Using Global Invariant Manifolds to Understand Metastability in the Burgers Equation With Small Viscosity

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Abstract

The large-time behavior of solutions to the Burgers equation with small viscosity is described using invariant manifolds. In particular, a geometric explanation is provided for a phenomenon known as metastability, which in the present context means that solutions spend a very long time near the family of solutions known as diffusive N-waves before finally converging to a stable self-similar diffusion wave. More precisely, it is shown that in terms of similarity, or scaling, variables in an algebraically weighted L^2 space, the self-similar diffusion waves correspond to a one-dimensional global center manifold of stationary solutions. Through each of these fixed points there exists a one-dimensional, global, attractive, invariant manifold corresponding to the diffusive N-waves. Thus, metastability corresponds to a fast transient in which solutions approach this "metastable" manifold of diffusive N-waves, followed by a slow decay along this manifold, and, finally, convergence to the self-similar diffusion wave.

1 Introduction

The study of stable, or stationary, states of a physical system is a well established field of applied mathematics. Less well known or understood are "metastable" states. Such states are not fixed points of the underlying equations of motion but are typically a family of states which emerge relatively quickly, dominate the evolution of the system for long times, and then ultimately give way to the asymptotic state of the system (from which they are typically distinct). Their presence is a signal that multiple time scales are important in the problem – for instance, one time scale might be associated with the emergence of the metastable state, one associated with the evolution along the family of such states, and one associated with the emergence of the asymptotic states. (There may, of course be additional intermediate time scales.)

An important class of physical systems in which metastable states play a significant role is two-dimensional fluid flows. In three dimensional turbulent fluids, energy typically flows from large scale features through increasingly smaller length scales until ultimately it is dissipated by viscous processes in the fluid. In two-dimensions this process is reversed and energy flows from small scales to large, leading to the so-called "inverse cascade" of energy. This leads rapidly to the emergence of large scale structures or "vortices" in the system, and the evolution and interaction of a relatively small number of these vortices then dominates the flow for a very long time until viscous effects finally lead to the emergence of the asymptotic state.

Two-dimensional viscous fluids are described the by Navier-Stokes equations, a system of nonlinear partial differential equations, which describe the evolution of the velocity of the fluid (or equivalently, the evolution

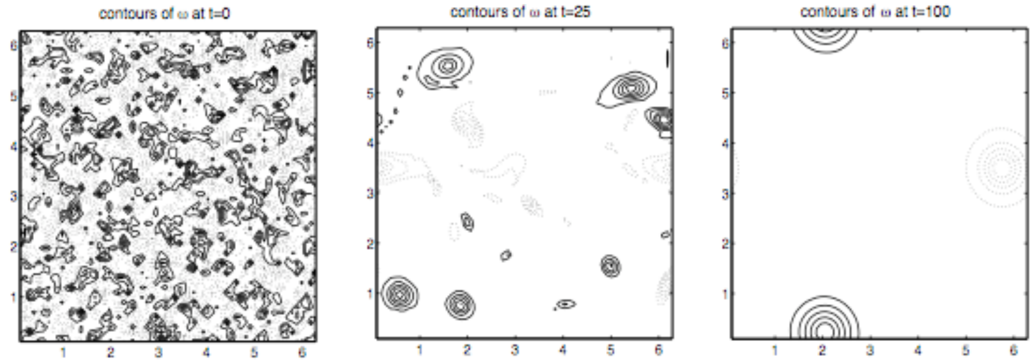

Figure 1: Evolution of the vorticity, ω, for the two-dimensional Navier-Stokes equation with periodic boundary conditions as computed by Yin, Montgomery and Clercx, [YMC03, Figure 5]. Reproduced with permission.

of its vorticity). The emergence of the metastable states in such flows is well illustrated in the numerical solution of these equations [MSM$^+$91, YMC03]. For instance, in Figure (1) (reproduced with permission from [YMC03, Figure 5]), one sees that already after a time $t = 100$, the system has converged to a set of two vortices of opposite sign which then persist for a very long time. Note that the time scale $t \sim \mathcal{O}(10^2)$ on which these vortices appear is much, much shorter that the viscous time scale determined by the viscosity parameter which in these computations was $\mu^{-1} = 5000$.

To date, there is no rigorous understanding of the emergence of these metastable states in the Navier-Stokes equation, nor an explanation for the time scale on which they appear. However, in this paper we will explore a similar phenomenon in Burgers equation. Burgers equation was proposed as a simplified model of turbulence by Burgers [Bur48] because, like the Navier-Stokes equation, it combines diffusive effects with a transport term. However, Burgers equation is much simpler both due to its one-dimensional character, as well as the fact that it is explicitly solvable via the Cole-Hopf transformation.

A key aspect of the development of metastable structures in equations like the Navier-Stokes equation is the interplay between viscous and inviscid effects and much work has gone into understanding the relationship between the solutions for zero and nonzero viscosities. For an overview, see, for example, [Daf05, Liu00]. With regard to the Burgers equation, one key property is the following. If $u^\mu = u^\mu(x,t)$ denotes the solution to the Burgers equation with viscosity μ and $u^0 = u^0(x,t)$ denotes the solution to the inviscid ($\mu = 0$) equation, then it is known that $u^\mu \to u^0$ in an appropriate sense for any fixed $t > 0$ as $\mu \to 0$. However, for fixed μ, the large-time behavior of u^μ and u^0 is quite different, and they converge to solutions known as diffusion waves and N-waves, respectively. Thus, the limits $\mu \to 0$ and $t \to \infty$ are not interchangeable.

Metastability has recently been investigated in the Burgers equation with small viscosity on an unbounded domain [KT01]. In [KT01], the authors observe numerically that solutions spend a very long time near a family of solutions known as "diffusive N-waves" before finally converging to the stable family of diffusion waves. See Figure (2) (reproduced with permission from [KT01, Figure 1]). The diffusive N-wave appears at time $\tau = 2$, whereas the diffusion wave does not emerge until time $\tau = 100$. Note that, for $\mu = 10^{-2}$, these times correspond to $\log(\mu)$ and $1/\mu$, respectively.

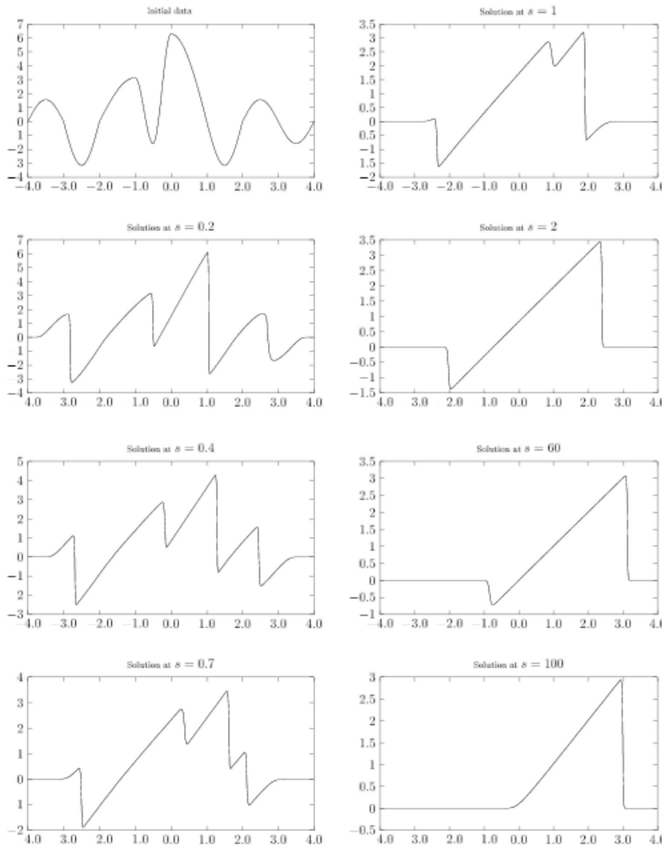

Figure 2: Evolution of the solution to Burgers equation in terms of similarity variables for $\mu = 0.01$ (see ((2.2) - (2.3)) below – the s above corresponds to τ in those equations) as computed by Kim and Tzavaras, [KT01, Figure 1]. Reproduced with permission.

This terminology[1] is due to the fact that the diffusive N-waves are close to inviscid N-waves. In [KT01] this is proven in a pointwise sense. Furthermore, in terms of scaling, or similarity, variables, they compute an asymptotic (in time) expansion for solutions to the Burgers equation with small viscosity. They find that the stable diffusion waves correspond to the first term in the expansion, whereas the diffusive N-waves correspond to taking the first two terms. Thus, by characterizing the metastability in terms of these diffusive N-waves, they provide a way of understanding the interplay between the limits $\mu \to 0$ and $t \to \infty$.

The theory of invariant manifolds has been a great aid in understanding the qualitative behavior of dynamical systems. The presence of stable, unstable or center manifolds in a system provides both a geometric picture of the possible asymptotic behaviors of the system, as well as being a powerful computational tool. For instance, the presence of a center manifold can often be used near a bifurcation point to reduce reduce the effective dimensionality of the system from infinity (in the case of a partial differential equation) to just one or two.

In this paper, we show that the metastable behavior in the viscous Burgers equation, described in [KT01], can be viewed as the approach to, and the motion along, a normally attractive, invariant manifold in the phase space of the equation. In terms of the similarity variables, we show that one has the following picture. There exists a global, one-dimensional center manifold of stationary solutions corresponding to the self-similar diffusion waves. Through each of these fixed points there exists a global, one-dimensional, invariant, normally attractive manifold corresponding to the diffusive N-waves. At the fixed point in the center manifold which represents the diffusion wave governing the long-time asymptotics of the solution, this "metastable manifold" is tangent to the eigenspace of the eigenvalue closest to zero (the zero eigenspace corresponds to motion along the center manifold) as would be expected for the leading order term in an asymptotic expansion of the long-time behavior of the solution. For almost any initial condition, the corresponding solution of the Burgers equation approaches one of the diffusive N-wave manifolds on a relatively fast timescale: $\tau = \mathcal{O}(|\log \mu|)$. Due to attractivity, the solution remains close to this manifold for all time and moves along it on a slower timescale, $\tau = \mathcal{O}(1/\mu)$, toward the fixed point which has the same total mass. Note that this corresponds to an extremely long timescale $t \approx \mathcal{O}(e^{1/\mu})$ in the original unscaled time variable. This scenario is illustrated in Figure (3).

In Burgers equation, the metastable states (diffusive N-waves) are closely related to solutions of the inviscid equation (N-waves). A similar scenario is generally believed to be true also for the 2D Navier–Stokes equation. The metastable states in that context, which are given by solutions consisting of multiple localized, well-separated vortices, are very similar to certain stationary solutions of the Euler equation, which is the inviscid limit of the Navier–Stokes equation. Although this has not been rigorously justified, there are results in that direction. See, for example, [CPR08, CPR09, Gal10].

Another similarity between Burgers equation and the 2D Navier–Stokes equation is apparent when one studies them from the point of view of dynamical systems. In [GW05] it was proved that there exists a one-dimensional center manifold of stationary solutions, similar to that depicted in Figure (3), for the 2D Navier–Stokes equation. (See section (6) for more details.) Thus, it is possible that one could obtain a similar geometric understanding of metastability in 2D Navier–Stokes.

