

Burgers Equation

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Synonyms

Burgers' equation; inviscid Burgers equation; viscous Burgers equation.

Mathematic Subject Classification

35K55; 35L60; 35L65; 35Q35.

Short Definition

Burgers equation is the scalar partial differential equation

$$u_t = \nu u_{xx} - uu_x, \tag{B}$$

where $x \in X \subseteq \mathbb{R}$, $t \geq 0$, and $u : X \times \mathbb{R}^+ \rightarrow \mathbb{R}$. The parameter $\nu \geq 0$ is typically referred to as the viscosity, due to the connection between this equation and the study of fluid dynamics. When $\nu > 0$, it is often referred to as the viscous Burgers equation, and when $\nu = 0$ it is often referred to as the inviscid Burgers equation. The constant -1 in front of the term uu_x is due to convention - its exact value is not important, as long as it is nonzero, since it can be adjusted by rescaling space and time.

Description

1 Origin and motivating application

Burgers equation was proposed as a model of turbulent fluid motion by J. M. Burgers in a series of several articles, the results of which are collected in [Burgers, 1948]. Although (B) is a special case of the system he originally described, it is this equation that has come to be known as Burgers equation. It is important in a variety of applications, perhaps most notably as a simplification of the Navier-Stokes equation, which models fluid dynamics. In addition, (B) is used as a prototypical PDE to rigorously develop, in a relatively simple setting, many of the fundamental tools used to analyze general classes of PDEs. For example, when $\nu = 0$ Burgers equation is one of the simplest nonlinear conservation laws [\[link\]](#), and when $\nu > 0$ it is one of the simplest nonlinear dissipative PDEs, due to the resulting decay of energy. With the addition of stochastic forcing, it has played an important role in the theoretical development of stochastic PDEs [\[link\]](#) [E et al., 2000].

Moreover, Burgers equation appears as a normal form, meaning that it describes the behavior, at least qualitatively, of a much larger class of equations. For example, it arises in the study of pattern formation [\[link\]](#), in the context of modulations of spatially-periodic waves [\[Doelman et al., 2009\]](#). Furthermore, the diffusion wave and viscous rarefaction wave, described below, can be used to characterize the large-time behavior of more general scalar viscous conservation laws [\[Liu, 2000\]](#). This is related to the fact that the term uu_x is critical, in the sense that it lies on the boundary between nonlinear terms that cause blow-up and those whose effect can be absorbed by the diffusive decay induced by the term u_{xx} [\[Bricmont et al., 1994\]](#).

2 Behavior of solutions

The behavior of solutions to (B) and the mathematical tools used in its analysis depend upon whether one considers the inviscid ($\nu = 0$) or viscous ($\nu > 0$) case. Only the key properties are summarized here. Technical details are avoided to the extent possible, and the focus is on the domain $X = \mathbb{R}$, which is the most widely studied. For a concise yet more detailed account of both the inviscid and viscous cases, within the context of conservation laws, see [\[Liu, 2000\]](#). For more information on the rigorous PDE theory that is relevant for the two cases, see [\[Evans, 1998\]](#) and [\[Henry, 1981\]](#), respectively.

2.1 Inviscid case

When $\nu = 0$, Burgers equation is a nonlinear hyperbolic conservation law [\[link\]](#). A key property of solutions is that they can develop discontinuities and, as a result, the derivatives that appear in equation (B) are not well-defined in the usual sense. Therefore, to make the following statements rigorous, the theory of weak solutions, meaning functions that solve an integral form of Burgers equation, is required.

For a large class of initial data, the resulting behavior is determined by phenomena referred to as shocks and rarefaction waves. The simplest such setting is if the initial data is $u(x, 0) = u_-$ if $x < 0$ and $u(x, 0) = u_+$ if $x > 0$, known as the Riemann problem [\[link\]](#). The Lax entropy condition then states that, if $u_- > u_+$, the solution is then given by the discontinuous shock

$$u_{\text{shock}}^0(x, t) = \begin{cases} u_- & \text{if } x < st \\ u_+ & \text{if } x > st \end{cases}, \quad s = \frac{1}{2}(u_+ + u_-),$$

where the speed s is determined in relation to the size of the discontinuity and the nonlinearity by the Rankine-Hugoniot condition. If instead $u_- < u_+$, the solution is the continuous rarefaction wave

$$u_{\text{rarefaction}}^0(x, t) = \begin{cases} u_- & \text{if } x < u_- t \\ x/t & \text{if } u_- t < x < u_+ t \\ u_+ & \text{if } u_+ t < x \end{cases}.$$

When $u_- = u_+$ the solution is constant. If the initial condition is more complicated, then the solution will evolve towards an appropriate combination of shocks and rarefaction waves, and may also involve another explicit solution known as an N-Wave, due to its resemblance of an (upside-down) N.

