Nonlinear convective stability of travelling fronts near Turing and Hopf instabilities

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Abstract

Reaction-diffusion equations on the real line that contain a control parameter are investigated. Of interest are travelling front solutions for which the rest state behind the front undergoes a supercritical Turing or Hopf bifurcation as the parameter is increased. This causes the essential spectrum to cross into the right half plane, leading to a linear convective instability in which the emerging pattern is pushed away from the front as it propagates. It is shown, however, that the wave remains nonlinearly stable in an appropriate sense. More precisely, using the fact that the instability is supercritical, it is shown that the amplitude of any pattern that emerges behind the wave saturates at some small parameter-dependent level and that the pattern is pushed away from the front interface. As a result, when considered in an appropriate exponentially weighted space, the travelling front remains stable, with an exponential in time rate of convergence.

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1. Introduction

Consider the reaction-diffusion equation

$$u_t = D\partial_x^2 u + f(u;\mu), \quad u \in \mathbb{R}^n, \quad x \in \mathbb{R}, \tag{1.1}$$

where D is a diagonal matrix with positive coefficients, μ is a real parameter, and f is smooth with $f(0;\mu) = 0$ for all μ. We assume that (1.1) has, for all μ near zero, a travelling front solution $u(x,t) = u_*(x - c(\mu)t;\mu)$ that moves with speed $c = c(\mu) > 0$ to the right and satisfies

$$\lim_{\xi\to-\infty} u_*(\xi;\mu) = 0, \qquad \lim_{\xi\to\infty} u_*(\xi;\mu) = u_+.$$

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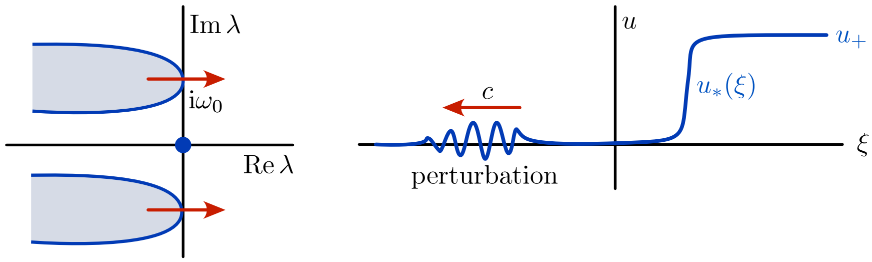

Figure 1: The left panel illustrates the rightmost spectrum of the linearization about the front in the comoving frame $\xi = x - ct$ at $\mu = 0$. As μ increases, we assume that essential spectrum crosses into the right half plane. The right panel illustrates how perturbations to the front profile are pushed, in the comoving frame, towards $\xi = -\infty$ just after bifurcation, whilst their amplitude saturates at $\mathrm{O}(\sqrt{|\mu|})$.

We are interested in such fronts for which the zero rest state behind the front undergoes a supercritical Turing or Hopf instability at $\mu = 0$. In the coordinate frame x, Turing bifurcations lead to small-amplitude patterns that are spatially-periodic and time-independent, while Hopf bifurcations lead to patterns that are spatially homogeneous and time-periodic. In the moving coordinate $\xi = x - c(\mu)t$, these bifurcations are generically caused by two locally parabolic curves of essential spectrum that cross into the right half plane at $\pm \mathrm{i}\omega_0$, for some critical temporal frequency $\omega_0 > 0$, as μ is increased through zero (see Figure 1).

Much previous work on travelling fronts and their stability in reaction-diffusion systems exists; see, for example, [12] and the references therein. In particular, the various ways in which a front can lose stability have been investigated. The spectrum can destabilize at $\pm \mathrm{i}\omega_0$ due to either a pair of isolated eigenvalues or the essential spectrum crossing into the right half plane. The former case is a classical Hopf bifurcation and was first analyzed by Henry [6, §6.4] using center-manifold theory. In the latter case, because the bifurcation is due to continuous spectrum, standard reduction techniques cannot be applied.

This type of destabilization, known as an essential instability, was first analyzed in [14]. The authors were interested in determining whether or not such a destabilization led to the creation of modulated fronts connecting the remaining stable rest state with the pattern that emerges. In their analysis they distinguished between two distinct cases: when the destabilization is caused by the rest state behind the front and when it is caused by the rest state ahead of the front. The reason for considering these cases separately can be seen by analyzing the Fredholm index of the operator $\mathcal{L} - \lambda$, where $\mathcal{L}$ is the linearization about the front. Roughly speaking, the Fredholm index is a measure of the solvability of the system $(\mathcal{L} - \lambda)u = h$ for a given h. When the instability is ahead of the front, the Fredholm index of $\mathcal{L} - \lambda$ increases from zero to one as λ moves from right to left through the essential spectrum. This leads to an underdetermined system of equations and the existence of bifurcating modulated waves. When the bifurcation is behind the front, the index changes from zero to minus one, leading to an overdetermined system. Therefore, the only nearby wave that exists after the bifurcation is the original, linearly unstable front. It is the nonlinear stability of this solution that we analyze here.

Previous stability analyses in the context of this type of bifurcation include the following. For the modulated waves that emerge when the bifurcation is ahead of the front, spectral stability was proved in [14]. Nonlinear stability, which does not immediately follow due to the presence of the essential spectrum on the imaginary axis, was proved for Turing bifurcations in [1] using weighted spaces and renormalization group techniques.

When the bifurcation is behind the front, and so no nearby modulated waves exist, one could expect some form of nonlinear stability for the original, linearly unstable front. This is because, in a comoving frame, the emergent pattern gets pushed towards minus infinity. In addition, because we assume the bifurcation is supercritical, the amplitude of perturbations behind the front should saturate at some small value that depends on μ. As a result, in a function space in which behavior at minus infinity is suppressed, the front should be stable. In this case, nonlinear stability has been proved for a specific system in which the rest state behind the front undergoes a Turing bifurcation [4]. However, the techniques used there include the maximum principle and energy methods, which are difficult to generalize. Our goal is to show that, for a general class of systems of the form (1.1), the front $u_*(\xi;\mu)$ is nonlinearly stable in an appropriate sense for all μ near zero when the rest state behind it experiences a supercritical Turing or Hopf instability. Our strategy for proving nonlinear stability is as follows. First, using weighted spaces, we show that the front is nonlinearly stable provided the amplitudes of perturbations to the front profile saturate. Afterwards, we utilize mode filters and the Ginzburg–Landau equation that governs the dynamics near the rest state behind the front to establish these a priori estimates. This second step relies heavily on results by Mielke and Schneider [9, 11, 16, 18].

The outline of the remainder of the paper is as follows. In §2 we precisely state our assumptions and results. A priori estimates for perturbations in the weighted space are given in §3. In §4, mode filters are used to show that perturbations in the unweighted space remain small. We briefly summarize our results in §5.

2. Set-up, assumptions and results

The function spaces in which we work need to contain functions that are not necessarily localized in space. Hence, we shall work with the so-called uniformly local spaces that are defined as follows.

We choose the weight function $\rho_{\rm ul}(x) = \mathrm{e}^{-|x|}$ and define $(T_y\rho_{\rm ul})(x) = \rho_{\rm ul}(x-y)$ to be the translation operator T_y applied to the weight function. The weighted L^2 norm and the uniformly local L^2 norm are then given respectively by

$$\|u\|_{\rho_{\rm ul}}^2 = \int_{\mathbb{R}} \rho_{\rm ul}(x)|u(x)|^2\,\mathrm{d}x \quad \text{and} \quad \|u\|_{L^2_{\rm ul}} = \sup_{y\in\mathbb{R}} \|u\|_{T_y\rho_{\rm ul}}.$$

The uniformly local space L^2_{ul} is then defined as

$$L^2_{\mathrm{ul}} = \text{closure of } C^\infty_{\mathrm{bdd}}(\mathbb{R}) \text{ in } \left\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}) : \|u\|_{L^2_{\mathrm{ul}}} < \infty \right\},$$

and the Sobolev spaces H^s_{ul} are defined similarly. The translation operator gives uniformity in space for the norm, and taking the closure of smooth functions ensures that the resulting space is complete and that the standard Sobolev embeddings still hold. Below we work in H^1_{ul} so that solutions are defined pointwise. For more information on these spaces, we refer the reader to [11].

Hypothesis (H1). *For $\mu = 0$, equation (1.1) has a travelling-wave solution $u^0_*(x - c_*t)$ for an appropriate wave speed $c_* > 0$, and the wave profile satisfies $\lim_{\xi\to-\infty} u^0_*(\xi) = 0$ and $\lim_{\xi\to\infty} u^0_*(\xi) = u_+$ for some $u_+ \in \mathbb{R}^n$.*

We now formulate the hypotheses on the spectral stability of the front u^0_*. Consider the linearized operator

$$\mathcal{L}_* = D\partial^2_\xi + c_*\partial_\xi + f_u(u^0_*(\xi); 0),$$

posed in the comoving frame $\xi = x - c_*t$, and the asymptotic operators

$$\mathcal{L}_-(\mu) = D\partial^2_x + f_u(0; \mu), \qquad \mathcal{L}_+(0) = D\partial^2_x + f_u(u_+; 0)$$

associated with the spatially homogeneous rest states $u = 0$ and $u = u_+$, formulated in the laboratory frame, on the space L^2_{ul} with domain H^2_{ul}. To capture transport, we let ρ_a be any smooth, monotone function that satisfies

$$\rho_a(\xi) = \begin{cases} 1 & \text{if } \xi \geq 1 \\ \mathrm{e}^{a\xi} & \text{if } \xi \leq -1 \end{cases} \tag{2.1}$$

with $a > 0$.