[1] These diffusive N-waves are also discussed in [Whi99, section 4.5], where they are referred to simply as N-waves. Here, as in [KT01], we reserve the term N-wave for solutions of the inviscid equation and diffusive N-wave for solutions of the viscous equation.

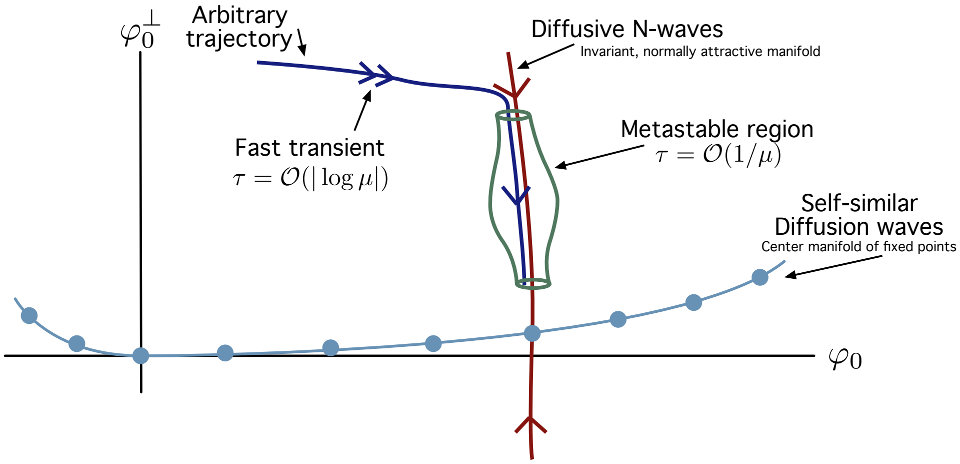

Figure 3: A schematic diagram of the invariant manifolds in the phase space of the Burgers equation, ((2.3)), and their role in the metastable behavior. The solution trajectory experiences an initial fast transient of $\tau = \mathcal{O}(|\log \mu|)$ before entering a neighborhood of the manifold of diffusive N-waves. It then remains in this neighborhood for all time as it approaches, on the slower timescale of $\tau = \mathcal{O}(1/\mu)$, a point on the manifold of stable stationary states. The horizontal axis corresponds to the eigenfunction, φ_0, associated with the zero eigenvalue, and the vertical axis corresponds to the directions orthogonal to it.

It is interesting to note that the timescales in Figure (3) are in some sense unexpected. Often metastability is associated with the presence of eigenvalues that are asymptotically small with respect to the small parameter - in this case the viscosity μ. However, the relevant linear operator that appears in the present analysis has spectrum independent of the viscosity, and the leading eigenvalues appear at $0, -1/2, -1, \ldots$. However, the asymptotic decay towards the center manifold is not given by terms like $\mathcal{O}(e^{-\tau/2})$, $\mathcal{O}(e^{-\tau})$, $\ldots$. Instead, the appropriate timescales are μ-dependent. We believe this is related to large coefficients that appear in the eigenfunction expansion of solutions. This could result from nonlinear effects, or possibly also from effects of the pseudo-spectrum, which is a property of many non-self-adjoint linear operators that can result in long transients in the dynamics of the system [TE05, GGN09].

The phenomenon of metastability occurs not just in the Burgers equation and the 2D Navier–Stokes equation, but in a wide variety of contexts. Some of the previous work that is closely related to that of this paper can be described as follows. In the setting of shocks in viscous conservation laws and fronts in gradient-type systems, such as Allen–Cahn and Cahn–Hilliard, all with small viscosity, the metastable behavior often manifests itself as the slow motion of thin transition layers. Much like the vortex patches for Navier–Stokes, these transition layers coalesce on very large timescales, until only a single transition for the shock/front solution remains. It was first noted in [KK86] that the convergence toward the viscous shock in the Burgers equation on a bounded domain can be very slow. Work toward understanding the dynamics of the transition layers in gradient systems on bounded domains was begun in [FH89, CP89, CP90], and a general result can be found in [OR07].

Further work includes asymptotic analysis for the motion of the transition layers and, in particular, the effect of the boundary conditions on this motion. For bounded domains, see [RW95b, RW95a, SW99, LO99]. On semi-infinite domains, asymptotic analysis can be found in [WR95], with rigorous justification in [LY97].

On unbounded domains, the analysis in [Che04] for gradient systems rigorously characterizes the metastable behavior by breaking it down into four stages. Particularly relevant for this paper is the result in [Che04] that the timescale of the initial transient is $\mathcal{O}(|\log \mu|)$. We obtain a similar result for the Burgers equation. See Theorem (1).

The Burgers equation on unbounded domains has been analyzed in [KT01], which was mentioned above, and also in [KN02]. In the latter, the authors focus largely on obtaining sharp decay rates for the convergence of solutions of the Burgers equation to the diffusion waves and diffusive N-waves. This is accomplished by adding some additional parameters into the diffusion waves and diffusive N-waves related to the center of mass of the initial data. Although the decay rates are not stated in terms of the size of the viscosity coefficient, in [KN02, sections 4 and 5] the authors do relate their results to the metastability observed in [KT01] through some illustrative calculations and numerical simulations. We also mention the related work [Sac87, Figure 5.9], in which slow decay of the diffusive N-waves was observed numerically.

On the surface, it seems that the mechanism creating the metastability in gradient systems, like Allen–Cahn, and in conservation laws, such as Burgers and Navier–Stokes, could be different. For example, in gradient systems, the stable, limiting states can be thought of as minimizers of some appropriate energy functional and metastable states as states that lie along the bottom of a steep-sided, but with gently sloping bottom, valley of that functional. Furthermore, in equations of the form $u_t = \mu u_{xx} - f'(u)$, like those studied in [CP89, Che04], the zero-viscosity equation is just an ODE. Thus, well-posedness of the limiting equation is not an issue. For conservation laws, it is not clear that there is an appropriate way to formulate the dynamics in terms of the minimization of energy. Furthermore, the corresponding inviscid equation is still a PDE and one that is not well-posed in the same sense as the viscous one.

In addition, as mentioned above, logarithmic timescales are observed in the metastability of the Burgers equation in this work and also in gradient systems [Che04]. However, we believe the mechanism is different. For the latter, an ODE governs the dynamics, to leading order, until gradients of $\mathcal{O}(\mu)$ develop. This process takes place exponentially fast in time, which leads to the logarithmic timescale. However, for the former, the leading order dynamics are governed by an inviscid conservation law, and so the formation of gradients is controlled only by motion along the characteristics, which is linear. The logarithmic timescale appears only in terms of the similarity variables, which induce exponential growth on the characteristics. Thus, in terms of the original (x, t) variables, the timescales for the Burgers equation and the gradient systems studied in [Che04] are actually different.

On the other hand, there are reasons to believe there could be important mathematical connections between the two settings. For example, in the gradient setting, the energy functional plays an important role. When the Burgers equation and Navier–Stokes are written in terms of similarity variables, in some sense there is an energy functional associated with those equations, as well. For Navier–Stokes it is the entropy functional, and it was used in [GW05] in analyzing the global stability of the Oseen vortex. See also [DM05]. A similar entropy functional exists for the Burgers equation, via the Cole–Hopf transformation. However, these functionals are defined only for everywhere positive (or negative) solutions. Since the metastable states are sign-varying, it is not clear that these functionals could be used to understand metastability for such systems.

A connection could also be made between metastability in Burgers and 2D Navier-Stokes and certain types of stochastic differential equations. For example, in that context, long time scales are associated with first order phase transitions. Roughly speaking, this refers the time it takes for a particle that is trapped in a local minimum of a potential to jump to another local minimum, when subjected to random perturbations. Under certain assumptions, logarithmic timescales appear in that setting, as well [BG10].

The remainder of the paper is organized as follows. In section (2) we state the equations and function spaces within which we will work, as well as some preliminary facts about the existence of invariant manifolds in the phase space of the Burgers equation. We also precisely formulate our results, Theorems (1) and (2). In sections (3), (4), and (5) we prove Lemma (2.3), Theorem (1), and Theorem (2), respectively. Concluding remarks are contained in section (6). Finally, the appendix contains a calculation that is referred to in section (2) and that originally appeared in [KT01].

2 Set-up and statement of results

We now explain the set-up for the analysis and some preliminary results on invariant manifolds. Our main results are precisely stated in section (2.3).

2.1 Equations and scaling variables

The scalar, viscous Burgers equation is the initial value problem

$$\begin{aligned} \partial_t u &= \mu \partial_x^2 u - u u_x, \\ u|_{t=0} &= h, \end{aligned} \tag{2.1}$$

where $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $u = u(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, and we assume the viscosity coefficient μ is small: $0 < \mu \ll 1$. For reasons described below, it is convenient to work in so-called similarity or scaling variables, defined as

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{1+t}} w\left(\frac{x}{\sqrt{1+t}}, \log(t+1)\right), \\ \xi &= \frac{x}{\sqrt{1+t}}, \qquad \tau = \log(t+1). \end{aligned} \tag{2.2}$$

In terms of these variables, ((2.1)) becomes

$$\begin{aligned} \partial_\tau w &= \mathcal{L}_\mu w - w w_\xi, \\ w|_{\tau=0} &= h, \end{aligned} \tag{2.3}$$

where $\mathcal{L}_\mu w = \mu \partial_\xi^2 w + \frac{1}{2}\partial_\xi(\xi w)$.

We will study the evolution of (2.3) in the algebraically weighted Hilbert space

$$L^2(m) = \left\{ w \in L^2(\mathbb{R}) : \|w\|^2_{L^2(m)} = \int (1+\xi^2)^m |w(\xi)|^2 d\xi < \infty \right\}.$$

It was shown in [GW02] that, in the spaces $L^2(m)$ with $m > 1/2$, the operator $\mathcal{L}_\mu$ generates a strongly continuous semigroup, and its spectrum is given by

$$\sigma(\mathcal{L}_\mu) = \left\{ -\frac{n}{2}, n \in \mathbb{N} \right\} \cup \left\{ \lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \le \frac{1}{4} - \frac{m}{2} \right\}. \tag{2.4}$$

This is exactly the reason why the similarity variables are so useful. Equation ((2.4)) shows that the operator $\mathcal{L}_\mu$ has a gap, at least for $m > 1/2$, between the continuous part of the spectrum and the zero eigenvalue. As m is increased, more isolated eigenvalues are revealed, allowing one to construct the associated invariant manifolds (see below for more details). In contrast, the linear operator in ((2.1)), in terms of the original variable x, has a spectrum given by $(-\infty, 0]$, which prevents the use of standard methods for constructing invariant manifolds.

For future reference, we remark that the eigenfunctions associated to the isolated eigenvalues $\lambda = -n/2$ are given by

$$\varphi_0(\xi) = \frac{1}{\sqrt{4\pi\mu}} e^{-\frac{\xi^2}{4\mu}}, \quad \varphi_n(\xi) = (\partial_\xi^n \varphi_0)(\xi). \tag{2.5}$$

See [GW02] for more details. Smoothness and well-posedness of ((2.1)) and ((2.3)) can be dealt with using standard methods, for example, information about the linear semigroups and nonlinear estimates using variation of constants.

