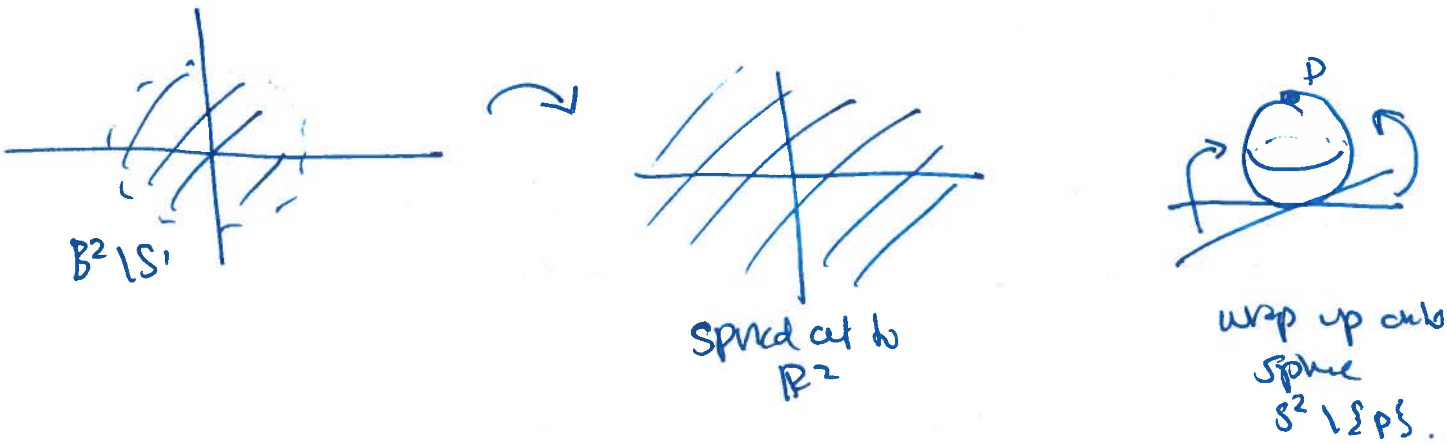
$$h_2(x) = \frac{x}{1 - |x|}$$

One can check that h_2 is a homeomorphism, too.

Consider the function

$$g: B^2 \to S^2$$

$$g(x) = \begin{cases} h_1 \circ h_2(x) & x \in B^2 \setminus S^1 \\ p & x \in S^1 \end{cases}$$



One can check that g is an identification map and so by a) $f: B^2/\sim \to S^2$ is a homeomorphism

$$\begin{array}{ccc} B^2 & & \\ \downarrow \pi & \searrow g & \\ B^2/\sim & \xrightarrow{f} & S^2 \end{array}$$

__Remark:__ Also true in higher dimensions: $B^n/\sim$ is homeomorphic to S^n, where $x_1 \sim x_2$ if I) $x_1 = x_2$ or if II) $x_1, x_2 \in S^{n-1}$.

→ Now we'll start discussing our final tool for deciding if/when spaces are homeomorphic: Fundamental Groups.

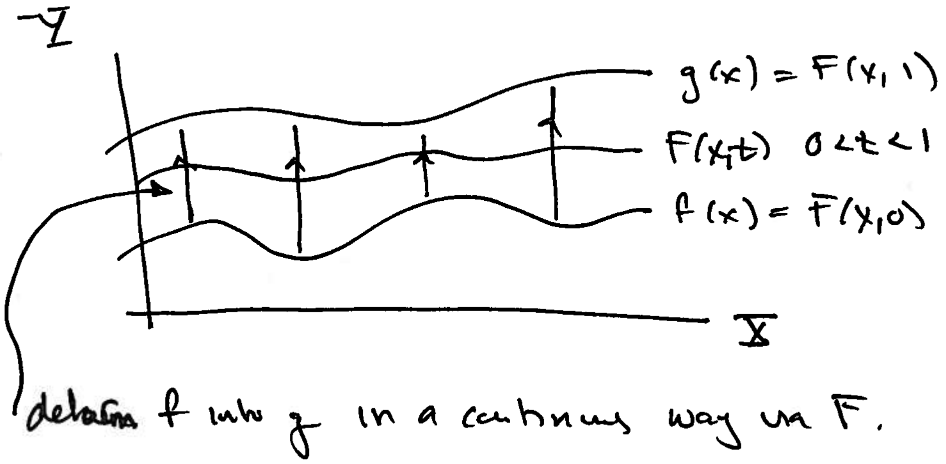
- **Def:** Let $f, g : X \to Y$ be continuous. We say f and g are homotopic if $\exists$ a continuous function