Hypothesis (H2). *We assume that the following is true:*

(i) *There is an $a_0 > 0$ such that the spectrum of the operator $\mathcal{L}^a_* := \rho_a\mathcal{L}_*\rho_a^{-1}$ on L^2_{ul} lies in the open left half plane for all $0 < a \leq a_0$ except for a simple eigenvalue at $\lambda = 0$.*

(ii) *For all μ close to zero, the spectrum of $\mathcal{L}_-(\mu)$ lies in the open left half plane except for two curves given by*

$$\lambda(k, \mu) = \lambda_0(\mu) - \lambda_2(\mu)(k - k_0)^2 + \mathrm{O}(|k - k_0|^3), \qquad |k - k_0| \ll 1$$

and its complex conjugate, where $\mathrm{Re}\,\lambda_2(0) > 0$, $\mathrm{Re}\,\lambda'_0(0) > 0$ and either
Turing: $k_0 > 0$ *and* $\lambda_0(0) = 0$, *or*
Hopf: $k_0 = 0$, $\lambda_0(0) = \mathrm{i}\omega_0$ *for some* $\omega_0 > 0$.

(iii) *The spectrum of $\mathcal{L}_+(0)$ lies in the open left half plane.*

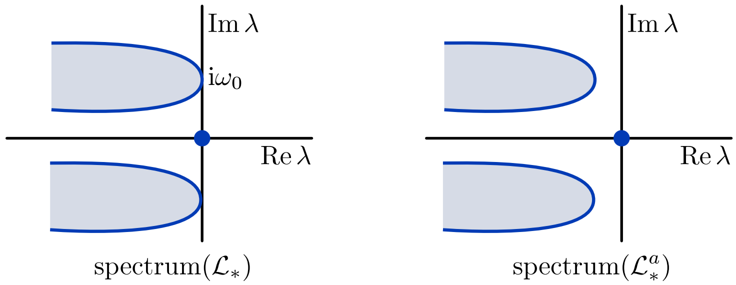

Figure 2: The rightmost parts of the $L^2_{\rm ul}$ spectra of the linearization $\mathcal{L}_*$ and the weighted operator $\mathcal{L}_*^a$ with $0 < a \ll 1$ are shown for the case where Hypothesis (H2) is met. Note that $\omega_0 = k_0 c_* > 0$ at Turing bifurcations.

As we shall now argue, Hypothesis (H2) implies that the spectra of the operators $\mathcal{L}_*$ and $\mathcal{L}_*^a$ on $L^2_{\rm ul}$ are as shown in Figure 2. In the laboratory frame x, the critical curves $\lambda(k,\mu)$ in the spectrum of $\mathcal{L}_-(\mu)$ correspond to neutral eigenmodes of the form $\mathrm{e}^{\mathrm{i}k_0 x} u_0$ for Turing bifurcations and $\mathrm{e}^{\mathrm{i}\omega_0 t} u_0$ for Hopf bifurcations. The dispersion curve $\lambda(k,\mu)$ in the laboratory frame x becomes $\lambda_*(k,\mu;c) = \lambda(k,\mu) + \mathrm{i}kc$ in the frame $\xi = x - ct$ that moves with speed c; see [15]. Since the spectrum of $\mathcal{L}_*$ contains the dispersion curve $\lambda_*(\cdot,0;c_*)$ (see, for instance, [6, Lemma 2 in the appendix to Chapter 5] or [12, §3.4.3]), we see that its spectrum indeed touches the imaginary axis at $\pm\mathrm{i}\omega_0$, where $\omega_0 = k_0 c_*$ at Turing bifurcations. Next, we note that the group velocity $-\frac{\mathrm{d}\,\mathrm{Im}\,\lambda}{\mathrm{d}k}(k_0,0)$ of the dispersion curve vanishes in the laboratory frame x, which implies that the essential spectrum of the weighted linearization $\mathcal{L}_*^a$ in the right-moving frame ξ will move into the left half plane for all sufficiently small $a > 0$ (see [13, 14]), and Hypothesis (H2)(i) ensures that no isolated eigenvalues of $\mathcal{L}_*^a$ are revealed on the imaginary axis when the essential spectrum is moved.

Our last assumption is that the bifurcation caused by the critical curve $\lambda(k,\mu)$ is supercritical. To make this precise, consider (1.1) near $u = 0$, and let $e(k_0)$ be an eigenvector of the matrix $-k_0^2 D + f_u(0;0)$ associated with the eigenvalue $\lambda_0(0)$. Upon substituting the ansatz

$$u(x,t) = \epsilon \mathrm{e}^{\mathrm{i}(k_0 x + \omega_0 t)} A(\epsilon x, \epsilon^2 t) e(k_0) + \text{c.c.}, \qquad \mu = \rho\epsilon^2 \tag{2.2}$$

into (1.1) and expanding in ϵ, we find that the amplitude $A(X,T)$ satisfies the Ginzburg–Landau equation

$$A_T = \lambda_2(0)\partial_X^2 A + \rho\lambda_0'(0)A - b|A|^2 A \tag{2.3}$$

for an appropriate value $b \in \mathbb{C}$; see [10] and references therein. The sign of the real part of the cubic coefficient b determines whether the bifurcation is subcritical or supercritical. We assume the latter:

Hypothesis (H3). *We assume that* $\mathrm{Re}\, b > 0$ *so that the bifurcation is supercritical.*

It is a consequence of Hypothesis (H2)(i) that the front $u_*^0(x - c_* t)$ persists for nearby values of μ. More precisely, there exist a unique profile $u_*(\xi;\mu)$ and a unique wave speed $c(\mu)$ with $u_*(\cdot;0) = u_*^0$ and $c(0) = c_*$ so that $u(x,t) = u_*(x - c(\mu)t;\mu)$ satisfies (1.1) for all μ near zero. The next theorem shows that these fronts remain nonlinearly stable, in an appropriate sense, for μ near zero and, in particular, the fronts do not feel the linear instability that occurs in their wake.

Theorem 1. *Suppose that Hypotheses (H1)-(H3) are satisfied, then there exist positive constants K, Λ_*, a_*, μ_*, and δ_* such that the following is true: for all $|\mu| \le \mu_*$ and any initial condition satisfying $\|v(\cdot,0)\|_{H^1_{\rm ul}} < \delta_*$, the solution of (1.1) with initial data $u(x,0) = u_*(x;\mu) + v(x,0)$ exists for all $t \ge 0$ and can be written as*

$$u(x,t) = u_*(x - c(\mu)t - p(t);\mu) + v(x - c(\mu)t, t)$$

for an appropriate real-valued function p; furthermore, there is a constant $p_* \in \mathbb{R}$ such that

$$\begin{aligned} \|v(\cdot,t)\|_{H^1_{\rm ul}} + |p(t)| &\le K\left(\|v(\cdot,0)\|_{H^1_{\rm ul}} + \sqrt{|\mu|}\right) \\ \|\rho_{a_*}(\cdot)v(\cdot,t)\|_{H^1_{\rm ul}} + |p(t) - p_*| &\le K\mathrm{e}^{-\Lambda_* t} \end{aligned}$$

for all $t \ge 0$. In other words, the perturbation $v(\xi,t)$ decays to zero exponentially in time in the weighted norm $\|\rho_{a_*}\cdot\|_{H^1_{\rm ul}}$ in the comoving frame $\xi = x - c(\mu)t$.

There are two steps in the proof of this theorem. The first, contained in §3, is to show that, if $\|v(\cdot,t)\|_{H^1_{\rm ul}}$ is small for all $t \ge 0$, then $\|\rho_a v(\cdot,t)\|_{H^1_{\rm ul}} \le K\mathrm{e}^{-\Lambda_* t}$ for some $\Lambda_* > 0$. This will follow using a method analogous to that in [4]. The second step, contained in §4, is to show that $\|v(\cdot,t)\|_{H^1_{\rm ul}}$ is, in fact, small for all $t \ge 0$. This will be proved using the mode filters.

3. A priori estimates in the weighted space

To prove that the fronts $u_*(x - c(\mu)t;\mu)$ are nonlinearly stable for all μ with $|\mu| \ll 1$, we write solutions to (1.1) in the form

$$u(x,t) = u_*(x - c(\mu)t - p(t);\mu) + v(x - c(\mu)t, t).$$

The real-valued function $p(t)$ will be defined in more detail below and will allow us to remove the neutral behavior due to the zero eigenvalue. From now on, we fix μ close to zero and consider the dynamics near the front $u_*(x - c(\mu)t;\mu)$. It is convenient to formulate all equations in the comoving frame $\xi = x - c(\mu)t$. To simplify notation, we define

$$h_0(\xi) := u_*(\xi;\mu), \qquad h_{p(t)}(\xi) := h_0(\xi - p(t)) = u_*(\xi - p(t);\mu)$$

and omit the dependence on μ. Similarly, we write c and $f(u)$ instead of $c(\mu)$ and $f(u;\mu)$, respectively, from now on. In the comoving frame $\xi = x - ct$, equation (1.1) becomes

$$u_t = D\partial_\xi^2 + c\partial_\xi u + f(u).$$

Substituting the ansatz

$$u(\xi,t) = h_0(\xi - p(t)) + v(\xi,t), \tag{3.1}$$

we obtain the system

$$v_t = \mathcal{L}_0 v + \dot{p}h_p' - [f_u(h_0) - f_u(h_p)]v + [f(h_p+v) - f(h_p) - f_u(h_p)v], \tag{3.2}$$

where we use the notation $' = \partial_\xi$, recall that $p = p(t)$, and set

$$\mathcal{L}_0 := D\partial_\xi^2 + c\partial_\xi + f_u(h_0).$$

We shall use the weighted function $w(\xi,t) := \rho_a(\xi)v(\xi,t)$, with ρ_a from (2.1), which then satisfies the equation

$$w_t = \mathcal{L}_0^a w + \dot{p}\rho_a h_p' - [f_u(h_0) - f_u(h_p)]w + \mathcal{N}(v)w, \tag{3.3}$$

where

$$\begin{aligned} \mathcal{L}_0^a &= \rho_a \mathcal{L}_0 \rho_a^{-1} \\ &= D\partial_\xi^2 + \left(c - \frac{2\rho_a'}{\rho_a}\right)\partial_\xi + \left(\frac{2(\rho_a')^2}{\rho_a^2} - \frac{c\rho_a'}{\rho_a} - \frac{\rho_a''}{\rho_a} + f_u(h_0)\right) \\ \mathcal{N}(v) &= \int_0^1 [f_u(h_p + \tau v) - f_u(h_p)]\,\mathrm{d}\tau = \mathrm{O}(|v|). \end{aligned}$$

We will consider $w \in H^1_{\mathrm{ul}}$ and choose the exponential rate a in the interval $(0, a_0]$, where a_0 is so small that Hypothesis (H2)(i) is met and so that there exists a $C > 0$ for which

$$\left|\frac{h_0(\xi)}{\rho_a(\xi)}\right| + \left|\frac{h_0'(\xi)}{\rho_a(\xi)}\right| \le C, \qquad x \in \mathbb{R} \tag{3.4}$$

for $0 < a \le a_0$. We call exponential rates $a \in (0, a_0]$ admissible. Throughout the remainder of this paper, we will denote by C any generic constant that does not depend on the initial data or on a and μ.