2.2 Invariant manifolds

We now present the construction, in the phase space of (2.3), of the explicit, global, one-dimensional center manifold that consists of self-similar stationary solutions. We remark that this is similar to the global manifold of stationary vortex solutions of 2D Navier–Stokes equations analyzed in [GW05]. First, note that stationary solutions satisfy

$$\partial_\xi \left(\mu w_\xi + \frac{1}{2}\xi w - \frac{1}{2} w^2 \right) = 0.$$

They can be found explicitly by integrating the above equation and rewriting it as

$$\frac{\partial_\xi (e^{\xi^2/(4\mu)} w)}{(e^{\xi^2/(4\mu)} w)^2} = \frac{1}{2\mu} e^{-\xi^2/(4\mu)}.$$

Integrating both sides of this equation from $-\infty$ to ξ leads to the following self-similar stationary solution for each $\alpha_0 \in \mathbb{R}$:

$$w(\xi) = \frac{\alpha_0 e^{-\xi^2/(4\mu)}}{1 - \frac{\alpha_0}{2\mu} \int_{-\infty}^{\xi} e^{-\eta^2/(4\mu)} d\eta}.$$

Note that (2.3) preserves mass, and we can therefore characterize these solutions by relating the parameter α_0 to the total mass M of the solution. We have

$$M = \int_{-\infty}^{\infty} w(\xi) d\xi = \alpha_0 \int_{-\infty}^{\infty} \frac{e^{-\xi^2/(4\mu)}}{1 - \frac{\alpha_0}{2\mu} \int_{-\infty}^{\xi} e^{-\eta^2/(4\mu)} d\eta} d\xi \ = -2\mu \log \left(1 - \alpha_0 \sqrt{\frac{\pi}{\mu}} \right),$$

where we have made the change of variables $\theta = 1 - \frac{\alpha_0}{2\mu} \int_{-\infty}^{\xi} e^{-\eta^2/(4\mu)} d\eta$. Therefore, we define

$$A_M(\xi) = \frac{\alpha_0 e^{-\xi^2/(4\mu)}}{1 - \frac{\alpha_0}{2\mu} \int_{-\infty}^{\xi} e^{-\eta^2/(4\mu)} d\eta}, \qquad \alpha_0 = \sqrt{\frac{\mu}{\pi}} (1 - e^{-M/(2\mu)}). \tag{2.6}$$

These solutions are often referred to as diffusion waves [Liu00].

For $m > 1/2$, the operator $\mathcal{L}_\mu$ has a spectral gap in $L^2(m)$. By applying, for example, the results of [CHT97], we can conclude that there exists a local, one-dimensional center manifold near the origin. In addition, because each member of the family of diffusion waves is a fixed point for (2.3), they must be contained in this center manifold. Thus, this manifold is in fact global, as indicated by Figure (3).

Remark 2.1. Another way to identify this family of asymptotic states is by means of the Cole–Hopf transformation, which works for the rescaled form of the Burgers equation as well as for the original form (2.1). If w is a solution of (2.3), define

$$W(\xi,\tau) = w(\xi,\tau)e^{-\frac{1}{2\mu}\int_{-\infty}^{\xi} w(y,\tau)dy} = -2\mu\partial_\xi \exp\left(-\frac{1}{2\mu}\int_{-\infty}^{\xi} w(y,\tau)dy\right). \tag{2.7}$$

A straightforward computation shows that W satisfies the *linear* equation

$$\partial_\tau W = \mathcal{L}_\mu W. \tag{2.8}$$

Conversely, let W be a solution of (2.8) for which $1 - \frac{1}{2\mu}\int_{-\infty}^{\xi} W(y,\tau)dy > 0$ for all $\xi \in \mathbb{R}$ and $\tau > 0$. Then the inverse of the above Cole–Hopf transformation is

$$w(\xi,\tau) = -2\mu\partial_\xi \log\left(1 - \frac{1}{2\mu}\int_{-\infty}^{\xi} W(y,\tau)dy\right). \tag{2.9}$$

The family of scalar multiples of the zero eigenfunction, $\beta_0\varphi_0(\xi)$, where φ_0 is given in ((2.5)), is an invariant manifold (in fact, an invariant subspace) of fixed points for (2.8). Thus, the image of this family under (2.9) must be an invariant manifold of fixed points for (2.3). Computing this image leads exactly to the family ((2.6)), where $\beta_0 = \sqrt{4\pi\mu}\alpha_0$.

Remark 2.2. One can prove that this self-similar family of diffusion waves is globally stable using the entropy functional

$$H[w](\tau) = \int_{\mathbb{R}} w(\xi,\tau)e^{-\frac{1}{2\mu}\int_{-\infty}^{\xi} w(y,\tau)dy} \log\left[\frac{w(\xi,\tau)e^{-\frac{1}{2\mu}\int_{-\infty}^{\xi} w(y,\tau)dy}}{e^{-\frac{\xi^2}{4\mu}}}\right] d\xi.$$

This is just the standard entropy functional for the linear equation ((2.8)) with potential $\xi^2/(4\mu)$, in combination with the Cole–Hopf transformation. For further details regarding these facts, see [DiF03].

We next construct the manifold of diffusive N-waves. Recall that, by ((2.4)), if $m > 3/2$, then both the eigenvalue at 0 and the eigenvalue at $-1/2$ are isolated. The latter will lead to a one-dimensional stable manifold at each stationary solution.

To see this, define $w = A_M + v$ and obtain

$$\begin{aligned} v_\tau &= \mathcal{A}_\mu^M v - v v_\xi, \\ \mathcal{A}_\mu^M v &= \mathcal{L}_\mu v - (A_M v)_\xi, \end{aligned} \tag{2.10}$$

where $\mathcal{A}_\mu^M$ is just the linearization of ((2.3)) about the diffusion wave with mass M. One can see explicitly, using the Cole–Hopf transformation, that the operators $\mathcal{L}_\mu$ and $\mathcal{A}_\mu^M$ are conjugate with the conjugacy operator given explicitly by

$$\mathcal{A}_\mu^M U = U\mathcal{L}_\mu,$$

$$U = \partial_\xi\left[\left(\int_{-\infty}^{\xi}\cdot\right)e^{\frac{1}{2\mu}\int_{-\infty}^{\xi} A_M(y)dy}\right], \qquad U^{-1} = \partial_\xi\left[\left(\int_{-\infty}^{\xi}\cdot\right)e^{-\frac{1}{2\mu}\int_{-\infty}^{\xi} A_M(y)dy}\right].$$

Thus, one can check that the spectra of the operators $\mathcal{A}_\mu^M$ and $\mathcal{L}_\mu$ are equivalent in $L^2(m)$. Furthermore, we can see explicitly that the eigenfunctions of $\mathcal{A}_\mu^M$ are given explicitly by

$$\Phi_n(\xi) = \partial_\xi \left(\frac{\int_{-\infty}^{\xi} \varphi_n(y)dy}{1 - \frac{\alpha}{2\mu} \int_{-\infty}^{\xi} e^{-\frac{\eta^2}{4\mu}} d\eta} \right),$$

where φ_n is an eigenfunction of $\mathcal{L}_\mu$. Notice that, up to a scalar multiple, $\Phi_1 = \partial_\xi A_M$. If we choose $m > 3/2$, by ((2.4)) we can then construct a local, 2D center-stable manifold near each diffusion wave. We wish to show that, if the mass is chosen appropriately, then this manifold is actually one-dimensional. Furthermore, we must show that this manifold is a global manifold.

To do this, we appeal to the Cole–Hopf transformation. Using ((2.7)), we define

$$V(\xi,\tau) = v(\xi,\tau)e^{-\frac{1}{2\mu}\int_{-\infty}^{\xi} v(y,\tau)dy}$$

and find that V solves the linear equation

$$\partial_\tau V = \mathcal{A}_\mu^M V.$$

Thus, the 2D center-stable subspace is given by $\operatorname{span}\{\Phi_0, \Phi_1\}$. The adjoint eigenfunction associated with Φ_0 is just a constant. Therefore, if we restrict ourselves to initial conditions that satisfy

$$\int_{\mathbb{R}} V(\xi,0)d\xi = 0, \tag{2.11}$$

then the subspace will be one-dimensional and will be given by solutions of the form

$$V(\xi,\tau) = \alpha_1 \Phi_1(\xi) e^{-\frac{\tau}{2}}.$$

One can check that condition ((2.11)) is equivalent to

$$\int_{\mathbb{R}} v(\xi,0)d\xi = 0.$$

Since $w = A_M + v$, we can ensure that this condition is satisfied by choosing the diffusion wave that satisfies

$$M = \int_{\mathbb{R}} A_M(\xi)d\xi = \int_{\mathbb{R}} w(\xi,0)d\xi.$$

Thus, near each diffusion wave of mass M, there exists a local invariant foliation of solutions with the same mass M that decay to the diffusion wave at rate $e^{-\frac{1}{2}\tau}$.

To extend this to a global foliation, we simply apply the inverse Cole–Hopf transformation ((2.9)), as in Remark (2.1), to the invariant subspace

$$\{V(\xi,\tau) = \alpha_1 \Phi_1(\xi)\} = \{V(\xi,\tau) = \alpha_1 \partial_\xi A_M\}.$$

This leads to the global stable invariant foliation consisting of solutions to ((2.10)) of the form

$$v_N(\xi,\tau) = \frac{\alpha_1 e^{-\frac{\tau}{2}} A_M'(\xi)}{1 - \frac{\alpha_1}{2\mu} e^{-\frac{\tau}{2}} A_M(\xi)}.$$

Using the relationship between v and w, this foliation leads to a family of solutions of ((2.3)) of the form

$$\tilde{w}_N(\xi,\tau) = A_M(\xi) + v_N(\xi,\tau) = A_M(\xi) + \frac{\alpha_1 e^{-\frac{\tau}{2}} A'_M(\xi)}{1 - \frac{\alpha_1}{2\mu} e^{-\frac{\tau}{2}} A_M(\xi)}. \tag{2.12}$$

Below it will be convenient to use a slightly different formulation of this family, which we now present.

The subspace span$\{\varphi_0, \varphi_1\}$, corresponding to the first two eigenfunctions in ((2.5)), is invariant for ((2.8)). Therefore, as in Remark (2.1), by using the inverse Cole–Hopf transformation ((2.9)) we immediately obtain the explicit, two parameter family

$$w_N(\xi,\tau) = \frac{\beta_0\varphi_0(\xi) + \beta_1 e^{-\frac{\tau}{2}}\varphi_1(\xi)}{1 - \frac{\beta_0}{2\mu}\int_{-\infty}^{\xi}\varphi_0(y)dy - \frac{\beta_1}{2\mu}e^{-\frac{\tau}{2}}\varphi_0(\xi)}, \tag{2.13}$$

where

$$\beta_0 = \sqrt{4\pi\mu}\alpha_0 = 2\mu(1 - e^{-\frac{M}{2\mu}}). \tag{2.14}$$

Based on the above analysis, ((2.13)) and ((2.12)) are equivalent. Note that, although the method used to produce ((2.13)) is much more direct than that of ((2.12)), we needed to use the operator $\mathcal{A}_M$ and its spectral properties to justify the claim that this family does in fact correspond to an invariant stable foliation of the manifold of diffusion waves.