$$F : X \times I \to Y \qquad I = [0,1]$$

such that $F(x,0) = f(x)$, $F(x,1) = g(x)$ $\forall\, x \in X$.

F is called a homotopy between f and g.

Intuitively:



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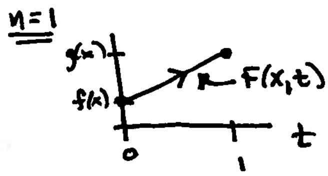
- **EX:** Let $f, g : X \to \mathbb{R}^n$ be continuous, where X is any space. Then the function



$$F(x,t) = (1-t)f(x) + t\,g(x)$$

is a homotopy between them, called a straight line homotopy. (ie for each fixed x, $F(x,t)$ is a straight line in $\mathbb{R}^n$ from $f(x)$ to $g(x)$.)

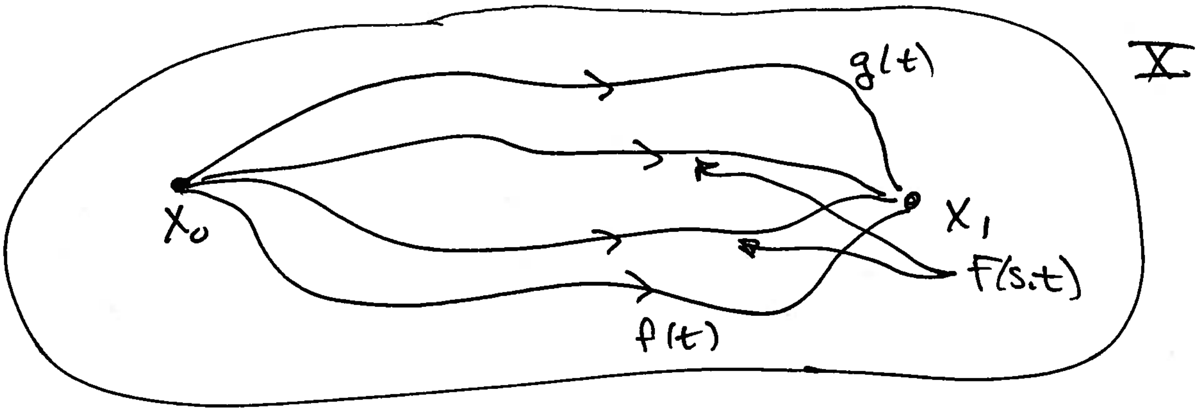
in lecture 22

- **Def:** If $f : X \to Y$ is homotopic to a constant function, $g : X \to Y$, $g(x) = y_0$ $\forall\, x \in X$, we say f is null-homotopic.

→ If f and g are both paths, there is a stronger version of homotopy.

<u>Recall:</u> f is a path from x_0 to x_1 if $f: I \to X$ is continuous with $f(0) = x_0$, $f(1) = x_1$

• <u>Def:</u> Two paths $f, g: I \to X$ are said to be <u>path-homotopic</u> if they have the same beginning and end points and if $\exists$ a function F, called a <u>path-homotopy</u>, such that $F: I \times I \to X$ is continuous with $F(s,0) = f(s)$, $F(s,1) = g(s)$ $\forall\, s \in I$ and $F(0,t) = x_0$, $F(1,t) = x_1$, $\forall\, t \in I$.