Hypothesis (H2) implies that the essential spectrum of $\mathcal{L}_0^a$ lies strictly to the left of the imaginary axis for all μ near zero and all admissible rates a. The only eigenvalue on or to the right of the imaginary axis is the simple eigenvalue $\lambda = 0$ with eigenfunction $\rho_a h_0'$. Therefore, it is possible to define a spectral projection P_a^{c} in H^1_{ul} onto the one-dimensional center subspace belonging to $\lambda = 0$. The rest of the spectrum will be bounded away from the imaginary axis, and so the complement of P_a^{c} is the projection P_a^{s} onto the generalized stable eigenspace. In particular, there exists a constant K_0 that is independent of μ so that

$$\|\mathrm{e}^{P_a^{\mathrm{s}}\mathcal{L}_0^a t}\|_{H^1_{\mathrm{ul}}} \le K_0 \mathrm{e}^{-\Lambda_0 t} \tag{3.5}$$

for $t \ge 0$. We combine this information in the following lemma.

Lemma 3.1. *There exist positive constants K_0, Λ_0, and μ_0 such that the following holds for any $|\mu| < \mu_0$. The spectral projection P_a^{c} is well defined and, in fact, given by*

$$P_a^{\mathrm{c}} w = \langle \psi_a, w \rangle_{L^2} \rho_a h_0',$$

where $\psi_a = \rho_a^{-1}\psi_0$, and ψ_0 spans the kernel of the operator adjoint to $\mathcal{L}_0$ with

$$\langle \psi_a, \rho_a h_0' \rangle_{L^2} = \langle \psi_0, h_0' \rangle_{L^2} = 1. \tag{3.6}$$

For $P_a^{\mathrm{s}} = I - P_a^{\mathrm{c}}$, we have (3.5).

In the decomposition (3.1),

$$u(\xi, t) = h_0(\xi - p(t)) + v(\xi, t),$$

the shift function $p(t)$ is not defined uniquely, even if $p(0) = 0$ is assumed. To avoid ambiguity we require that $w(\cdot, t)$ lies in the range of the projection P_a^{s} for all $t \geq 0$ for which the decomposition (3.1) exists. In other words, we require that $w(\xi, t)$ satisfies

$$P_a^{\mathrm{c}} w(\cdot, t) = 0 \tag{3.7}$$

for all $t \geq 0$, and, applying the projections P_a^{s} and P_a^{c} to (3.3), we obtain the system

$$\begin{aligned} w_t &= P_a^{\mathrm{s}} \mathcal{L}_0^a w + P_a^{\mathrm{s}} \left(\dot{p} \rho_a h_p' - [f_u(h_0) - f_u(h_p)] w + \mathcal{N}(v) w \right) \\ \dot{p} &= \frac{\langle \psi_a, [f_u(h_0) - f_u(h_p)] w - \mathcal{N}(v) w \rangle_{L^2}}{\langle \psi_a, \rho_a h_p' \rangle_{L^2}} \end{aligned} \tag{3.8}$$

for (w, p). Conversely, solutions (w, p) of (3.8) automatically satisfy (3.7). Using the initial data $p(0) = 0$, the function $p(t)$ is then defined uniquely. Note that as long as $p(t)$ remains sufficiently small, the denominator $\langle \psi_a, \rho_a h_p' \rangle_{L^2} = \langle \psi_0, h_0'(\cdot - p(t)) \rangle_{L^2}$ in (3.8) is bounded away from zero due to (3.6).

We consider the system consisting of (3.8) coupled to (3.2). Because the nonlinearity f is smooth, there exists a constant K_1 so that the nonlinearity $\mathcal{N}$ and the difference in the linearization about h_0 and h_p satisfy

$$\begin{aligned} \|f_u(h_0) - f_u(h_p)\|_{H^1_{\mathrm{ul}}} + \|\mathcal{N}(v)\|_{H^1_{\mathrm{ul}}} &\leq K_1 \left(|p| + \|v\|_{H^1_{\mathrm{ul}}} \right) \\ |\dot{p}| &\leq K_1 \left(|p| + \|v\|_{H^1_{\mathrm{ul}}} \right) \|w\|_{H^1_{\mathrm{ul}}} \end{aligned} \tag{3.9}$$

for μ close to zero. The second estimate follows from the first and the equation governing $\dot{p}$ in (3.8). By [11, Lemma 3.3], the linear operators in (3.2) and (3.8) are sectorial operators on H^1_{ul}, and the arguments in [6, 11] therefore imply that solutions exist locally in time, are unique, and depend continuously on the initial conditions. This proves the local existence and uniqueness of the decomposition (3.1). Moreover, the continuous dependence of the solutions on the initial conditions implies that, for any given $0 < \eta_0 \leq 1$, there exists a $\gamma_0 > 0$ and $T > 0$, such that

$$\sup_{t \in [0,T]} \left(|p(t)| + \|v(\cdot, t)\|_{H^1_{\mathrm{ul}}} \right) \leq \eta_0 \quad \text{provided} \quad \|v(\cdot, 0)\|_{H^1_{\mathrm{ul}}} \leq \gamma_0. \tag{3.10}$$

The maximal time T such that the above holds is denoted by $T_{\max}(\eta_0)$.

The following lemma states that, as long as the solutions v in the unweighted space and p remain small, the solution w in the weighted space will decay exponentially fast in time and control the behavior of p. This is the main result of this section and is analogous to [4, Lemma 3.2].

Lemma 3.2. *Pick Λ such that $0<\Lambda<\Lambda_0$, then there exists an $\hat{\eta}_0$ with $0<\hat{\eta}_0\le 1$ so that the following is true. If $0<\eta_0<\hat{\eta}_0$ and w is a solution of (3.3) for which the corresponding solutions v and p satisfy (3.10), then*

$$\|w(\cdot,t)\|_{H^1_{\mathrm{ul}}}\le K\mathrm{e}^{-\Lambda t}\|w(\cdot,0)\|_{H^1_{\mathrm{ul}}},\qquad |p(t)|\le K\|w(\cdot,0)\|_{H^1_{\mathrm{ul}}}$$

for all $0\le t\le T_{\max}(\eta_0)$, for some positive constant K that is independent of μ and η_0. If $T_{\max}(\eta_0)=\infty$, then there is a $p_*\in\mathbb{R}$ with

$$|p(t)-p_*|\le K\mathrm{e}^{-\Lambda t}\|w(\cdot,0)\|_{H^1_{\mathrm{ul}}}\tag{3.11}$$

for all $t\ge 0$.

Once this lemma has been established, the proof of Theorem 1 will follow if we can prove that $T_{\max}(\hat{\eta}_0)=\infty$ for our particular choice of $\hat{\eta}_0$.

Proof. Consider the equation for w in (3.8) for $t\in[0,T_{\max})$, rewritten here for convenience:

$$w_t=P_a^{\mathrm{s}}\mathcal{L}_0^a w+P_a^{\mathrm{s}}\left(\dot{p}\rho_a h_p'-[f_u(h_0)-f_u(h_p)]w+\mathcal{N}(v)w\right).$$

Applying the variation-of-constants formula to this equation, we obtain

$$\begin{aligned} w(\cdot,t) \quad &= \quad \mathrm{e}^{P_a^{\mathrm{s}}\mathcal{L}_0^a t}w(\cdot,0)+\int_0^t \mathrm{e}^{P_a^{\mathrm{s}}\mathcal{L}_0^a(t-s)}P_a^{\mathrm{s}}\Big(\dot{p}(s)\rho_a h_{p(s)}'(s)\\ &\qquad -[f_u(h_0)-f_u(h_{p(s)})]w(\cdot,s)+\mathcal{N}(v(\cdot,s))w(\cdot,s)\Big)\,\mathrm{d}s,\end{aligned}$$

where we recall that $h_{p(t)}=h_0(\xi-p(t))$ depends on t through $p(t)$. From (3.5) and (3.10), it follows that

$$\begin{aligned}\|w(\cdot,t)\|_{H^1_{\mathrm{ul}}} \quad &\le \quad K_0\mathrm{e}^{-\Lambda_0 t}\|w(\cdot,0)\|_{H^1_{\mathrm{ul}}}+K_0\int_0^t \mathrm{e}^{-\Lambda_0(t-s)}\Big(|\dot{p}(s)|\|\rho_a h_{p(s)}'\|_{H^1_{\mathrm{ul}}}\\ &\qquad +K_1\left(|p(s)|+\|v(\cdot,s)\|_{H^1_{\mathrm{ul}}}\right)\|w(\cdot,s)\|_{H^1_{\mathrm{ul}}}\Big)\,\mathrm{d}s.\end{aligned}$$