We now explain why solutions of the form ((2.13)) are referred to as the family of diffusive N-waves. As mentioned in section 1, this terminology was justified in [KT01] by showing that each solution w_N is close to an inviscid N-wave pointwise in space. Since we are working in $L^2(m)$, we need to prove a similar result in that space.

Recall some facts about the N-waves, which can be found, for example, in [Liu00]. Define

$$p = -2\inf_y \int_{-\infty}^{y} u(x)dx, \quad \text{and} \quad q = 2\sup_y \int_{y}^{\infty} u(x)dx, \tag{2.15}$$

which are invariant for solutions of ((2.1)) when $\mu = 0$. (Note that our definitions of p and q differ from those in [KT01] by a factor of 2.) The mass satisfies $M = (q-p)/2$. We will refer to q as the "positive mass" of the solution and p as the "negative mass" of the solution. The associated N-wave is given by

$$\mathcal{N}_{p,q}(x,t) = \begin{cases} \frac{x}{t+1} & \text{if } -\sqrt{p(t+1)} < x < \sqrt{q(t+1)}, \\ 0 & \text{otherwise,} \end{cases}$$

which is a weak solution of ((2.1)) only when $\mu = 0$. When $0 < \mu \ll 1$ it is only an approximate solution because the necessary jump condition associated with weak solutions is not satisfied. One can check that its positive and negative masses are given by q and p. In terms of the similarity variables ((2.2)), this gives a two parameter family of *stationary* solutions

$$N_{p,q}(\xi) = \begin{cases} \xi & \text{if } -\sqrt{p} < \xi < \sqrt{q}, \\ 0 & \text{otherwise} \end{cases}$$

of ((2.3)) when $\mu = 0$.

We now relate the quantities β_0 and β_1 in ((2.13)) to the quantities p and q. These calculations closely follow those of [KT01, section 5]. Using ((2.14)) and the fact that $M = (q-p)/2$, we see that

$$\beta_0 = 2\mu(1 - e^{-\frac{(q-p)}{4\mu}}) = \begin{cases} 2\mu + \mathbf{exp} & \text{if } q > p, \\ -2\mu e^{-\frac{(q-p)}{4\mu}} + \mathcal{O}(\mu) & \text{if } q < p, \end{cases} \tag{2.16}$$

where $\mathbf{exp} = \mathcal{O}(e^{-C/\mu})$ for some $C > 0$. Using the calculation in the appendix, one can relate the quantity β_1 in ((2.13)), for any fixed τ, to the quantities p and q via

$$\beta_1 e^{-\frac{\tau}{2}} = -4\mu^{3/2}\sqrt{\pi}e^{p/(4\mu)} - \frac{1\sqrt{\mu}}{\sqrt{\pi}} + \mathcal{O}(\mu) \quad \text{for} \quad 0 < q < p, \tag{2.17}$$

and a similar result holds for $q > p > 0$. Two key consequences of this, which will be used below, are as follows:

- The quantities β_0 and β_1 are related via
$$\frac{\beta_0}{\beta_1} = \mathbf{exp}.$$
- The values of p and q for the diffusive N-waves change on a timescale of $\tau = \mathcal{O}(\frac{1}{\mu})$. (Recall that they are invariant only for $\mu = 0$.)

This second property, which can be seen by differentiating ((2.17)) with respect to τ, will lead to the slow drift along the manifold of diffusive N-waves (see below for more details).

The following lemma, which will be proved in section (3), states precisely that there exists an N-wave that is close in $L^2(m)$ to each member of the family w_N, at least if the viscosity is sufficiently small, thus justifying the terminology "diffusive N-wave."

Lemma 2.3. *Given any positive constants δ, p, and q, let $w_N(\xi,\tau)$ be a member of the family ((2.13)) of diffusive N-waves such that, at time $\tau = \tau_0$, the positive mass of $w_N(\cdot,\tau_0)$ is q and the negative mass is p. There exists a $\mu_0 > 0$ sufficiently small such that, if $0 < \mu < \mu_0$, then*

$$\|w_N(\cdot,\tau_0) - N_{p.q}(\cdot)\|_{L^2(m)} < \delta.$$

2.3 Statement of main results

We have seen above that the phase space of ((2.3)) does possess the global invariant manifold structure that is indicated in Figure (3). To complete the analysis, we must prove our two main results, which provide the fast timescale on which solutions approach the family of diffusive N-waves and the slow timescale on which solutions decay, along the metastable family of diffusive N-waves, to the stationary diffusion wave.

Theorem 1 (the initial transient). *Fix $m > 3/2$. Let $w(\xi,\tau)$ denote the solution to the initial value problem ((2.3)) whose initial data has mass M, and let $N_{p,q}(\xi)$ be the inviscid N-wave with values p and q determined by the initial data $w(\xi,0) = h(\xi) \in L^2(m)$. Given any $\delta > 0$, there exist a $T > 0$, which is $\mathcal{O}(|\log\mu|)$, and a μ sufficiently small so that*

$$||w(T) - N_{p,q}||_{L^2(m)} \leq \delta.$$

This theorem states that, although the quantities p and q are determined by the initial data $w(\xi, 0)$, w is close to the associated N-wave, $N_{p,q}$, at a time $\tau = T = \mathcal{O}(|\log \mu|)$. The reason for this is that $p = p(\tau)$ and $q = q(\tau)$ change on a timescale of $\mathcal{O}(1/\mu)$, which can be seen using ((2.17)) and is slower than the initial evolution of w. The rate of change of p and q also determines the rate of motion of solutions along the manifold of diffusive N-waves, as illustrated in Figure (3). Note that this theorem states that the solution will be close to an inviscid N-wave after a time $T = \mathcal{O}(|\log \mu|)$. By combining this with Lemma (2.3), we see that the solution is also close to a diffusive N-wave.

We remark that the timescale $\mathcal{O}(|\log \mu|)$ is similar to the timescale obtained in [Che04] when analyzing metastability in gradient systems. Furthermore, this timescale corresponds well with the numerical observations of [KT01, Figure 1], where one can see that, for $\mu = 0.01$, the solution looks like a diffusive N-wave at time 2 and a diffusion wave at time 100.

Remark 2.4. To some extent, this fast approach to the manifold of N-waves can be thought of in terms of the Cole–Hopf transformation ((2.7)), which depends on μ. For small μ, this nonlinear coordinate change can reduce the variation in the solution for $|\xi|$ large. This is illustrated in Figure 5.1 of [KN02]. If μ is small enough, the Cole–Hopf transformation can make the initial data look like an N-wave even before any evolution has taken place.

Theorem 2 (local attractivity). *There exists a c_0 sufficiently small such that, for any solution $w(\cdot, \tau)$ of the viscous Burgers equation ((2.3)) for which the initial conditions satisfy*

$$w|_{\tau=0} = w_N^0 + \phi^0,$$

where w_N^0 is a diffusive N-wave and $\|\phi^0\|_{L^2(m)} \leq c_0$, there exists a constant C_ϕ such that

$$w(\cdot, \tau) = w_N(\cdot, \tau) + \phi(\cdot, \tau),$$

with w_N the corresponding diffusive N-wave solution and

$$\|\phi(\cdot, \tau)\|_{L^2(m)} \leq C_\phi e^{-\tau}.$$

By combining these results, we obtain a geometric description of metastability. Theorem (1) and Lemma (2.3) tell us that, for any solution, there exists a $T = \mathcal{O}(|\log \mu|)$ at which point the solution is near a diffusive N-wave. By using Theorem (2) with this solution at time T as the "initial data," we see that the solution must remain near the family of diffusive N-waves for all time. As remarked above, the timescale of $\mathcal{O}(1/\mu)$ on which the solution decays to the stationary diffusion wave is then determined by the rates of change of $p(\tau)$ and $q(\tau)$ within the family of diffusive N-waves. In other words, near the manifold of diffusive N-waves, $w(\xi, \tau) = w_N(\xi, \tau) + \phi(\xi, \tau)$, where $\phi(\xi, \tau) \sim e^{-\tau}$ and $w_N(\xi, \tau)$ is approaching a self-similar diffusion wave on a timescale determined by the rates of change of $p(\tau)$ and $q(\tau)$, which are $\mathcal{O}(1/\mu)$.

We remark that it is not the spectrum of $\mathcal{L}_\mu$ that determines, with respect to μ, the rate of metastable motion. Instead, this is given by the sizes of the coefficients β_0 and β_1 in the spectral expansion and their relationship to the quantities p and q.

3 Proof of Lemma (2.3)

We now prove Lemma (2.3).

In [KT01], Kim and Tzavaras prove that the inviscid N-wave is the pointwise limit, as $\mu \to 0$, of the diffusive N-wave. Here we extend their argument to show that one also has convergence in the $L^2(m)$ norm. For simplicity we will check explicitly the case in which $0 < q < p$; the case in which q is larger than p follows in an analogous fashion. However, we note that we do require that both p and q be nonzero, which is why we stated in the introduction that our results hold only for "almost all" initial conditions. See Remark (3.1) below.

Using ((2.12)) we can write the diffusive N-wave with positive and negative mass given by q and p at time τ_0 as

$$w_N(\xi,\tau_0) = \frac{\beta_0\varphi_0(\xi) + \beta_1 e^{-\tau_0/2}\varphi_1(\xi)}{1 - \frac{\beta_0}{2\mu}\int_{-\infty}^{\xi}\varphi_0(y)dy - \frac{\beta_1}{2\mu}e^{-\tau_0/2}\varphi_0(\xi)} = \frac{\beta_0\varphi_0(\xi) + \tilde\beta_1\varphi_1(\xi)}{1 - \frac{\beta_0}{2\mu}\int_{-\infty}^{\xi}\varphi_0(y)dy - \frac{\tilde\beta_1}{2\mu}\varphi_0(\xi)},$$

where for notational simplicity we have defined $\tilde\beta_1 = \beta_1 e^{-\tau_0/2}$. If we now recall that $\varphi_1(\xi) = -\frac{\xi}{2\mu}\varphi_0(\xi)$, we can rewrite the expression for the w_N as

$$w_N(\xi,\tau_0) = \frac{\xi - \frac{2\mu\beta_0}{\tilde\beta_1}}{1 - \frac{2\mu}{\tilde\beta_1\varphi_0(\xi)}\{1 - \frac{\beta_0}{2\mu}\int_{-\infty}^{\xi}\varphi_0(y)dy\}}. \tag{3.1}$$

We need to prove that

$$\int_{-\infty}^{\infty}(1+\xi^2)^m(w_N(\xi,\tau_0) - N_{p,q}(\xi))^2 < \delta^2.$$

We will give the details for

$$\int_{0}^{\infty}(1+\xi^2)^m(w_N(\xi,\tau_0) - N_{p,q}(\xi))^2 < \delta^2/2.$$

The integral over the negative half axis is entirely analogous.