<u>Notation:</u> $f \simeq g$ f and g are homotopic
$f \simeq_p g$ f and g are path-homotopic.

• <u>Lemma:</u> Both $\simeq$ and $\simeq_p$ are equivalence relations.

<u>Proof:</u> (for $\simeq_p$ only)

1) Need $f \simeq_p f$. Use $F(s,t) = f(s)$ $\forall\, t \in I$.

2) Need $f \simeq_p g$ implies $g \simeq_p f$. If $F(s,t)$ is a path homotopy from f to g, then $F(s,1-t)$ is a path homotopy from g to f.

3) Need $f \simeq_p g$ and $g \simeq_p h$ implies $f \simeq_p h$. Let F_1 go from f to g and F_2 go from g to h. Define

$$G(s,t) = \begin{cases} F_1(s,2t) & 0 \le t \le 1/2 \\ F_2(s,2t-1) & 1/2 \le t \le 1 \end{cases}$$

which ~~goes~~ is a path homotopy from f to h.

$\blacksquare$

- Notation: We'll use $\langle f \rangle$ to denote the path-homotopy equivalence class of f: $\langle f \rangle = \{g : f \simeq_p g\}$.

→ We'd like to think of the collection of all path-homotopy equivalence classes as a group.

- Recall: A group is a nonempty set G together with an operation $\cdot : G \times G \to G$ such that

(Identity) 1) $\exists$ an identity element $e \in G$ s.t. $g \cdot e = e \cdot g = g \quad \forall g \in G$

(Inverses) 2) $\forall g \in G$, $\exists g^{-1} \in G$ s.t. $g \cdot g^{-1} = g^{-1} \cdot g = e$

(Associative) 3) $\forall f, g, h \in G$, $f \cdot (g \cdot h) = (f \cdot g) \cdot h$

→ To construct our group of path-equivalence classes, we'll focus on a particular type of path called a loop.

- **Def:** A loop is a path that begins and ends at the same point.

$f: I \to X$

$x_0 = x_1$

- **Def:** Given two path-homotopic classes $\langle f \rangle$ and $\langle g \rangle$, where f and g are both loops in X at x_0, define an operation $\bullet$ to be

$$\langle f \rangle \bullet \langle g \rangle = \langle f \cdot g \rangle$$

where $f \cdot g$ is the loop defined by

$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

- **Theorem:** Let G be the collection of all path-homotopy equivalence classes of loops based at a point x_0 in a space X. Then G is a group under the operation $\bullet$.

Proof:

0) Does $\bullet$ map $G \times G$ to G? Yes, by construction.

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1) (Identity) Define $e : I \to X$, $e(s) = x_0 \ \forall s \in I$.

We need to check

$$\langle e \cdot f \rangle = \langle f \cdot e \rangle = \langle f \rangle$$

$$\langle e \rangle \cdot \langle f \rangle \qquad \langle f \rangle \cdot \langle e \rangle$$

So, is $e \cdot f \simeq_p f$?

$$(e \cdot f)(s) = \begin{cases} e(2s) & 0 \le s \le 1/2 \\ f(2s-1) & 1/2 \le s \le 1 \end{cases}$$

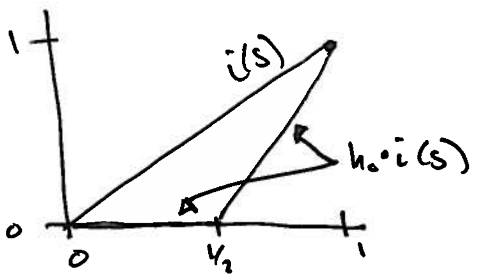
We must construct a homotopy F from $e \cdot f$ to f.