Since ρ_a and $h_p'(\xi)=h_0'(\xi-p)$ are bounded uniformly in p, there exists a constant $K_2>0$ such that

$$\|\rho_a h_p'\|_{H^1_{\mathrm{ul}}}\le K_2.$$

Therefore, using (3.10) we obtain

$$\|w(\cdot,t)\|_{H^1_{\mathrm{ul}}} \quad \le \quad K_0\mathrm{e}^{-\Lambda_0 t}\|w(\cdot,0)\|_{H^1_{\mathrm{ul}}}+K_0\int_0^t \mathrm{e}^{-\Lambda_0(t-s)}K_1(K_2+1)$$

$$
\begin{aligned}
& \times \Big(|p(s)| + \|v(\cdot,s)\|_{H^1_{\mathrm{ul}}} \Big) \|w(\cdot,s)\|_{H^1_{\mathrm{ul}}} \,\mathrm{d}s \\
\leq \quad & K_0 \mathrm{e}^{-\Lambda_0 t} \|w(\cdot,0)\|_{H^1_{\mathrm{ul}}} \qquad\qquad (3.12) \\
& + K_0 K_1 (K_2+1)\eta_0 \int_0^t \mathrm{e}^{-\Lambda_0 (t-s)} \|w(\cdot,s)\|_{H^1_{\mathrm{ul}}} \,\mathrm{d}s.
\end{aligned}
$$

For $0 < \Lambda < \Lambda_0$ and $0 \leq t \leq T_{\max}$, define

$$
M(t) := \sup_{0 \leq s \leq t} \mathrm{e}^{\Lambda s} \|w(\cdot,s)\|_{H^1_{\mathrm{ul}}}.
$$

After multiplying (3.13) by $\mathrm{e}^{\Lambda t}$ we find that

$$
\begin{aligned}
\mathrm{e}^{\Lambda t} \|w(\cdot,t)\|_{H^1_{\mathrm{ul}}} \quad \leq \quad & K_0 \mathrm{e}^{-(\Lambda_0 - \Lambda)t} \|w(\cdot,0)\|_{H^1_{\mathrm{ul}}} \\
& + K_0 K_1 (K_2+1)\eta_0 \int_0^t \mathrm{e}^{-(\Lambda_0 - \Lambda)(t-s)} \mathrm{e}^{\Lambda s} \|w(\cdot,s)\|_{H^1_{\mathrm{ul}}} \,\mathrm{d}s \\
\leq \quad & K_0 \|w(\cdot,0)\|_{H^1_{\mathrm{ul}}} + K_0 K_1 (K_2+1)\eta_0 M(t) \int_0^t \mathrm{e}^{-(\Lambda_0 - \Lambda)(t-s)} \,\mathrm{d}s \\
\leq \quad & K_0 \|w(\cdot,0)\|_{H^1_{\mathrm{ul}}} + \frac{K_0 K_1 (K_2+1)\eta_0}{\Lambda_0 - \Lambda} M(t),
\end{aligned}
$$

and therefore

$$
M(t) \leq K_0 \|w(\cdot,0)\|_{H^1_{\mathrm{ul}}} + \frac{K_0 K_1 (K_2+1)\eta_0}{\Lambda_0 - \Lambda} M(t).
$$

If $\hat{\eta}_0$ is chosen so that

$$
1 - \frac{K_0 K_1 (K_2+1)}{\Lambda_0 - \Lambda} \hat{\eta}_0 \geq \frac{1}{2} \qquad \text{or, equivalently,} \qquad \hat{\eta}_0 \leq \frac{\Lambda_0 - \Lambda}{2 K_0 K_1 (K_2+1)},
$$

then we have

$$
M(t) \leq 2K_0 \|w(\cdot,0)\|_{H^1_{\mathrm{ul}}}
$$

for $0 < \eta_0 < \hat{\eta}_0$, and therefore

$$
\|w(\cdot,t)\|_{H^1_{\mathrm{ul}}} \leq 2K_0 \mathrm{e}^{-\Lambda t} \|w(\cdot,0)\|_{H^1_{\mathrm{ul}}}. \qquad (3.13)
$$

From (3.13), (3.9), (3.10) and the assumption that $\eta_0 < 1$, we then obtain

$$
|\dot{p}(t)| \leq 2K_0 K_1 \eta_0 \mathrm{e}^{-\Lambda t} \|w(\cdot,0)\|_{H^1_{\mathrm{ul}}} \leq 2K_0 K_1 \mathrm{e}^{-\Lambda t} \|w(\cdot,0)\|_{H^1_{\mathrm{ul}}} \qquad (3.14)
$$

for $0 \leq t \leq T_{\max}$. On the other hand,

$$
|p(t)| \leq \int_0^t |\dot{p}(t)| \,\mathrm{d}s \leq 2K_0 K_1 \|w(\cdot,0)\|_{H^1_{\mathrm{ul}}} \int_0^t \mathrm{e}^{-\Lambda s} \,\mathrm{d}s \leq \frac{2K_0 K_1}{\Lambda} \|w(\cdot,0)\|_{H^1_{\mathrm{ul}}}.
$$

If $T_{\max} = \infty$, then (3.9), and consequently (3.14) and (3.13), hold for any t. According to (3.14), $p_* = \int_0^\infty \dot{p}(s) \,\mathrm{d}s$ exists, and we have

$$
p(t) - p_* = \int_\infty^t \dot{p}(s) \,\mathrm{d}s
$$

for all $t \geq 0$. To estimate the convergence rate, we use (3.14) and obtain

$$|p(t) - p_*| \leq 2K_0K_1\|w(\cdot,0)\|_{H^1_{\mathrm{ul}}} \int_t^\infty \mathrm{e}^{-\Lambda s}\,\mathrm{d}s \leq \frac{2K_0K_1}{\Lambda}\|w(\cdot,0)\|_{H^1_{\mathrm{ul}}}\mathrm{e}^{-\Lambda t}.$$

The statements of the lemma then follow with $K = \max\{\frac{K_0}{2}, \frac{K_0K_1}{2\Lambda}\}$; recall that the constants K_0, K_1 and K_2 are independent of η_0 and T. □

4. Estimates in the unweighted space via mode filters

In this section, we prove that, in fact, $T_{\max}(\hat{\eta}_0) = \infty$. In particular, we will prove the following proposition.

Proposition 4.1. *There exist positive constants K, δ_* and μ_* such that, if $\|v(\cdot,0)\|_{H^1_{\mathrm{ul}}} < \delta_*$, then the solution to (3.2) exists for each μ with $|\mu| \leq \mu_*$ and satisfies*

$$\|v(\cdot,t)\|_{H^1_{\mathrm{ul}}} + |p(t)| \leq K\left(\|v(\cdot,0)\|_{H^1_{\mathrm{ul}}} + \sqrt{|\mu|}\right)$$

for all $t \geq 0$. In particular, $T_{\max}(\hat{\eta}_0) = \infty$ for $\hat{\eta}_0 = K(\delta_* + \sqrt{\mu_*})$ and all μ with $|\mu| \leq \mu_*$.

The proof will consist of two steps. First, we will show that the behavior of solutions to (3.2) is really governed by the bifurcation at $-\infty$. Then we will show that, because this bifurcation is supercritical, the amplitude of perturbations saturates eventually at $\mathrm{O}(\sqrt{|\mu|})$ in H^1_{ul}. The combination of these results, which leads to the above proposition, is contained in Lemma 4.2 below.

4.1. Reduction to behavior at minus infinity

In order to prove Proposition 4.1, we first show that the evolution of v in the comoving frame is controlled by the dynamics near $\xi = -\infty$. To do this, we will use a method similar to that of [1, §5], which we now describe.

We write equation (3.2) as

$$v_t = \mathcal{L}_- v + \mathcal{N}_-(v) + \Delta(p,v), \tag{4.1}$$

where

$$\begin{aligned}
\mathcal{L}_- &= D\partial_\xi^2 + c\partial_\xi + f_u(0) \\
\mathcal{N}_-(v) &= f(v) - f_u(0)v \\
\Delta(p,v) &= \dot{p}h_p' + [f_u(h_0) - f_u(0)]v - [f_u(h_0) - f_u(h_p)]v \\
&\quad + [f(h_p+v) - f(h_p) - f_u(h_p)v] - \mathcal{N}_-(v).
\end{aligned}$$

Notice that $\mathcal{L}_-$ and $\mathcal{N}_-$ are respectively the linearization about the unstable state $u = 0$ in the comoving frame and the corresponding nonlinearity, where we recall that we assumed that $f(0;\mu) = 0$ for all μ. The function Δ consists of the drift term $\dot{p}h_p'$, the difference between the linearization about the front and about 0, and the difference between the nonlinearity evaluated at the front and at 0. On account of Lemma 3.2, we can view $p(t)$ as a given function. Our goal is to use the information in §3 about exponential decay in a weighted space and information about p to obtain the following estimates.

Lemma 4.1. *There exists a constant C, depending only on Λ_0 and $\hat{\eta}_0$, such that*

$$\|\Delta(p,v)(\cdot,t)\|_{H^1_{\mathrm{ul}}} \le C\|w(\cdot,t)\|_{H^1_{\mathrm{ul}}} \le C\mathrm{e}^{-\Lambda t}\|w(\cdot,0)\|_{H^1_{\mathrm{ul}}}$$

for all $0 \le t \le T_{\max}$.

This lemma implies that we can think of the evolution of v as being governed by the evolution near the unstable state plus the effects of an inhomogeneity that is exponentially decaying in time.

Proof. We will deal with each term inside Δ separately and use that any function in H^1_{ul} is defined pointwise by Sobolev embedding.