Break the integral over the positive axis into three pieces—the integral from $[0,\sqrt{q}-\epsilon]$, the integral from $[\sqrt{q}-\epsilon,\sqrt{q}+\epsilon]$, and the integral from $[\sqrt{q}+\epsilon,\infty)$. Here ϵ is a small constant that will be fixed in the discussion below. We refer to the integrals over each of these subintervals as I, II, and III, respectively, and bound each of them in turn.

The simplest one to bound is the integral II. Note that, using ((2.14)) and ((2.17)), the denominator in ((3.1)) can be bounded from below by $1/2$ and, thus, the integrand can be bounded by $C(1+(\sqrt{q}+\epsilon)^2)^m q$. Therefore, if $\epsilon < \sqrt{q}$, we have the elementary bound

$$II \le C\epsilon q(1+4q)^m.$$

Next consider term III. For $\xi > \sqrt{q}$, $N_{p,q}(\xi) = 0$, and so

$$III = \int_{\sqrt{q}+\epsilon}^{\infty}(1+\xi^2)^m(w_{p,q}(\xi,\tau_0))^2 d\xi.$$

To estimate this term we begin by considering the denominator in (3.1). Using ((2.16)) and ((2.17)), we have

$$1 - \frac{\beta_0}{2\mu}\int_{-\infty}^{\xi}\varphi_0(y)dy = e^{\frac{1}{4\mu}(p-q)} + \frac{\beta_0}{2\mu}\int_{\xi}^{\infty}\varphi_0(y)dy.$$

Thus, the full denominator in (3.1) has the form

$$\begin{aligned}1 - \frac{2\mu}{\tilde{\beta}_1 \varphi_0(\xi)} &\left\{ e^{\frac{1}{4\mu}(p-q)} + \frac{\beta_0}{2\mu} \int_{\xi}^{\infty} \varphi_0(y)dy \right\} \\ &= 1 - \frac{4\sqrt{\pi}\mu^{3/2}}{\tilde{\beta}_1} e^{\frac{1}{4\mu}(p-q)} e^{\xi^2/(4\mu)} + \frac{\beta_0}{\tilde{\beta}_1} \frac{\int_{\xi}^{\infty} \varphi_0(y)dy}{\varphi_0(\xi)} \\ &= 1 + \frac{4\sqrt{\pi}\mu^{3/2}}{1\sqrt{\pi}\mu^{3/2} e^{p/(4\mu)} + \mathcal{O}(\sqrt{\mu})} e^{\frac{1}{4\mu}(p-q)} e^{\xi^2/(4\mu)} + \mathbf{exp} \\ &= 1 + \left(\frac{e^{\frac{1}{4\mu}(\xi^2-q)}}{1 + \mathcal{O}(\mu^{-1} e^{-p/(4\mu)})} \right) + \mathbf{exp},\end{aligned}$$

where $\mathbf{exp}$ denotes terms that are exponentially small in μ (i.e., contain terms of the form $e^{-p/(4\mu)}$ or $e^{-q/(4\mu)}$), uniformly in ξ. Note that, in the above, the term $\int_{\xi}^{\infty} \varphi_0(y)dy/\varphi_0(\xi)$ was bounded uniformly in μ using the estimate

$$\int_x^{\infty} e^{-\frac{s^2}{2}} ds \leq \frac{1}{x} e^{-\frac{x^2}{2}} \quad \text{for} \quad x > 0,$$

which can be found in [KS91, Problem 9.22]. But with this estimate on the denominator of w_N, we can bound the integral III by

$$III \leq C \int_{\sqrt{q}+\epsilon}^{\infty} (1+\xi^2)^m \left(\xi - \frac{2\mu\beta_0}{\tilde{\beta}_1} \right)^2 (1 + e^{\frac{1}{4\mu}(\xi^2-q)})^{-2} d\xi,$$

where the constant C can be chosen independently of μ for $\mu < \mu_0$ if μ_0 is sufficiently small. This integral can now be bounded by elementary estimates, and we find

$$III \leq C e^{-\epsilon\sqrt{q}/2\mu},$$

where the constant C depends on q but can be chosen independently of μ.

Finally, we bound the integral I. Note that for $0 < \xi < \sqrt{q}$, $N_{p,q}(\xi) = \xi$, and so

$$w_{p,q}(\xi, \tau_0) - N_{p,q}(\xi) = \frac{-\frac{2\mu\beta_0}{\tilde{\beta}_1} + \xi e^{\frac{1}{4\mu}(\xi^2-q)} + \xi\mathbf{exp}}{(1 + e^{(\xi^2-q)/4\mu} + \mathbf{exp})}.$$

However, using our expressions for $\beta_{0,1}$ in terms of p and q and the fact that in term I $\xi^2 - q < -\epsilon\sqrt{q}$, we see that all of these terms are exponentially small. Since the length of the interval of integration is bounded by $\sqrt{q}$, we have the bound

$$I \leq Cq e^{-\epsilon\sqrt{q}/(4\mu)} + \mathbf{exp}.$$

Combining the estimates on the terms I, II, and III, we see that if we first choose $\epsilon \ll \delta$ and then $\mu \ll \epsilon$, the estimate in the lemma follows. This completes the proof of Lemma (2.3).

Remark 3.1. The calculation in the appendix shows that $\beta_1 = 0$ if and only if $p = 0$ or $q = 0$. Therefore, in that case, w_N is really just a diffusion wave, and so there is no metastable period in which it looks like an inviscid N-wave.

4 Proof of Theorem (1)

In this section we show that for arbitrary initial data in $L^2(m)$, $m > 3/2$, the solution approaches an inviscid N-wave in a time of $\mathcal{O}(|\log\mu|)$, thus proving Theorem (1).

Remark 4.1. Here we will use a different form of the Cole–Hopf transformation than that given in ((2.7)). In particular, we will use

$$U(x,t) = e^{-\frac{1}{2\mu}\int_{-\infty}^{x} u(y,t)dy}. \tag{4.1}$$

Equation ((2.7)) is essentially the derivative of ((4.1)), and both transform the nonlinear Burgers equation into the linear heat equation. Each is useful to us in different ways. Equation ((2.7)) preserves the localization of functions, for example, whereas ((4.1)) leads to a slightly simpler inverse, which will be easier to work with in the current section.

Using the Cole–Hopf transformation ((4.1)) and the formula for the solution of the heat equation, we find that the solution of (2.1) can be written as

$$u(x,t) = \frac{\int \frac{(x-y)}{t} e^{-\frac{1}{2\mu}\left(\frac{1}{2t}(x-y)^2 + H(y)\right)} dy}{\int e^{-\frac{1}{2\mu}\left(\frac{1}{2t}(x-y)^2 + H(y)\right)} dy},$$

where $H(y) = \int_{-\infty}^{y} h(z)dz$. If we change to the rescaled variables ((2.2)), this gives the solution to (2.3) in the form

$$w(\xi,\tau) = \frac{\int (\xi-\eta) e^{-\frac{1}{2\mu}\left(\frac{1}{2}(\xi-\eta)^2 + H(e^{\tau/2}\eta)\right)} d\eta}{\int e^{-\frac{1}{2\mu}\left(\frac{1}{2}(\xi-\eta)^2 + H(e^{\tau/2}\eta)\right)} d\eta}. \tag{4.2}$$

We will prove that, if we fix $\delta > 0$, then there exist a μ sufficiently small and a T sufficiently large ($\mathcal{O}(|\log\mu|)$ as $\mu \to 0$) such that

$$\|w(\cdot,T) - N_{p,q}(\cdot)\|_{L^2(m)} < \delta.$$

We estimate the norm by breaking the corresponding integral into subintegrals using $(-\infty, -\sqrt{p}-\epsilon)$, $(-\sqrt{p}-\epsilon, -\sqrt{p}+\epsilon)$, $(-\sqrt{p}+\epsilon, -\epsilon)$, $(-\epsilon, \epsilon)$, $(\epsilon, \sqrt{q}-\epsilon)$, $(\sqrt{q}-\epsilon, \sqrt{q}+\epsilon)$, and $(\sqrt{q}+\epsilon, \infty)$. Note that the integrals over the "short" intervals can all be bounded by $C\epsilon$, so we ignore them. We estimate the integrals over $(\epsilon, \sqrt{q}-\epsilon)$ and $(\sqrt{q}+\epsilon, \infty)$; the remaining two are very similar.

First, consider the region where $\xi > \sqrt{q}+\epsilon$. In this region, $N(\xi) \equiv 0$, so we need only to show that, given any $\delta > 0$, there exist a μ sufficiently small and a $T > 0$, of $\mathcal{O}(|\log\mu|)$, such that

$$\int_{\sqrt{q}+\epsilon}^{\infty} (1+\xi^2)^m |w(\xi,\tau)|^2 d\xi < \delta.$$

Consider the formula for w given in ((4.2)). To bound this, we must bound the denominator from below and the numerator from above. We will first focus on the denominator.

We will write the denominator as

$$\begin{aligned}\int e^{-\frac{1}{2\mu}\left(\frac{1}{2}(\xi-\eta)^2+H(e^{\tau/2}\eta)\right)}d\eta &= \int_{-\infty}^{-Re^{-\tau/2}} e^{-\frac{1}{2\mu}\left(\frac{1}{2}(\xi-\eta)^2+H(e^{\tau/2}\eta)\right)}d\eta \\ &\quad + \int_{Re^{-\tau/2}}^{\infty} e^{-\frac{1}{2\mu}\left(\frac{1}{2}(\xi-\eta)^2+H(e^{\tau/2}\eta)\right)}d\eta \\ &\quad + \int_{-Re^{-\tau/2}}^{Re^{-\tau/2}} e^{-\frac{1}{2\mu}\left(\frac{1}{2}(\xi-\eta)^2+H(e^{\tau/2}\eta)\right)}d\eta \\ &\equiv I_1+I_2+I_3\end{aligned}$$

for some $R>0$ that will be determined later. Consider the first integral, I_1. In this region

$$\begin{aligned}|H(e^{\frac{\tau}{2}}\eta)| &= \left|\int_{-\infty}^{e^{\frac{\tau}{2}}\eta} \frac{1}{(1+y^2)^{\frac{m}{2}}}(1+y^2)^{\frac{m}{2}}h(y)dy\right| \\ &\leq ||h||_m \int_{-\infty}^{-R} \frac{1}{(1+y^2)^m}dy \\ &\leq C(R,||h||_m),\end{aligned} \tag{4.3}$$

where the constant $C(R,||h||_m)\to 0$ as $R\to\infty$ or $||h||_m\to 0$. In addition, note that the error function satisfies the bounds

$$\frac{z}{1+z^2}e^{-\frac{z^2}{2}} \leq \int_z^\infty e^{-\frac{s^2}{2}}ds \leq \frac{1}{z}e^{-\frac{z^2}{2}}$$

for $z>0$ [KS91, p. 112, Problem 9.22]. Therefore, we have that

$$\begin{aligned}I_1 &\geq e^{-\frac{C(R,||h||_m)}{2\mu}} \int_{-\infty}^{-Re^{\frac{\tau}{2}}} e^{-\frac{1}{4\mu}(\xi-\eta)^2}d\eta \\ &\geq e^{\frac{-C(R,||h||_m)}{2\mu}} \frac{2\mu(\xi+Re^{-\tau/2})}{2\mu+(\xi+Re^{-\frac{\tau}{2}})^2} e^{-\frac{(\xi+Re^{-\frac{\tau}{2}})^2}{4\mu}} \\ &\geq C\frac{\mu}{\sqrt{q}}e^{\frac{-C(R,||h||_m)}{2\mu}}e^{-\frac{(\xi+Re^{-\frac{\tau}{2}})^2}{4\mu}} \\ &\geq C\frac{\mu}{\sqrt{q}}e^{\frac{-C(R,||h||_m)}{2\mu}}e^{-\frac{R^2e^{-\tau}}{2\mu}}e^{-\frac{\xi^2}{2\mu}},\end{aligned} \tag{4.4}$$

where we have used the fact that $(a+b)^2\leq 2a^2+2b^2$. In order to bound I_2, we will use that, for $\eta > Re^{\frac{\tau}{2}}$, similar to ((4.3)),

$$\left|\int_{-\infty}^{e^{\frac{\tau}{2}}\eta} h(y)dy\right| = \left|M - \int_{e^{\frac{\tau}{2}}\eta}^{\infty} h(y)dy\right| \leq M + C(R,||h||_m).$$