We need to rescale the s-variable.

Let $h_0 : I \to I$, $h_0(s) = 0 \ \forall s \in I$

$i : I \to I$, $i(s) = s \ \forall s \in I$

and so $h_0 \cdot i = \begin{cases} h_0(2s) & 0 \le s \le 1/2 \\ i(2s-1) & 1/2 \le s \le 1 \end{cases}$



Define

$$G(s,t) = (1-t)(h_0 \cdot i)(s) + t\, i(s)$$

which is a path homotopy from $h_0 \cdot i$ to i.

Now define $F(s,t) = f(G(s,t))$. Notice

$$F(s,0) = f(G(s,0)) = f(h_0 \cdot i(s)) = \begin{cases} f(0) = x_0 & 0 \le s \le 1/2 \\ f(2s-1) & 1/2 \le s \le 1 \end{cases}$$

$$= (e \cdot f)(s)$$

$$F(s,1) = f(G(s,1)) = f(i(s)) = f(s)$$

$\therefore F$ is a path-homotopy from $e \cdot f$ to f.

The construction to go from f to $e \cdot f$ is similar.

→ We're in the middle of proving:

- **Thm:** Let G be the collection of all path-~~equivalence~~ homotopy equivalence classes of loops based at a pt x_0 in X. Then G is a group under the operation

$$\langle f \rangle \cdot \langle g \rangle = \langle f \cdot g \rangle$$

$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s-1) & 1/2 \le s \le 1 \end{cases}$$

__Proof.__

1) Identity ✓ last time.

2) __Inverses__: Let $f^{-1}(s) = f(1-s)$. We'll show

$$\langle f \rangle \cdot \langle f^{-1} \rangle = \langle f^{-1} \rangle \cdot \langle f \rangle = \langle e \rangle$$

__Check__: $f \cdot f^{-1} \simeq_p e$

$$(f \cdot f^{-1})(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ f^{-1}(2s-1) = f(2-2s) & 1/2 \le s \le 1 \end{cases}$$

Consider $H(s,t) = (1-t)h_0(s) + t(i \cdot i^{-1})(s)$

And define $F(s,t) = f(H(s,t))$. One can check

$$F(s,0) = f(h_0(s)) = x_0 = e(s)$$

$$F(s,1) = f(i \cdot i^{-1}(s)) = f \cdot f^{-1}(s)$$

$\boxed{\begin{array}{l} \text{recall} \\ h_0(s) \equiv 0 \\ i(s) = s \end{array}}$

$\boxed{i \cdot i^{-1}(s) = \begin{cases} 2s \\ 1-(2s-1) \end{cases}}$

Similar proofs work for $f^{-1} \cdot f$.

3) __Associativity__: Need $(f \cdot g) \cdot h \simeq_p f \cdot (g \cdot h)$.

This proof is similar using the function

$$p(s) = \begin{cases} 2s & 0 \le s \le 1/4 \\ s + 1/4 & 1/4 \le s \le 1/2 \\ \frac{s+1}{2} & 1/2 \le s \le 1 \end{cases}$$

And one can check that

$$(f \cdot g) \cdot h = (f \cdot (g \cdot h)) \circ p$$

composition as functions

$$\simeq_p f \cdot (g \cdot h)$$

∎

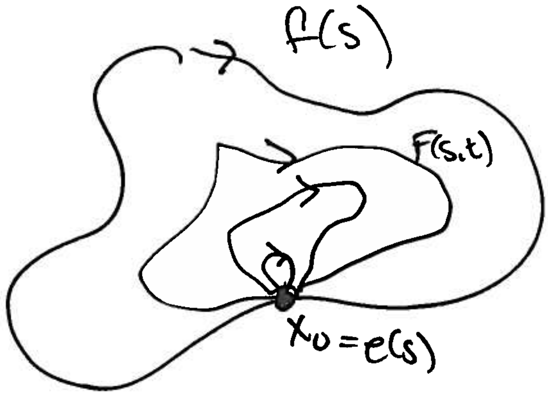
- **Def:** The group G, defined in the above theorem, is denoted by $\pi_1(X, x_0)$ and called the fundamental group of X at x_0.