Consider the term $\dot{p}h_p'$. Since the underlying wave is smooth, we know that $\|h_p'\|_{H^1_{\mathrm{ul}}} \le K_3$ for some constant K_3. Furthermore, equation (3.9) and Lemma 3.2 imply that

$$|\dot{p}| \le K_1\left(|p| + \|v\|_{H^1_{\mathrm{ul}}}\right)\|w\|_{H^1_{\mathrm{ul}}} \le K_4\mathrm{e}^{-\Lambda t}\|w(\cdot,0)\|_{H^1_{\mathrm{ul}}} \tag{4.2}$$

for $0 \le t \le T_{\max}$ and some constant K_4, which gives the desired estimate for the first term. Next, we write

$$\begin{aligned} |[f_u(h_0) - f_u(0)]v(\xi)| &\le \frac{|f_u(h_p(\xi)) - f_u(0)|}{\rho_a(\xi)}|w(\xi)| \\ &\le \left[\sup_{|u|\le|h_0|_\infty} |f_{uu}(u)|\right]\frac{|h_p(\xi)|}{\rho_a(\xi)}|w(\xi)|. \end{aligned}$$

Equation (3.4) and smoothness of f imply that

$$\|[f_u(h_0) - f_u(0)]v\|_{H^1_{\mathrm{ul}}} \le C\|w\|_{H^1_{\mathrm{ul}}} \le C\mathrm{e}^{-\Lambda t}\|w(\cdot,0)\|_{H^1_{\mathrm{ul}}}$$

for $0 < a \le a_0$ and $0 \le t \le T_{\max}$. Similarly,

$$\begin{aligned} |[f_u(h_0) - f_u(h_p)]v(\xi)| &\le \left[\sup_{|u|\le|h_0|_\infty} |f_{uu}(u)|\right]\frac{|h_0(\xi) - h_0(\xi - p(t))|}{\rho_a(\xi)}|w(\xi)| \\ &\le C\left[\sup_{|\zeta|\le\sup|p(t)|} |h_0'(\xi+\zeta)|\right]\frac{|p(t)|}{\rho_a(\xi)}|w(\xi)|. \end{aligned}$$

Appealing again to (3.4), this gives

$$\|[f_u(h_0) - f_u(h_p)]v\|_{H^1_{\mathrm{ul}}} \le C\mathrm{e}^{-\Lambda t}\|w(\cdot,0)\|_{H^1_{\mathrm{ul}}}.$$

It remains to estimate the expression

$$\begin{aligned}&[f(h_p+v)-f_u(h_p)v-f(h_p)]-\mathcal{N}_-(v)\\ &\quad = \quad [f(h_p+v)-f_u(h_p)v-f(h_p)]-[f(v)-f_u(0)v].\end{aligned}$$

Rearranging the terms and using that $f(0;\mu)=0$ for all μ, we get

$$\begin{aligned}&|[f(h_p+v)-f(v)]-[f(h_p)-f(0)]+[f_u(0)-f_u(h_p)]v|\\ &= \left|\int_0^1[f_u(v+sh_p)-f_u(sh_p)]\,\mathrm{d}s\;h_p-\int_0^1 f_{uu}(sh_p)[h_p,v]\,\mathrm{d}s\right|\\ &\le C|h_p||v|=\frac{|h_p|}{\rho_a}|w|\le C|w|.\end{aligned}$$

Therefore,

$$\|[f(h_p+v)-f_u(h_p)v-f(h_p)]-\mathcal{N}_-(v)\|_{H^1_{\mathrm{ul}}}\le C\mathrm{e}^{-\Lambda t}\|w(\cdot,0)\|_{H^1_{\mathrm{ul}}},$$

which proves the lemma. □

4.2. Boundedness of solutions via mode filters

Since the estimate for Δ in Lemma 4.1 is in H^1_{ul} and does therefore not depend on the underlying reference frame, we can write equation (4.1) for the evolution of the small perturbation v in the original frame x and obtain

$$v_t=D\partial_x^2v+f(v;\mu)+\Delta(p,v)=D\partial_x^2v+f(v;\mu)+\mathrm{O}(\mathrm{e}^{-\Lambda t}). \tag{4.3}$$

In particular, for sufficiently large times, the dynamics of $v(x,t)$ ought to be governed by the dynamics of the reaction-diffusion system

$$v_t=D\partial_x^2v+f(v;\mu). \tag{4.4}$$

Since v is small, this suggests that we use the Ginzburg–Landau formalism for Turing or Hopf bifurcations, which describes the dynamics of the envelopes $A(X,T)$ of modulated waves of the form

$$v(x,t)=\tilde{\mathcal{V}}_\delta(A):=\delta\mathrm{e}^{\mathrm{i}k_0x+\mathrm{i}\omega_0t}A(\delta x,\delta^2t)e(k_0)+\mathrm{c.c.}, \tag{4.5}$$

for $0\le|\mu|\le\delta^2\ll 1$ and with $e(k_0)$ given via $\lambda(k_0,0)e(k_0)=\hat{\mathcal{L}}_-(\mathrm{i}k_0,0)e(k_0)$, by the Ginzburg–Landau equation

$$A_T=\lambda_2(0)\partial_X^2A+\frac{\mu\lambda_0'(0)}{\delta^2}A-b|A|^2A. \tag{4.6}$$

For $\mathrm{Re}\,b>0$, which we assumed in Hypothesis (H3), equation (4.6) has a bounded attractor, and solutions are therefore bounded uniformly in time; see [9]. Furthermore, it was shown in [18, 11] that these properties for the Ginzburg–Landau equation imply that solutions of (4.4) belonging to sufficiently small initial data stay small for all times. The proof of the latter assertion is based upon the use of mode filters, which separate the neutral part of the continuous spectrum near the imaginary axis from the rest of the spectrum and allow one to decompose the solution of (4.4) into a center-unstable component, governed by the Ginzburg–Landau equation, and a stable component, governed by an exponentially decaying semigroup.

The additional $\mathrm{O}(\mathrm{e}^{-\Lambda t})$ term that is present in (4.3) prevents us from applying the results of [18, 11] directly to equation (4.3). However, the ideas and techniques developed in those papers are still applicable, and by establishing the so-called approximation and attractivity properties introduced there, we will show that solutions to (4.3) with sufficiently small initial data remain small in H^1_{ul} for all $t \geq 0$. These two properties will be stated precisely below but, essentially, the approximation property says that, given any solution A of (4.6), there is a solution v of (4.3) that is close to $\tilde{\mathcal{V}}_\delta(A)$ in an appropriate sense for large but finite times. In other words, any small solution v that looks like a modulated wave at a given initial time will continue to look like a modulated wave for large finite times. The attractivity property states that, given any solution v of (4.3), there is a solution A of (4.6) such that $\tilde{\mathcal{V}}_\delta(A)$ is close to v, again in an appropriate sense, for large but finite times. In other words, all small solutions v will eventually look like a modulated wave for a large finite time. Proving this is more difficult and requires the use of the mode filters developed in [18]. These two properties together will then give the proof of Proposition 4.1. We state the details for Turing bifurcations only and remark that the modifications for the Hopf case require the use of the results of [19].

First, we define precisely the spaces in which we will work. Let $Y_v = H^1_{\mathrm{ul}}(\mathbb{R}, \mathbb{R}^n)$ and $\mathcal{S}_t$ be the semiflow associated with equation (4.3) in Y_v. In addition, let $\mathcal{G}_T$ be the semiflow associated with (4.6) in $Y_A = H^1_{\mathrm{ul}}(\mathbb{R}, \mathbb{C})$. Local existence for both these semiflows follows from standard arguments. As mentioned above, it is known that the flow for (4.6) is globally bounded [9]. We will also need to use the fact that solutions to (4.6) remain in a ball of radius $\mathrm{O}(\sqrt{|\mu|}/\delta)$ in H^1_{ul} for all $t \geq 0$. This result, for the space H^1_{ul} rather than L^2_{ul} or L^∞, follows from the energy estimates of [9, §3].

To define the mode filters, we introduce a cutoff function $\hat{\chi} \in C^\infty(\mathbb{R}, [0, 1])$ that satisfies

$$\hat{\chi}(k) = \begin{cases} 1 & \text{if} \quad k \in [-\gamma, \gamma] \\ 0 & \text{if} \quad k \notin (-2\gamma, 2\gamma), \end{cases}$$

where γ is some small constant that is independent of μ and δ. Let

$$\mathcal{L}_-(\mu) = D\partial_x^2 + f_u(0; \mu)$$

be the linearized operator at $u = 0$ in the original frame x and denote its Fourier transform and associated adjoint by $\hat{\mathcal{L}}_- = \hat{\mathcal{L}}_-(k, \mu)$ and $\hat{\mathcal{L}}_-^* = \hat{\mathcal{L}}_-^*(k, \mu)$, respectively. For k close to k_0, these operators have the eigenvalue $\lambda(k, \mu)$ from Hypothesis (H2)(ii), plus its complex conjugate, and we denote by $\hat{e}(k, \mu)$ and $\hat{e}^*(k, \mu)$ the respective eigenvectors with $\langle \hat{e}, \hat{e}^* \rangle = 1$ for all (k, μ). We now define the operators $\widehat{P}^{\pm}$ in Fourier space as the multiplication operators

$$(\widehat{P}^{\pm}\hat{u})(k) = \hat{\chi}(k \mp k_0)\langle \hat{e}^*(k, \mu), \hat{u}(k)\rangle \hat{e}(k, \mu)$$

and the associated complex-valued versions $\hat{p}^{\pm}$ via

$$\hat{p}^{\pm}\hat{u} = \hat{\chi}(k \mp k_0)\langle \hat{e}^*(k,\mu), \hat{u}(k)\rangle.$$

These multipliers, and the ones to follow, depend on μ, but we shall suppress this dependence in our notation. Note also that the functions $\hat{e}(k,\mu)$ and $\hat{e}^*(k,\mu)$ need only be defined in balls of radius 2γ around $\pm k_0$ for the above definitions to be meaningful. We also define

$$(\widehat{P}^{\pm}_{\mathrm{mf}}\hat{u})(k) = \hat{\chi}(2(k \mp k_0))\langle \hat{e}^*(k,\mu), \hat{u}(k)\rangle \hat{e}(k,\mu),$$

and similarly $\hat{p}^{\pm}_{\mathrm{mf}}$, which have smaller support in Fourier space. Using these operators, we can define the multipliers

$$\widehat{P}^{\mathrm{c}} := \widehat{P}^{+} + \widehat{P}^{-}, \qquad \widehat{P}^{\mathrm{s}} := 1 - \widehat{P}^{\mathrm{c}}, \qquad \widehat{P}^{\mathrm{c}}_{\mathrm{mf}} := \widehat{P}^{+}_{\mathrm{mf}} + \widehat{P}^{-}_{\mathrm{mf}}, \qquad \widehat{P}^{\mathrm{s}}_{\mathrm{mf}} := 1 - \widehat{P}^{\mathrm{c}}_{\mathrm{mf}},$$

which filter either the stable or center modes, and the associated complex-valued operators $\hat{p}^{\mathrm{c}}$, $\hat{p}^{\mathrm{s}}$, $\hat{p}^{\mathrm{c}}_{\mathrm{mf}}$, and $\hat{p}^{\mathrm{s}}_{\mathrm{mf}}$. It was shown in [17, Lemma 5] that the operators $\widehat{P}^{j}_{\mathrm{mf}}$ and $\widehat{P}^{j}$ defined above in Fourier space correspond to bounded linear operators P^{j}_{mf} and P^{j} from $H^{\mathrm{s}}_{\mathrm{ul}}$ into itself for any $s \geq 0$. Furthermore, P^{j}_{mf} and P^{j} commute with $\mathcal{L}_{-}$ and map their ranges into itself. Though the operators P^{j}_{mf} and P^{j} are not projections, we have $P^{\mathrm{c}}P^{\mathrm{c}}_{\mathrm{mf}} = P^{\mathrm{c}}_{\mathrm{mf}}$, which we shall use below.