Therefore,

$$\begin{aligned}I_2 &\geq e^{-\frac{1}{2\mu}(M+C(R,||h||_m))} \int_{Re^{-\frac{\tau}{2}}}^{\infty} e^{-\frac{1}{4\mu}(\xi-\eta)^2}d\eta \\ &= e^{-\frac{1}{2\mu}(M+C(R,||h||_m))}\sqrt{4\mu}\int_{-\infty}^{\frac{\xi-Re^{-\frac{\tau}{2}}}{\sqrt{4\mu}}} e^{-s^2}ds \\ &\geq Ce^{-\frac{1}{2\mu}(M+C(R,||h||_m))}\sqrt{4\mu}.\end{aligned} \tag{4.5}$$

Note that in making the above estimate, we have chosen τ large enough so that $|Re^{-\frac{\tau}{2}}|<\epsilon/2$, and so $\xi - Re^{-\tau/2}>0$. Thus, the error function is bounded from below by $\sqrt{\pi}/2$.

Consider I_3. We can bound

$$H(e^{\frac{\tau}{2}}\eta) = M - \int_{e^{\frac{\tau}{2}}\eta}^{\infty} h(y)dy \leq M + \frac{q}{2}.$$

Therefore,

$$\begin{aligned} I_3 &\geq e^{-\frac{1}{2\mu}(M+\frac{q}{2})} \int_{Re^{-\frac{\tau}{2}}}^{Re^{\frac{\tau}{2}}} e^{-\frac{1}{4\mu}(\xi-\eta)^2} d\eta \\ &\geq e^{-\frac{1}{2\mu}(M+\frac{q}{2})} e^{-\frac{\xi^2}{2\mu}} \int_{-Re^{-\frac{\tau}{2}}}^{Re^{-\frac{\tau}{2}}} e^{-\frac{\eta^2}{2\mu}} d\eta \\ &\geq e^{-\frac{1}{2\mu}(M+\frac{q}{2})} e^{-\frac{\xi^2}{2\mu}} 2Re^{-\frac{\tau}{2}} e^{-\frac{R^2 e^{-\tau}}{2\mu}}. \end{aligned} \tag{4.6}$$

Taking the largest of ((4.4)), ((4.5)), and ((4.6)), we obtain

$$\int e^{-\frac{1}{2\mu}\left(\frac{1}{2}(\xi-\eta)^2 + H(e^{\tau/2}\eta)\right)} \geq C\sqrt{\mu} e^{-\frac{1}{2\mu}(M+C(R,||h||_m))}. \tag{4.7}$$

Now, we will bound the numerator in ((4.2)) from above. We will split the integral up into the same three regions as above and denote the resulting terms by J_1, J_2, and J_3. First, we have

$$\begin{aligned} |J_1| &= \left| \int_{-\infty}^{-Re^{-\frac{\tau}{2}}} (\xi-\eta) e^{-\frac{1}{4\mu}(\xi-\eta)^2} e^{-\frac{1}{2\mu}H(e^{\frac{\tau}{2}}\eta)} d\eta \right| \\ &\leq \left| e^{\frac{C(R,||h||_m)}{2\mu}} \int_{-\infty}^{-Re^{-\frac{\tau}{2}}} (\xi-\eta) e^{-\frac{1}{4\mu}(\xi-\eta)^2} d\eta \right| \\ &= e^{\frac{C(R,||h||_m)}{2\mu}} 2\mu e^{-\frac{(\xi+Re^{-\frac{\tau}{2}})^2}{4\mu}} \\ &\leq 2\mu e^{\frac{C(R,||h||_m)}{2\mu}} e^{-\frac{\xi^2}{4\mu}}. \end{aligned} \tag{4.8}$$

Next, consider J_2. We have

$$|J_2| = \left| \int_{Re^{-\frac{\tau}{2}}}^{\infty} (\xi-\eta) e^{-\frac{1}{4\mu}(\xi-\eta)^2} e^{-\frac{1}{2\mu}H(e^{\frac{\tau}{2}}\eta)} d\eta \right|.$$

If we now integrate by parts inside the integral and use the fact that $H(e^{\frac{\tau}{2}}\eta) \geq M - (q/2)$, we obtain

$$\begin{aligned} |J_2| &\leq C\mu e^{-\frac{M}{2\mu}} e^{\frac{1}{4\mu}[q-(\xi-Re^{-\tau/2})^2]} \\ &\quad + e^{-\frac{M}{2\mu}} \left| \int_{Re^{\tau/2}}^{\infty} e^{-\frac{1}{4\mu}(\xi-\eta)^2} h(e^{\tau/2}\eta) e^{+\frac{1}{2\mu}(M-H(e^{\frac{\tau}{2}}\eta))} d\eta \right|. \end{aligned} \tag{4.9}$$

We now turn to J_3. Again we use the fact that $H(e^{\frac{\tau}{2}}\eta) \geq M - (q/2)$. Then

$$|J_3| \leq 2CRe^{-\frac{\tau}{2}} e^{-\frac{M}{2\mu}} (\xi + Re^{-\frac{\tau}{2}}) e^{\frac{1}{4\mu}[q-(\xi-Re^{-\tau/2})^2]}. \tag{4.10}$$

Combining the estimates for the J_i's, ((4.8))–((4.10)), and the estimate for the denominator ((4.7)), we have

$$\int_{\sqrt{q}+\epsilon}^{\infty}(1+\xi^2)^m|w(\xi,\tau)|^2 d\xi \le C\int_{\sqrt{q}+\epsilon}^{\infty}(1+\xi^2)^m \mu e^{\frac{M}{\mu}} e^{\frac{2}{\mu}C(R,||h||_m)} e^{-\frac{\xi^2}{2\mu}} d\xi$$
$$+C\int_{\sqrt{q}+\epsilon}^{\infty}(1+\xi^2)^m \frac{1}{\mu} e^{\frac{1}{\mu}C(R,||h||_m)} (Re^{-\frac{\tau}{2}})^2(\xi+Re^{-\frac{\tau}{2}})^2 e^{\frac{1}{2\mu}[q-(\xi-Re^{-\tau/2})^2]} d\xi$$
$$+C\int_{\sqrt{q}+\epsilon}^{\infty}(1+\xi^2)^m \mu e^{\frac{1}{\mu}C(R,||h||_m)} e^{\frac{1}{2\mu}[q-(\xi-Re^{-\tau/2})^2]} d\xi$$
$$+C\int_{\sqrt{q}+\epsilon}^{\infty}(1+\xi^2)^m \frac{1}{\mu} e^{\frac{1}{\mu}C(R,||h||_m)} \left|\int_{Re^{-\tau/2}}^{\infty} e^{-\frac{1}{4\mu}(\xi-\eta)^2} h(e^{\tau/2}\eta) e^{+\frac{1}{2\mu}(M-H(e^{\frac{\tau}{2}}\eta))} d\eta\right|^2 d\xi$$
$$=: I+II+III+IV.$$

We now estimate term II. Terms I and III are similar. Define $z=\xi-\sqrt{q}-\epsilon\in(0,\infty)$. Recalling that τ has been chosen sufficiently large so that $Re^{-\frac{\tau}{2}}<\epsilon/2$, we have

$$e^{\frac{1}{2\mu}[q-(\xi-Re^{-\tau/2})^2]} \le e^{-\frac{1}{2\mu}z^2} e^{-\frac{1}{8\mu}\epsilon^2} e^{-\frac{1}{2\mu}\epsilon\sqrt{q}}.$$

Therefore, we have

$$|II| \le \frac{C}{\mu} e^{\frac{1}{\mu}C(R,||h||_m)} \int_0^{\infty}(1+(z+\sqrt{q}+\epsilon)^2)^m \left(z+\sqrt{q}+\frac{3}{2}\epsilon\right) e^{-\frac{1}{2\mu}z^2} e^{-\frac{1}{8\mu}\epsilon^2} e^{-\frac{1}{2\mu}\epsilon\sqrt{q}} dz$$
$$\le C(\sqrt{q})^{2m+1} e^{-\frac{1}{2\mu}\epsilon\sqrt{q}} \frac{1}{\sqrt{\mu}} e^{\frac{1}{\mu}C(R,||h||_m)} e^{-\frac{1}{8\mu}\epsilon^2}.$$

This can be made as small as we like (for any q) by choosing R large enough so that $C(R,||h||_m)<\epsilon^2/16$ and μ is sufficiently small.

Term IV can be bound by

$$|IV| \le C\frac{1}{\mu} e^{\frac{2}{\mu}C(R,||h||_m)} ||f(z)*g(z)||_{L^2}^2,$$

where $f(z)=(1+|z|^m)h(e^{\tau/2}z)$ and $g(z)=(1+|z|^m)e^{-\frac{1}{4\mu}z^2}$, and we have used the fact that $(1+|\xi|^m)\le C(1+|\xi-\eta|^m)(1+|\eta|^m)$. Estimating the convolution by the L^1 norm of g and the L^2 norm of f, we arrive at

$$|IV| \le Ce^{-\tau/2} e^{\frac{2}{\mu}C(R,||h||_m)} ||h||_m^2.$$

In order to make this term small, we must choose R so that $C(R,||h||_m)\sim\mu$ as $\mu\to 0$. One can check that $C(R,||h||_m)\le C||h||_m/R^{2m-1}$. Since we have required that $Re^{-\tau/2}<\epsilon/2$, this means we must choose τ large enough so that

$$\tau \ge \frac{C}{2m-1}|\log(\mu)|.$$

Term IV will then be small because $e^{-\tau/2}$ is.

Next consider the part of the integral contributing to $\|w(\cdot,\tau)-N(\cdot)\|_m$ for $\epsilon<\xi<\sqrt{q}-\epsilon$. We assume, as above, that $Re^{-\tau/2}<\epsilon/2$. From (4.2) and the fact that $N(\xi)=\xi$ for ξ in this range, we have

$$w(\xi,\tau)-N(\xi) = -\frac{\int \eta e^{-\frac{1}{2\mu}\left(\frac{1}{2}(\xi-\eta)^2+H(e^{\tau/2}\eta)\right)} d\eta}{\int e^{-\frac{1}{2\mu}\left(\frac{1}{2}(\xi-\eta)^2+H(e^{\tau/2}\eta)\right)} d\eta}.$$

As above we will split the denominator up into three pieces, I_1–I_3, to bound it from below, and split the numerator up into three pieces, J_1–J_3, to bound it from above. Many of the estimates are similar to those above, and so we omit some of the details.