2.2 Viscous case

When $\nu > 0$, (B) is an example of a nonlinear dissipative equation. For a large class of initial data solutions exist and are smooth. Roughly speaking, their behavior will be determined by whether or not the initial data is localized: $\lim_{x \rightarrow \pm\infty} u(x, 0) = 0$. If this holds with sufficiently fast convergence, the solution will approach as $t \rightarrow \infty$ the explicit solution known as the Burgers kernel, or diffusion wave

$$G(x, t; M) = \frac{\frac{M}{\sqrt{4\pi\nu t}} e^{-\frac{x^2}{4\nu t}}}{1 - \frac{1}{2\nu} \int_{-\infty}^x \frac{M}{\sqrt{4\pi\nu t}} e^{-\frac{y^2}{4\nu t}} dy}, \quad M = 2\nu \left(1 - e^{-\frac{1}{2\nu} \int_{\mathbb{R}} u(x, 0) dx} \right),$$

which is essentially a nonlinear Gaussian. Similarly if $\lim_{x \rightarrow \pm\infty} u(x, 0) = u_\infty$, then the solution will approach the sum of a diffusion wave and the constant u_∞ . If instead $\lim_{x \rightarrow \pm\infty} u(x, 0) = u_\pm$ with sufficiently fast convergence, the solution will approach a smooth version of the rarefaction wave or the shock, with the Lax entropy condition again determining which will emerge. The viscous shock is given explicitly by

$$u_{\text{shock}}^\nu(x, t; x_0) = (u_+ + u_-)/2 - ((u_+ - u_-)/2) \tanh[(u_+ - u_-)(x - st - x_0)/(4\nu)], \quad u_- > u_+,$$

where the speed s is as defined above and the position x_0 is chosen so that $\int_{\mathbb{R}} [u(x, 0) - u_{\text{shock}}^\nu(x, 0; x_0)] dx = 0$. An explicit formula for the viscous rarefaction wave also exists, but it is more involved [Liu, 2000]. In all cases, the fact that mass is conserved, $\int_{\mathbb{R}} u(x, t) dx = \int_{\mathbb{R}} u(x, 0) dx$, plays an important role in the dynamics.

One way to derive these, as well as other, results is via the change of variables

$$U(x, t) = e^{-\frac{1}{2\nu} \int_{-\infty}^x u(y, t) dy}, \quad u(x, t) = -2\nu \partial_x \log[U(x, t)],$$

which is referred to as the Hopf-Cole (or Cole-Hopf) transformation [Hopf, 1950, Cole, 1951]. As long as the transformation is well defined, U will solve the heat equation, $U_t = \nu U_{xx}$, and thus have the explicit solution $U(x, t) = (4\pi\nu t)^{-1/2} \int_{\mathbb{R}} \exp[-(x - y)^2/(4\nu t)] U_0(y) dy$. Inverting the transformation leads to an explicit formula for the solution to (B). In some cases, it may be useful to alter the change of variables slightly, for example by adjusting the domain of integration in the definition of U or using the related transformation $U(x, t) = u(x, t) e^{-\frac{1}{2\nu} \int_{-\infty}^x u(y, t) dy}$.

2.3 Vanishing viscosity limit

In certain situations it is of interest to determine how solutions to the viscous equation are related to those of the inviscid equation. For example, if $u^\nu(x, t)$ denotes the solution to (B) for viscosity ν , in what sense, if at all, does $\lim_{\nu \rightarrow 0} u^\nu(x, t) = u^0(x, t)$? This is potentially relevant because solutions to the viscous equation are unique, whereas they are not in the inviscid case. Since any real system would have at least some dissipation, the physically relevant inviscid solutions should be those that can be approximated by viscous solutions [Renardy and Rogers, 1993]. In addition, when ν is positive but small, the qualitative behavior of solutions is initially determined by the inviscid equation, and the viscous dynamics in some sense only appears after an exponentially long time [Kim and Tzavaras, 2001].

References and recommended reading

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