Next, define the scaling operator $\mathcal{R}_\delta$ by $(\mathcal{R}_\delta u)(x) = u(\delta x)$ and the multiplication operator Θ by $(\Theta u)(x) = \mathrm{e}^{\mathrm{i}k_0 x}u(x)$, which is just a translation operator in Fourier space. To relate the reaction-diffusion system and the Ginzburg–Landau equation, we use the modified ansatz

$$v(x) = \mathcal{V}_\delta(A) := \delta\Theta\mathcal{F}^{-1}\left[\hat{\chi}(k)\hat{e}(k+k_0,\mu)\mathcal{F}(\mathcal{R}_\delta A)\right] + \mathrm{c.c.} : Y_A \to Y_v,$$

where $\mathcal{F}$ denotes the Fourier transform. The map that sends a solution to its approximation by extracting its critical modes is given by

$$\mathcal{A}_\delta : Y_v \to Y_A, \qquad \mathcal{A}_\delta u = \frac{1}{\delta}\mathcal{R}_{1/\delta}\Theta^{-1}p^{+}u. \tag{4.7}$$

We may now state, in terms of $\mathcal{V}_\delta$ and $\mathcal{A}_\delta$, the attractivity property that will be used to connect the flows for equations (4.3) and (4.6). Let B^{Z}_{R} denote the ball of radius R in Z centered at 0.

Proposition 4.2 (Attractivity). *For each $r_0 > 0$, there exist positive constants r_1, $\tilde{r}_1$, R_0, T_0 and δ_0 such that*

$$\sup_{v \in B^{Y_v}_{\delta r_0}} \inf_{A \in B^{Y_A}_{R_0}} |\mathcal{S}_{T_0/\delta^2}(v) - \mathcal{V}_\delta(A)|_{Y_v} \quad \leq \quad r_1\delta^{5/4} \tag{4.8}$$

$$\bigcup_{t \in [0, T_0/\delta^2]} \mathcal{S}_t(B^{Y_v}_{\delta r_0}) \quad \subset \quad B^{Y_v}_{\delta \tilde{r}_1} \tag{4.9}$$

for all δ and μ with $0 < \delta < \delta_0$ and $|\mu| \leq \delta^2$.

Remark 4.1. *Proposition 4.2 asserts that we can extend the time interval on which $v(\cdot,t)$ is defined and remains small from $0 \le t \le T_{\max}$ to $0 \le \delta^2 t \le T_0$, for some T_0. Technically, we cannot extend this time interval for v without also doing so for p. However, one can see that, on any time interval where $\|v(\cdot,t)\|_{H^1_{\mathrm{ul}}}$ remains small, $\|w(\cdot,t)\|_{H^1_{\mathrm{ul}}}$ decays exponentially, and $|p(t)|$ must therefore also remain small due to equation (4.2). We will use this fact below.*

Proof (of Proposition 4.2). Since the emphasis in this proof is on the temporal behavior of solutions, we write throughout $v(t)$ and $w(t)$ instead of $v(\cdot,t)$ and $w(\cdot,t)$. All norms are taken with respect to the spatial variable for fixed time t.

The method of proof is similar to that of [18, Lemma 10], except that we need to take the term $\Delta(p,v)$ into account. We will show the following: there exists a T_0 with $0 < T_0 \le 1$ and a constant C, independent of μ and δ, such that, for any v_0 satisfying $\|v_0\|_{Y_v} \le \delta r_0$, the corresponding solution to (4.3) satisfies $\|v(t)\|_{Y_v} \le C\delta$ for all $0 \le \delta^2 t \le T_0$. Furthermore, we can write $v(T_0/\delta^2) = \delta u_{\mathrm{c}} + \delta^2 u_{\mathrm{s}}$, where $\|u_j\|_{Y_v} \le C$ and $P^j u_j = u_j$ for $j = \mathrm{c},\mathrm{s}$.

To see why this leads to the statement of the proposition, define

$$A(T_0) = \mathcal{A}_\delta(v(T_0/\delta^2)).$$

We claim that this defines the element of $\mathcal{V}_\delta(B^{Y_A}_{R_1})$ to which $\mathcal{S}_t(v_0)$ is attracted, in the sense of the proposition. Indeed, it is straightforward to show that $P^{\mathrm{c}}u = \mathcal{V}_\delta\mathcal{A}_\delta(u)$ for each u with $P^{\mathrm{c}}u = u$, and we therefore have

$$\begin{aligned}
\|\mathcal{S}_{T_0/\delta^2}(v_0) - \mathcal{V}_\delta(A(T_0))\|_{Y_v} &= \|\delta u_{\mathrm{c}} + \delta^2 u_{\mathrm{s}} - \mathcal{V}_\delta[\mathcal{A}_\delta(\delta u_{\mathrm{c}} + \delta^2 u_{\mathrm{s}})]\|_{Y_v} \\
&\le \delta\|P^{\mathrm{c}}u_{\mathrm{c}} - \mathcal{V}_\delta\mathcal{A}_\delta u_{\mathrm{c}}\|_{Y_v} + C\delta^2 \\
&\le C\delta^2.
\end{aligned}$$

Hence, it suffices to show how solutions of (4.3) can be controlled using the mode filters.

Using Lemmas 4.1 and 3.2, we can write $\Delta(p,v)(x,t) = \mathcal{H}(x,t)w(x,t)$, where $\mathcal{H}(x,t)$ is smooth and bounded uniformly for $0 \le t \le T_{\max}$, and $w(x,t)$ denotes, with a slight abuse of notation, the function $w(\xi,t)$ written in the frame (x,t). We recall that, as we shall use only H^1_{ul} estimates in the remainder of this section, the frame does not matter. Next, we set $v_{\mathrm{c}}(\cdot,0) = \delta^{-1}P^{\mathrm{c}}_{\mathrm{mf}}v_0$ and $v_{\mathrm{s}}(\cdot,0) = \delta^{-1}P^{\mathrm{s}}_{\mathrm{mf}}v_0$, and substitute $v = \delta v_{\mathrm{c}} + \delta v_{\mathrm{s}}$ into equation (4.3) to get

$$\delta\left(\partial_t v_{\mathrm{c}} + \partial_t v_{\mathrm{s}}\right) = \delta\left(\mathcal{L}_- v_{\mathrm{c}} + \mathcal{L}_- v_{\mathrm{s}}\right) + \mathcal{N}_-(\delta v_{\mathrm{c}} + \delta v_{\mathrm{s}}) + \mathcal{H}(t)w(t).$$

We now define v_{c} and v_{s} to be solutions to the following integral equations

$$\begin{aligned}
v_{\mathrm{c}}(t) &= \mathrm{e}^{\mathcal{L}_- t}v_{\mathrm{c}}(0) \qquad (4.10)\\
&\quad + \frac{1}{\delta}P^{\mathrm{c}}_{\mathrm{mf}}\int_0^t \mathrm{e}^{\mathcal{L}_-(t-\tau)}\left[\mathcal{N}_-(\delta v_{\mathrm{c}}(\tau) + \delta v_{\mathrm{s}}(\tau)) + \mathcal{H}(\tau)w(\tau)\right](\tau)\,\mathrm{d}\tau \\
v_{\mathrm{s}}(t) &= \mathrm{e}^{\mathcal{L}_- t}v_{\mathrm{s}}(0) \qquad (4.11)\\
&\quad + \frac{1}{\delta}P^{\mathrm{s}}_{\mathrm{mf}}\int_0^t \mathrm{e}^{\mathcal{L}_-(t-\tau)}\left[\mathcal{N}_-(\delta v_{\mathrm{c}}(\tau) + \delta v_{\mathrm{s}}(\tau)) + \mathcal{H}(\tau)w(\tau)\right](\tau)\,\mathrm{d}\tau.
\end{aligned}$$

Using that the mode filters and the semigroup commute, we can prove as in [18, Lemma 4] that, for each fixed T_0, there are constants K_1, K_2 and $\kappa > 0$ such that

$$\|P^{\mathrm{c}}_{\mathrm{mf}} \mathrm{e}^{\mathcal{L}_- t} u\|_{H^1_{\mathrm{ul}}} \leq K_1 \|u\|_{H^1_{\mathrm{ul}}}, \qquad \|P^{\mathrm{s}}_{\mathrm{mf}} \mathrm{e}^{\mathcal{L}_- t} u\|_{H^1_{\mathrm{ul}}} \leq K_2 \mathrm{e}^{-\kappa t} \|u\|_{H^1_{\mathrm{ul}}}$$

for all t with $0 \leq \delta^2 t \leq T_0$. The constant K_1 in the above estimate will, in general, depend on T_0 due to the growth of order $\mathrm{e}^{\mu t} \leq \mathrm{e}^{\mu T_0/\delta^2}$ in the center directions. However, as long as $|\mu| \leq \delta^2$, this will not affect our result, because below we choose $0 < T_0 \leq 1$.