- **EX:** For any $x_0 \in \mathbb{R}^n$, $\pi_1(\mathbb{R}^n, x_0) = \{\langle e \rangle\}$, ie it is the trivial group consisting only of the identity element. This is b/c, given any loop f at x_0,

$$F(s,t) = (1-t)f(s) + t x_0$$

is a path-homotopy from f to e. So $f \simeq_p e$ $\forall$ loops @ x_0

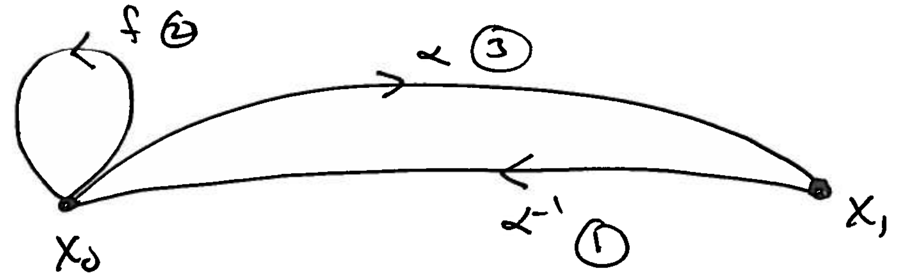
Intuitively: Shrink any loop to a point :



$\boxed{?}$ Given $x_0, x_1 \in X$, how are $\pi_1(X, x_0)$, $\pi_1(X, x_1)$ related?

- **Def:** Given a path α in X from x_0 to x_1, define

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$$
$$\hat{\alpha}(\langle f \rangle) = \langle \alpha^{-1} \rangle \cdot \langle f \rangle \cdot \langle \alpha \rangle$$



→ $\hat{\alpha}$ concatenates the three paths in this order.

→ We want to use $\hat{\alpha}$ to show that $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are "the same".

- **Recall:** If $(G_1, \cdot)$ and $(G_2, *)$ are two groups and $h : G_1 \to G_2$ s.t.

$$h(g_1 \cdot g_2) = h(g_1) * h(g_2) \quad \text{(respects group operations)}$$

then h is called a group homomorphism. If, in addition, h is a bijection, then h is called a group isomorphism.

- **Thm:** $\hat{\alpha}$ is a group isomorphism.

Proof:

① Show the operation is preserved.

$$\hat{\alpha}(\langle f_1\rangle)\cdot\hat{\alpha}(\langle f_2\rangle)$$

$$= \left(\langle\alpha^{-1}\rangle\cdot\langle f_1\rangle\cdot\langle\alpha\rangle\right)\cdot\left(\langle\alpha^{-1}\rangle\cdot\langle f_2\rangle\cdot\langle\alpha\rangle\right)$$

associativity and inverse
$$= \langle\alpha^{-1}\rangle\cdot\left(\langle f_1\rangle\cdot\langle f_2\rangle\right)\cdot\langle\alpha\rangle$$

$$= \hat{\alpha}(\langle f_1\rangle\cdot\langle f_2\rangle)$$

② bijection: we can check that $\hat{\alpha}^{-1}$ exists and is defined by

$$\hat{\alpha}^{-1}(\langle g\rangle) = \langle\alpha\rangle\cdot\langle g\rangle\cdot\langle\alpha^{-1}\rangle$$

which implies $\hat{\alpha}$ is a bijection. ∎

- Corollary: If X is path-connected, then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.

(Can also split up into path components ...)

- Definition: A space X is called simply connected if it is path connected and $\pi_1(X, x_0)$ is trivial for some (and hence all) x_0.

Ex: $\mathbb{R}^n$ is simply connected.