For the denominator, we have

$$\begin{aligned}|I_1| &\geq e^{-\frac{1}{2\mu}C(R,||h||_m)} \int_{-\infty}^{-Re^{-\tau/2}} e^{-\frac{1}{2\mu}\left(\frac{1}{2}(\xi-\eta)^2\right)} d\eta \\ &\geq C\mu e^{-\frac{1}{2\mu}C(R,||h||_m)} e^{-\frac{\xi^2}{4\mu}}.\end{aligned}$$

Also,

$$|I_2| \geq C\sqrt{\mu} e^{-\frac{1}{2\mu}(M+C(R,||h||_m))},$$

where, as above, we have used the fact that $\xi - Re^{-\tau/2} > 0$. Finally, we have

$$|I_3| \geq e^{-\frac{M}{2\mu}-\frac{q}{4\mu}} 2Re^{-\tau/2} e^{-\frac{1}{4\mu}(\xi-Re^{-\tau/2})^2}.$$

For the numerator, we have

$$\begin{aligned}|J_1| &\leq e^{\frac{C(R,||h||_m)}{2\mu}} \int_{-\infty}^{-Re^{-\tau/2}} -\eta e^{-\frac{1}{4\mu}(\xi-\eta)^2} d\eta \\ &\leq e^{\frac{C(R,||h||_m)}{2\mu}} \int_{\xi+Re^{-\tau/2}}^{\infty} (z-\xi) e^{-\frac{z^2}{4\mu}} dz \\ &\leq C\mu e^{\frac{C(R,||h||_m)}{2\mu}} e^{-\frac{(\xi+Re^{-\tau/2})^2}{4\mu}} + C\mu\xi e^{-\frac{(\xi+Re^{-\tau/2})^2}{4\mu}} \\ &\leq C\mu\xi e^{-\frac{\xi^2}{4\mu}},\end{aligned}$$

where we have used the fact that $\xi > 0$. Next,

$$\begin{aligned}|J_2| &\leq e^{-\frac{M}{2\mu}} e^{\frac{C(R,||h||_m)}{2\mu}} \sqrt{4\mu} \int_{-\infty}^{\frac{\xi-Re^{-\tau/2}}{\sqrt{4\mu}}} (\xi - \sqrt{4\mu}z) e^{-z^2} dz \\ &\leq C\mu\xi e^{-\frac{M}{2\mu}} e^{\frac{C(R,||h||_m)}{2\mu}} e^{-\frac{(\xi-Re^{-\tau/2})^2}{4\mu}}.\end{aligned}$$

Finally,

$$|J_3| \leq Ce^{-\frac{M}{2\mu}} e^{\frac{C(R,||h||_m)}{2\mu}} (Re^{-\tau/2})^2 e^{-\frac{1}{4\mu}(\xi-Re^{-\tau/2})^2}.$$

Therefore, we have

$$\begin{aligned}&\int_{\epsilon}^{\sqrt{q}-\epsilon} (1+\xi^2)^m |w(\xi,\tau) - N(\xi)|^2 d\xi \\ &\quad\leq C\int_{\epsilon}^{\sqrt{q}-\epsilon} (1+\xi^2)^m \left(\frac{\xi e^{-\frac{M}{2\mu}} e^{\frac{C(R,||h||_m)}{2\mu}} e^{-\frac{(\xi-Re^{-\tau/2})^2}{4\mu}}}{\sqrt{\mu} e^{-\frac{1}{2\mu}(M+C(R,||h||_m))}} \right)^2 d\xi \\ &\quad\leq C\frac{1}{\mu}\int_{\epsilon}^{\sqrt{q}-\epsilon} (1+\xi^2)^{m+1} e^{\frac{2}{\mu}C(R,||h||_m)} e^{-\frac{(\xi-Re^{-\tau/2})^2}{2\mu}} d\xi \\ &\quad\leq C\frac{1}{\mu} e^{\frac{2}{\mu}C(R,||h||_m)} \int_0^{\sqrt{q}-2\epsilon} (1+(z+\epsilon)^2)^m e^{-\frac{\epsilon^2}{8\mu}} e^{-\frac{z\epsilon}{2\mu}} e^{-\frac{z^2}{2\mu}} dz \\ &\quad\leq C\frac{1}{\sqrt{\mu}} e^{\frac{2}{\mu}C(R,||h||_m)} e^{-\frac{\epsilon^2}{8\mu}},\end{aligned}$$

where we have used the change of variables $\xi = z + \epsilon$. This can be made small by first choosing R large enough so that $2C(R,||h||_m) < \epsilon^2/8$, and then taking μ small.

5 Proof of Theorem (2)

In the previous section we saw that in a time $\tau = \mathcal{O}(|\log \mu|)$ we end up in an arbitrarily small (but $\mathcal{O}(1)$ with respect to μ) neighborhood of the manifold of diffusive N-waves. In the present section we show that this manifold is locally attractive by proving Theorem (2).

Remark 5.1. An additional consequence of the proof of this theorem is that the manifold of diffusive N-waves is attracting in a Lyapunov sense because the rate of approach to it, $\mathcal{O}(e^{-\tau})$, is faster than the decay along it, $\mathcal{O}(e^{-\tau/2})$. Note that this is not immediate from just spectral considerations since this manifold does not consist of fixed points. Therefore, the eigendirections at each point on the manifold can change as the solution moves along it.

Remark 5.2. In [KN02, section 5], the authors make a numerical study of the metastable asymptotics of the Burgers equation. Their numerics indicate that, while the rate of convergence toward the diffusive N-wave ($e^{-\tau}$ in our formulation) seems to be optimal, the constant in front of the convergence rate (C_ϕ in our formulation) can be very large for some initial conditions. In fact, our proof indicates that this constant can be as large as $\mathcal{O}(1/\mu)\max\{1, e^{M/2\mu}\}$.

To prove the theorem note that by the Cole–Hopf transformation we know that if $w(\xi,\tau) = w_N(\xi,\tau) + \phi(\xi,\tau)$ solves the rescaled Burgers equation, where w_N is given in ((2.13)), then

$$W(\xi,\tau) = (w_N(\xi,\tau) + \phi(\xi,\tau))e^{-\frac{1}{2\mu}\int_{-\infty}^{\xi}(w_N(y,\tau)+\phi(y,\tau))dy}$$

is a solution of the (rescaled) heat equation:

$$\partial_\tau W = \mathcal{L}_\mu W.$$

We now write $W = V_N + \Psi$, where $V_N = w_N \exp(-\frac{1}{2\mu}\int_{-\infty}^{\xi} w_N(y,\tau)dy) = \beta_0\varphi_0(\xi) + \beta_1 e^{-\tau/2}\varphi_1(\xi)$. That is, V_N is the heat equation representation of the diffusive N-wave, which we know is a linear combination of the Gaussian, φ_0, and φ_1.

With the aid of the Cole–Hopf transformation we can show that ϕ decreases with the rate claimed in Theorem (2). To see this, first note that if we choose the coefficients β_0 and β_1 appropriately, we can ensure that

$$\int \Psi(\xi,0) = \int \xi\Psi(\xi,0)d\xi = 0.$$

This follows from the fact that the adjoint eigenfunctions corresponding to the eigenfunctions φ_0 and φ_1, respectively, are just 1 and $-\xi$. This in turn means that there exists a constant C_ψ such that

$$\|\Psi(\cdot,\tau)\|_{L^2(m)} \le C_\psi e^{-\tau},$$

at least if $m > 5/2$. Integrating the Cole–Hopf transformation, we find

$$\int_{-\infty}^{\xi}(V_N(y,\tau) + \Psi(y,\tau))dy = -2\mu\left\{e^{-\frac{1}{2\mu}\int_{-\infty}^{\xi}(w_N(y,\tau)+\phi(y,\tau))dy} - 1\right\} \tag{5.1}$$

and, in the case $\phi = 0$, $\int_{-\infty}^{\xi} V_N = -2\mu\{e^{-\frac{1}{2\mu}\int_{-\infty}^{\xi} W_N} - 1\}$. For later use we note the following easy consequence of (5.1).

Lemma 5.3. *There exists a constant $\delta_N > 0$ such that for all $\tau \geq 0$ we have*

$$1 - \int_{-\infty}^{\xi} \frac{1}{2\mu} \left(V_N(y,\tau) + \Psi(y,\tau) \right) dy = e^{-\frac{1}{2\mu}\int_{-\infty}^{\xi} (w_N(y,\tau)+\phi(y,\tau))dy} \geq \delta_N.$$

Proof. For any finite τ the estimate follows immediately because of the exponential. The only thing we have to check is that the right-hand side does not tend to zero as $\tau \to \infty$. However, this follows from the fact that we know (from a Lyapunov function argument, for example) that $w_N(\xi,\tau) + \phi(\xi,\tau) \to A_M(\xi)$ as $\tau \to \infty$, where A_M is one of the self-similar solutions constructed in section (1), and hence

$$\begin{aligned} e^{-\frac{1}{2\mu}\int_{-\infty}^{\xi}(w_N(y,\tau)+\phi(y,\tau))dy} &\to e^{-\frac{1}{2\mu}\int_{-\infty}^{\xi} A_M(y)} \\ &= 1 - (1 - e^{-M/2\mu}) \int_{-\infty}^{\xi} \varphi_0(y)dy \geq \min\{1, e^{-M/2\mu}\} > 0. \end{aligned}$$

$\square$

Next note that by rearranging (5.1) we find

$$\int_{-\infty}^{\xi} \phi(y,\tau)dy = -2\mu \log \left\{ \frac{1 - \frac{1}{2\mu}\int_{-\infty}^{\xi} \left(V_N(y,\tau) + \Psi(y,\tau) \right) dy}{1 - \frac{1}{2\mu}\int_{-\infty}^{\xi} V_N(y,\tau)dy} \right\}.$$

Differentiating, we obtain the corresponding formula for ϕ, namely,

$$\phi(\xi,\tau) = -\frac{1}{2\mu} \frac{\Psi(\xi,\tau)\int_{-\infty}^{\xi} V_N(y,\tau)dy - V_N(\xi,\tau)\int_{-\infty}^{\xi} \Psi(y,\tau)dy - 2\mu\Psi(\xi,\tau)}{\left(1 - \frac{1}{2\mu}\int_{-\infty}^{\xi}(V_N(y,\tau) + \Psi(y,\tau))dy\right)\left(1 - \frac{1}{2\mu}\int_{-\infty}^{\xi} V_N(y,\tau)dy\right)}.$$

But now, by Lemma (5.3) the denominator of the expression for ϕ can be bounded from below by δ_N^2, while in the numerator $\int_{-\infty}^{\xi} V_N$ and V_N are bounded in time while $\int_{-\infty}^{\xi} \Psi$ and Ψ are each bounded by $C_{\psi,\Psi}e^{-\tau}$, which leads to the bound asserted in Theorem (2).