We proceed now as follows: First, the system (4.3) has a unique solution (p, v) on the interval $0 \leq t \leq T_{\max}$, which we use to define the inhomogeneity $\mathcal{H}(t)w(t)$. Using this information and the estimate from Lemma 4.1, we see that the system (4.10) of integral equations defines a contraction in the ball of radius $2R_0$ centered at $(v_{\mathrm{c}}(0), v_{\mathrm{s}}(0))$ in $C^0_{\mathrm{bdd}}([0, \delta^{-\frac{1}{4}}], H^1_{\mathrm{ul}} \times H^1_{\mathrm{ul}})$, provided $\delta^{-\frac{1}{4}} \leq T_{\max}$. We now estimate v_{c} and v_{s} for $0 \leq t \leq \delta^{-\frac{1}{4}}$ to show that the components $v_{\mathrm{c,s}}$ indeed remain bounded on this time interval from which we can then infer, via the relation $v = \delta v_{\mathrm{c}} + \delta v_{\mathrm{s}}$ and by Lemma 3.2 and Remark 4.1, that $T_{\max}$ must be at least as large as $\delta^{-\frac{1}{4}}$. Using (4.10) and the fact that $\|w(0)\|_{Y_v} \leq \delta$, we find for $0 \leq t \leq \delta^{-\frac{1}{4}}$

$$\begin{aligned}
\|v_{\mathrm{c}}(t)\|_{Y_v} &\leq C\|v_{\mathrm{c}}(0)\|_{Y_v} + C\delta \int_0^t (\|v_{\mathrm{c}}(\tau)\|_{Y_v} + \|v_{\mathrm{s}}(\tau)\|_{Y_v})^2 \,\mathrm{d}\tau \\
&\quad + \frac{C}{\delta} \int_0^t \mathrm{e}^{-\Lambda\tau} \|w(0)\|_{Y_v} \,\mathrm{d}\tau \\
&\leq CR_0 + C\delta^{\frac{3}{4}} R_0^2 + \frac{C\|w(0)\|_{Y_v}}{\delta} \qquad (\text{since } 0 \leq t \leq \delta^{-\frac{1}{4}}) \\
&\leq C_1
\end{aligned}$$

and, again for $0 \leq t \leq \delta^{-\frac{1}{4}}$,

$$\begin{aligned}
\|v_{\mathrm{s}}(t)\|_{Y_v} &\leq C\mathrm{e}^{-\kappa t}\|v_{\mathrm{s}}(0)\|_{Y_v} + C\delta \int_0^t \mathrm{e}^{-\kappa(t-\tau)} (\|v_{\mathrm{c}}(\tau)\|_{Y_v} + \|v_{\mathrm{s}}(\tau)\|_{Y_v})^2 \,\mathrm{d}\tau \\
&\quad + \frac{C}{\delta} \int_0^t \mathrm{e}^{-\kappa(t-\tau)-\Lambda\tau} \|w(0)\|_{Y_v} \,\mathrm{d}\tau \\
&\leq CR_0\mathrm{e}^{-\kappa t} + C\delta R_0^2 + \frac{C\|w(0)\|_{Y_v}}{\delta} \left(\mathrm{e}^{-\Lambda/\delta^{\frac{1}{4}}} - \mathrm{e}^{-\kappa/\delta^{\frac{1}{4}}} \right) \\
&\leq \frac{C_2}{2} (\mathrm{e}^{-\kappa t} + \delta).
\end{aligned}$$

Choosing $t = \delta^{-\frac{1}{4}}$, we can conclude from the last estimate that

$$\|v_{\mathrm{s}}(\delta^{-\frac{1}{4}})\|_{Y_v} \leq \frac{C_2}{2} (\mathrm{e}^{-\kappa/\delta^{\frac{1}{4}}} + \delta) \leq C_2 \delta.$$

We now exploit this better estimate to bound the solution v over the longer time interval $[0, T_0/\delta^2]$. We define $u_{\mathrm{c}}(0) = v_{\mathrm{c}}(1/\delta^{1/4})$ and $u_{\mathrm{s}}(0) = \delta^{-1} v_{\mathrm{s}}(1/\delta^{1/4})$ and substitute $v = \delta u_{\rm c} + \delta^2 u_{\rm s}$ into equation (4.3) to arrive at the integral equation

$$\begin{aligned} u_{\rm c}(t) &= \mathrm{e}^{\mathcal{L}_- t} u_{\rm c}(0) + \frac{1}{\delta} P^{\rm c}_{\rm mf} \int_0^t \mathrm{e}^{\mathcal{L}_-(t-\tau)} \Big[\mathcal{N}_-(\delta u_{\rm c}(\tau) + \delta^2 u_{\rm s}(\tau)) \\ &\qquad + \mathcal{H}(\tau + \delta^{-\frac14}) w(\tau + \delta^{-\frac14}) \Big] \,\mathrm{d}\tau \\ u_{\rm s}(t) &= \mathrm{e}^{\mathcal{L}_- t} u_{\rm s}(0) + \frac{1}{\delta^2} P^{\rm s}_{\rm mf} \int_0^t \mathrm{e}^{\mathcal{L}_-(t-\tau)} \Big[\mathcal{N}_-(\delta u_{\rm c}(\tau) + \delta^2 u_{\rm s}(\tau)) \\ &\qquad + \mathcal{H}(\tau + \delta^{-\frac14}) w(\tau + \delta^{-\frac14}) \Big] \,\mathrm{d}\tau. \end{aligned} \tag{4.12}$$

Local existence of solutions is clear, and we therefore need to show that solutions $(u_{\rm c}, u_{\rm s})$ remain bounded on the interval $[0, T_0/\delta^2]$, uniformly in δ. The key observation that allows us to obtain the necessary estimates of the right-hand side of (4.12) is due to Schneider [16] who proved that $P^{\rm c}_{\rm mf}\mathcal{B}[P^{\rm c}_{\rm mf}u, P^{\rm c}_{\rm mf}u] = 0$ for any bilinear form $\mathcal{B}$ of the form $\mathcal{B}[u,v] = u^T B v$, where $B \in \mathbb{C}^{n\times n}$. We therefore write

$$\mathcal{N}_-(v) = \mathcal{B}[v,v] + \tilde{\mathcal{N}}(v), \qquad \tilde{\mathcal{N}}(v) = \mathrm{O}(|v|^3)$$

and note that cubic nonlinearities do not pose any problems for estimates of the below type [8]. Thus, to keep the analysis simple, we focus from now on on the quadratic terms and remark that the analysis to follow can be extended easily to account for the nonlinearity $\tilde{\mathcal{N}}(v)$ using $\|\tilde{\mathcal{N}}(v_1+v_2)\| \le C(\|v_1\|^3 + \|v_2\|^3)$. From (4.12), we obtain

$$\begin{aligned} \|u_{\rm c}(t)\|_{Y_v} &\le \|\mathrm{e}^{\mathcal{L}_- t} u_{\rm c}(0)\|_{Y_v} + C\delta^2 \int_0^t \left(\|u_{\rm c}(\tau)\|_{Y_v} \|u_{\rm s}(\tau)\|_{Y_v} + \delta \|u_{\rm s}(\tau)\|^2_{Y_v} \right) \mathrm{d}\tau \\ &\quad + \frac{C\mathrm{e}^{-\frac{\Lambda}{\delta^{1/4}}}}{\delta} \|w(0)\|_{Y_v} \\ \|u_{\rm s}(t)\|_{Y_v} &\le \|\mathrm{e}^{\mathcal{L}_- t} u_{\rm s}(0)\|_{Y_v} \\ &\quad + C\left(\|u_{\rm c}(t)\|^2_{Y_v} + \delta \|u_{\rm c}(t)\|_{Y_v} \|u_{\rm s}(t)\|_{Y_v} + \delta^2 \|u_{\rm s}(t)\|^2_{Y_v} \right) \\ &\quad + \frac{C\mathrm{e}^{-\frac{\Lambda}{\delta^{1/4}}}}{\delta} \|w(0)\|_{Y_v}. \end{aligned}$$

For $j = \mathrm{c}, \mathrm{s}$, we introduce the variables

$$U_j(t) := \sup_{0\le\tau\le t} \|u_j(\tau)\|_{Y_v}, \qquad 0 \le t \le T_0/\delta^2$$

and get

$$\begin{aligned} U_{\rm c}(t) &\le C\|u_{\rm c}(0)\|_{Y_v} + C\delta^2 \int_0^t \left[U_{\rm c}(\tau) U_{\rm s}(\tau) + \delta U_{\rm s}(\tau)^2 \right] \mathrm{d}\tau \\ &\quad + \frac{C\mathrm{e}^{-\frac{\Lambda}{\delta^{1/4}}}}{\delta} \|w(0)\|_{Y_v} \\ U_{\rm s}(t) &\le C\|u_{\rm s}(0)\|_{Y_v} + C\left[U_{\rm c}(t)^2 + \delta U_{\rm c}(t) U_{\rm s}(t) + \delta^2 U_{\rm s}(t)^2 \right] \\ &\quad + \frac{C\mathrm{e}^{-\frac{\Lambda}{\delta^{1/4}}}}{\delta} \|w(0)\|_{Y_v}. \end{aligned} \tag{4.13}$$

The constant C that appears in (4.13) does not depend on the initial data or on δ or T_0 with $T_0 \leq 1$. We now choose constants K_c and K_s so that

$$K_\mathrm{c} \leq C\|u_\mathrm{c}(0)\|_{Y_v} + C\mathrm{e}^{-\frac{\Lambda}{\delta^{1/4}}}, \qquad K_\mathrm{s} \leq C\|u_\mathrm{s}(0)\|_{Y_v} + C\mathrm{e}^{-\frac{\Lambda}{\delta^{1/4}}}$$

for all relevant initial data and values of δ, and define

$$\tilde{K}_\mathrm{s} := 4(K_\mathrm{s} + 16CK_\mathrm{c}^2).$$

Next, we pick T_0 so that $0 < T_0 < \min\{1, \log 2/[4CK_\mathrm{s}]\}$. As long as $U_\mathrm{c}(t) \leq 4K_\mathrm{c}$ and $U_\mathrm{s}(t) \leq \tilde{K}_\mathrm{s}$ for $0 \leq t \leq T_0/\delta^2$, we have

$$U_\mathrm{s}(t) \leq K_\mathrm{s} + 16CK_\mathrm{c}^2 + C[4\delta K_\mathrm{c} U_\mathrm{s}(t) + \delta^2 U_\mathrm{s}(t)^2].$$

Therefore, if δ_0 is chosen sufficiently small so that

$$4C\delta_0 K_\mathrm{c} + C\delta_0^2 4(K_\mathrm{s} + 16CK_\mathrm{c}^2) \leq \frac{1}{2},$$

then, in fact,

$$U_\mathrm{s}(t) \leq \frac{1}{2}\tilde{K}_\mathrm{s}.$$

Furthermore, substituting this bound into the equation for U_c, we have

$$U_\mathrm{c}(t) \leq K_\mathrm{c} + C\delta T_0 \tilde{K}_\mathrm{s}^2 + 4C\delta^2 \tilde{K}_\mathrm{s} \int_0^t U_\mathrm{c}(\tau)\,\mathrm{d}\tau$$

for all t with $0 \leq \delta^2 t \leq T_0$. Gronwall's inequality then implies that

$$U_\mathrm{c}(t) \leq (K_\mathrm{c} + C\delta T_0 \tilde{K}_\mathrm{s}^2)\mathrm{e}^{4CK_\mathrm{s}T_0} \leq \frac{3}{4}K_\mathrm{c}$$

due to our choice of T_0, provided δ_0 is so small that $C\delta T_0 K_\mathrm{s}^2 \leq K_\mathrm{c}/4$. This means that the preceding estimates hold true for all t in the interval $[0, T_0/\delta^2]$ as claimed. □

Proposition 4.3 (Approximation). *For all positive R_0, T_1, and r_1, there exist positive constants r_2 and δ_0 such that the following is true for all δ and μ with $\sqrt{|\mu|} \leq \delta \leq \delta_0$. If $A_0 \in B_{R_0}^{Y_A}$ and $\mathcal{S}_{T_0/\delta^2}(v_0) \in Y_v$ with $\|\mathcal{S}_{T_0/\delta^2}(v_0) - \mathcal{V}_\delta(A_0)\|_{Y_v} \leq r_1\delta^{5/4}$, then*

$$\sup_{0 \leq t \leq T_1/\delta^2} \left\|\mathcal{S}_t(\mathcal{S}_{T_0/\delta^2}(v_0)) - \mathcal{V}_\delta(\mathcal{G}_{\delta^2 t}(A_0))\right\|_{Y_v} \leq r_2\delta^{5/4}. \tag{4.14}$$

Proof. This statement is a generalization, in two ways, of the standard Ginzburg–Landau approximation theorems found, for example, in [5, 18] or the review [10]. First, the parameter δ that measures the size of the solutions is typically taken to be $\mathrm{O}(\sqrt{|\mu|})$. However, we need to allow for solutions that do not necessarily shrink to zero as μ does. This type of extension has been discussed in [11], and a similar technique can be used here.

Second, we need to account for the term $\Delta(p, v)$, which we are viewing as an inhomogeneity. To deal with this, the statement of the proposition has been slightly modified so that the approximation does not occur until a time T_0/δ^2. To see why this works, suppose we allow the solution with initial data v_0 to evolve up to the time T_0/δ^2. The proof of the preceding proposition shows the flow is well defined for this long time and that the solution remains bounded. From the time T_0/δ^2 onwards, the inhomogeneity $\Delta(p, v)$ will therefore be exponentially small in δ, even when we replace its argument x by x/δ. As a result, one can follow the arguments in [18, §3.2] to prove approximation by subsuming the inhomogeneity into the residual terms. $\square$

By combining these results with the fact that solutions of (4.6) with initial data of size $\mathrm{O}(\sqrt{|\mu|}/\delta)$ remain small, we can now prove that v stays small in Y_v for all time.

Lemma 4.2. *Under the assumptions of Propositions 4.2 and 4.3, there exist positive constants T_0, T_1 and δ_0 such that, for all δ, μ and r_0 with $\sqrt{|\mu|} \le \delta < \delta_0$ and r_0 sufficiently large, we have $\mathcal{S}_{(T_0+T_1)/\delta^2}(B^{Y_v}_{\delta r_0}) \subset B^{Y_v}_{\delta r_0}$. In particular, solutions $\mathcal{S}_t(v_0)$ with initial conditions in $B^{Y_v}_{\delta r_0}$ stay bounded and exist for all time.*

Proof. This is essentially the same proof as in [18, §1], but we restate it here for convenience. Since the bifurcation is supercritical, the results of [9] imply that all solutions of (4.6) satisfy $\limsup_{T\to\infty} \|A(T)\|_{Y_A} \le C\sqrt{|\mu|}/\delta$. Furthermore, they imply that there exists an R_∞ such that, for each $R_0 > 0$, there exist positive constants R_1 and T_1 so that

$$\mathcal{G}_{T_1}(B^{Y_A}_{R_0}) \subset B^{Y_A}_{R_\infty} \qquad \text{and} \qquad \bigcup_{T\in[0,T_1]} \mathcal{G}_T(B^{Y_A}_{R_0}) \subset B^{Y_A}_{R_1}. \tag{4.15}$$

Choose r_0 independent of δ and sufficiently large so that

$$\mathcal{V}_\delta(B^{Y_A}_{R_\infty}) \subset B^{Y_v}_{\delta r_0/2}, \tag{4.16}$$

which can be done simply by the definition of $\mathcal{V}_\delta$. By attractivity, we know that there are positive constants R_0, T_0 and r_1 so that

$$\mathcal{S}_{T_0/\delta^2}(B^{Y_v}_{\delta r_0}) \subset \mathcal{U}_{r_1\delta^{5/4}}(\mathcal{V}_\delta(B^{Y_A}_{R_0}))$$

uniformly in δ and μ, where $\mathcal{U}_r$ denotes the neighborhood of size r. For this value of R_0, there is then a T_1 so that (4.15) holds. Furthermore, by the approximation property (4.14), there is an r_2 so that

$$\begin{aligned} \mathcal{S}_{T_1/\delta^2}\left[\mathcal{S}_{T_0/\delta^2}(B^{Y_v}_{\delta r_0})\right] &\overset{(4.14)}{\subset} \mathcal{U}_{r_2\delta^{5/4}}(\mathcal{V}_\delta(\mathcal{G}_{T_1}(B^{Y_A}_{R_0}))) \\ &\overset{(4.15)}{\subset} \mathcal{U}_{r_2\delta^{5/4}}(\mathcal{V}_\delta(B^{Y_A}_{R_\infty})) \overset{(4.16)}{\subset} B^{Y_v}_{\delta r_0}. \end{aligned}$$

Furthermore, it follows from (4.9), (4.14) and (4.15) that these solutions are bounded by $\max\{\tilde{r}_1\delta, 2R_1\delta\}$ on the interval $[0, (T_0+T_1)/\delta^2]$. We can now repeat and iterate the preceding argument with $r_0 = 2R_\infty$, which proves the result. $\square$

Proposition 4.1, and therefore Theorem 1, follows now from the above results upon taking $\delta := \sqrt{|\mu|}$ when $|v(\cdot,0)|_{H^1_{\rm ul}} \le \sqrt{|\mu|}$, while taking $\delta := |v(\cdot,0)|_{H^1_{\rm ul}}$ when $|v(\cdot,0)|_{H^1_{\rm ul}} > \sqrt{|\mu|}$.

5. Discussion

We proved nonlinear stability of fronts near a supercritical Hopf or Turing bifurcation of the rest state left behind by the front. Specifically, we proved that small bounded perturbations stay bounded for all times and are pushed away from the front interface towards the wake of the front. Similar results to the ones obtained here for general reaction-diffusion systems were previously obtained in [4] for a specific model problem in which the front undergoes a Turing bifurcation. To prove our stability result, we combined the approach from [4] with techniques from [9, 16, 18], where a priori bounds of small-amplitude solutions were established using the Ginzburg–Landau formalism.

Following the arguments in [11], it should be possible to show that the dynamics in the wake of the front is governed by the associated Ginzburg–Landau equation. For Hopf bifurcations, the dynamics of the Ginzburg–Landau equation depends strongly on the coefficients $\lambda_2(0)$ and b discussed in §2. Depending on these coefficients, the prevalent dynamics may consist of stable oscillatory waves or of spatio-temporally complex patterns, which will appear with small but finite amplitude in the wake of the front. Our results show that the front will ultimately outrun these structures in its wake. We mention that this result was previously derived by Sherratt [21] through a formal analysis for fronts near supercritical Hopf bifurcations in the case when these can be described by λ-ω systems. We refer the reader to [7, 21, 20] for numerical simulations and applications to predator-prey systems.

We did not consider Turing–Hopf bifurcations, where both k_0 and ω_0 are nonzero in the original frame x. In this case, the dynamics near the destabilizing rest state can be captured formally by a system of coupled Ginzburg–Landau equations that describe small left- and right-travelling waves of the form $e^{i(k_0 x \pm \omega_0 t)}$. No rigorous approximation or validity results are known in this case.

Finally, we mention that the ideas from [4] have recently been used in [2], see also [3], to prove nonlinear stability of combustion fronts. The key difficulties in the situation discussed in [2] are that there are multiple fronts that decay algebraically to the same rest state as $x \to -\infty$ and that both rest states have essential spectrum up to the imaginary axis. The approach discussed in [4] allowed the author to obtain a priori estimates that guarantee nonlinear stability.

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