6 Concluding remarks

Our main motivation for the above analysis was the numerical observation of metastable behavior in the vorticity formulation of the 2D Navier–Stokes equations [YMC03]. Although there are, of course, many differences between the 2D Navier–Stokes equations and the Burgers equation, there are also many similarities between the two, when analyzed from the point of view of dynamical systems. For example, in [GW05], the dynamical systems viewpoint was successfully used to prove global stability in 2D Navier–Stokes of a one parameter family of self-similar solutions known as the Oseen vortices. In addition, it was shown that this family can be thought of as a one-dimensional center manifold in the phase space of the equation. This is analogous to the globally stable family of self-similar diffusion waves in the Burgers equation.

In order to carry out the above metastability analysis for the 2D Navier–Stokes equations, one would need to adapt the analysis so as not to rely on the Cole–Hopf transformation. Cole-Hopf was utilized to extend the stable foliation of the center manifold to a global foliation and also to give an explicit representation of solutions that was used in some of the estimates. It is not at all clear how to construct a global folliation of the center manifold for Navier–Stokes. Regarding the estimates of the metastable timescales, however, one possible way to do this without Cole–Hopf is via energy estimates. In other words, prove that if the solution to 2D Navier–Stokes is near one of the metastable states, which correspond to solutions of the inviscid Euler equations, then the evolution cannot occur at a rate faster than some slow rate that is given by energy estimates.

Other than 2D Navier–Stokes, another interesting future direction could be to look at variants of the Burgers equation, such as the nonplanar Burgers equation [RSR02] or the complex Burgers equation [LZ95]. Both of these equations are relatively simple and can potentially be analyzed via the self-similar variables used here. However, the Cole–Hopf transformation does not linearize these equations. We remark that slow decay of solutions has been observed for the planar Burgers equation in [Sac87, Figure 5.9].

Finally, we remark that in [Liu00] it was shown that the large-time behavior of solutions to a general class of conservation laws is governed by that of solutions to the Burgers equation. Roughly speaking, this is due to the marginality of the nonlinearity in the case of the Burgers equation and the fact that any higher order nonlinear terms in other conservation laws are irrelevant. Therefore, the present analysis for the Burgers equation could also potentially be used to predict and understand metastability in other conservation laws with small viscosity, i.e., equations of the form $u_t = \mu u_{xx} - f(u)_x$.

Appendix

We now give the calculation that leads to ((2.17)) and, in a slightly different form, was originally presented in [KT01]. Using the definition of p in ((2.15)), we find that

$$\begin{aligned} p &= -2 \inf_y \int_{-\infty}^{y} w_N(\xi,\tau) d\xi \\ &= 4\mu \sup_y \int_{-\infty}^{y} \partial_\xi \log \left[1 - \frac{\beta_0}{2\mu} \int_{-\infty}^{\xi} \varphi_0(y) dy - \frac{\beta_1}{2\mu} e^{-\frac{\tau}{2}} \varphi_0(\xi) \right] d\xi \\ &= 4\mu \sup_y \log \left[1 - \frac{\beta_0}{2\mu} \int_{-\infty}^{y} \varphi_0(z) dz - \frac{\beta_1}{2\mu} e^{-\frac{\tau}{2}} \varphi_0(y) \right]. \end{aligned}$$

A direct calculation shows that the supremum is achieved at

$$y^* = \frac{2\mu\beta_0}{\beta_1 e^{-\frac{\tau}{2}}}.$$

Substituting this value and rearranging terms, we find

$$-\frac{\beta_1}{2\mu} e^{-\frac{\tau}{2}} \varphi_0(y^*) = e^{\frac{p}{4\mu}} - \left(1 - \frac{\beta_0}{2\mu} \int_{-\infty}^{y^*} \varphi_0(z) dz \right).$$

Since $\int_{-\infty}^{y^*} \varphi_0(z) dz \in (0,1)$, the value of β_0 in ((2.14)) implies that the right-hand side satisfies

$$e^{\frac{p}{4\mu}} - 1 \leq e^{\frac{p}{4\mu}} - \left(1 - \frac{\beta_0}{2\mu} \int_{-\infty}^{y^*} \varphi_0(z) dz \right) \leq e^{\frac{p}{4\mu}} - e^{-\frac{M}{2\mu}}$$

if $M > 0$, and

$$e^{\frac{p}{4\mu}} - e^{-\frac{M}{2\mu}} \leq e^{\frac{p}{4\mu}} - \left(1 - \frac{\beta_0}{2\mu}\int_{-\infty}^{y^*} \varphi_0(z)dz\right) \leq e^{\frac{p}{4\mu}} - 1$$

if $M < 0$. Because $\varphi_0(y) \geq 0$ and $p \geq 2|M|$, we see that, if μ is sufficiently small, then $\beta_1 \leq 0$. Furthermore, $\beta_1 = 0$ if $p = -M/2$; i.e., $M < 0$ and $q = 0$. Using the fact that $\varphi(y) \leq 1/\sqrt{4\pi\mu}$, we see that

$$-\beta_1 e^{-\frac{\tau}{2}} \geq 2\mu\sqrt{4\pi\mu}\left[e^{\frac{p}{4\mu}} - \left(1 - c\frac{\beta_0}{2\mu}\right)\right],$$

where $c \in (0,1)$. This leads to the estimate ((2.17)), at least when both $q \neq 0$ and $p \neq 0$.

We remark that $\beta_1 \leq 0$, and the fact that

$$1 - \frac{\beta_0}{2\mu}\int_{-\infty}^{y} \varphi_0(z)dz = 1 - (1 - e^{-\frac{M}{2\mu}})\int_{-\infty}^{y} \varphi_0(z)dz \in \begin{cases} (e^{-\frac{M}{2\mu}}, 1) & \text{if } M > 0, \\ (1, e^{-\frac{M}{2\mu}}) & \text{if } M < 0 \end{cases}$$

implies that the denominator in the definition of w_N ((2.13)) is never zero.

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References

[BG10] Nils Berglund and Barbara Gentz. The Eyring-Kramers law for potentials with nonquadratic saddles. *To appear in Markov Processes and Related Fields*, 2010.

[Bur48] J. M. Burgers. A mathematical model illustrating the theory of turbulence. In *Advances in Applied Mechanics*, pages 171–199. Academic Press Inc., New York, N. Y., 1948. edited by Richard von Mises and Theodore von Kármán,.

[Che04] Xinfu Chen. Generation, propagation, and annihilation of metastable patterns. *J. Differential Equations*, 206(2):399–437, 2004.

[CHT97] X. Chen, J. K. Hale, and B. Tan. Invariant foliations for c^1 semigroups in banach spaces. *Journal of Differential Equations*, 139:283–318, 1997.

[CP89] J. Carr and R. L. Pego. Metastable patterns in solutions of $u_t = \epsilon^2 u_{xx} - f(u)$. *Comm. Pure Appl. Math.*, 42(5):523–576, 1989.

[CP90] Jack Carr and Robert Pego. Invariant manifolds for metastable patterns in $u_t = \epsilon^2 u_{xx} - f(u)$. *Proc. Roy. Soc. Edinburgh Sect. A*, 116(1-2):133–160, 1990.

[CPR08] E. Caglioti, M. Pulvirenti, and F. Rousset. The 2D constrained Navier-Stokes equation and intermediate asymptotics. *J. Phys. A*, 41(34):344001, 9, 2008.

[CPR09] E. Caglioti, M. Pulvirenti, and F. Rousset. On a constrained 2-D Navier-Stokes equation. *Comm. Math. Phys.*, 290(2):651–677, 2009.

[Daf05] Constantine M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2005.

[DiF03] Marco DiFrancesco. *Diffusive behavior and asymptotic self similarity for fluid models*. PhD thesis, University of Rome, 2003.

[DM05] Jochen Denzler and Robert J. McCann. Fast diffusion to self-similarity: complete spectrum, long-time asymptotics, and numerology. *Arch. Ration. Mech. Anal.*, 175(3):301–342, 2005.

[FH89] G. Fusco and J. K. Hale. Slow-motion manifolds, dormant instability, and singular perturbations. *J. Dynam. Differential Equations*, 1(1):75–94, 1989.

[Gal10] T. Gallay. Interaction of vortices in weakly viscous planar flows. *To appear in Arch. Ration. Mech. Anal.*, 2010.

[GGN09] Isabelle Gallagher, Thierry Gallay, and Francis Nier. Special asymptotics for large skew-symmetric perturbations of the harmonic oscillator. *Int. Math. Res. Not. IMRN*, (12):2147–2199, 2009.

[GW02] T. Gallay and C. E. Wayne. Invariant manifolds and the long-time asymptotics of the navier-stokes and vorticity equations on $\mathbb{R}^2$. *Arch. Rational Mech. Anal.*, 163:209–258, 2002.

[GW05] Thierry Gallay and C. Eugene Wayne. Global stability of vortex solutions of the two-dimensional Navier-Stokes equation. *Comm. Math. Phys.*, 255(1):97–129, 2005.

[KK86] Gunilla Kreiss and Heinz-Otto Kreiss. Convergence to steady state of solutions of Burgers' equation. *Appl. Numer. Math.*, 2(3-5):161–179, 1986.

[KN02] Yong-Jung Kim and Wei-Ming Ni. On the rate of convergence and asymptotic profile of solutions to the viscous Burgers equation. *Indiana Univ. Math. J.*, 51(3):727–752, 2002.

[KS91] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[KT01] Yong Jung Kim and Athanasios E. Tzavaras. Diffusive N-waves and metastability in the Burgers equation. *SIAM J. Math. Anal.*, 33(3):607–633 (electronic), 2001.

[Liu00] Tai-Ping Liu. *Hyperbolic and Viscous Conservation Laws.* Number 72 in CBMS-NSF Regional Conference Series in Applied Mathematics. SIAM, 2000.

[LO99] Jacques G. L. Laforgue and Robert E. O'Malley, Jr. Exponential asymptotics, the viscid Burgers' equation, and standing wave solutions for a reaction-advection-diffusion model. *Stud. Appl. Math.*, 102(2):137–172, 1999.

[LY97] Tai-Ping Liu and Shih-Hsien Yu. Propagation of a stationary shock layer in the presence of a boundary. *Arch. Rational Mech. Anal.*, 139(1):57–82, 1997.

[LZ95] Tai-Ping Liu and Kevin Zumbrun. On nonlinear stability of general undercompressive viscous shock waves. *Comm. Math. Phys.*, 174(2):319–345, 1995.

[MSM$^+$91] W. H. Matthaeus, W. T. Stribling, D. Martinez, S. Oughton, and D. Montgomery. Decaying, two-dimensional, Navier-Stokes turbulence at very long times. *Physica D*, 51(531-538), 1991.

[OR07] Felix Otto and Maria G. Reznikoff. Slow motion of gradient flows. *J. Differential Equations*, 237(2):372–420, 2007.

[RSR02] Ch. Srinivasa Rao, P. L. Sachdev, and Mythily Ramaswamy. Analysis of the self-similar solutions of the nonplanar Burgers equation. *Nonlinear Anal.*, 51(8):1447–1472, 2002.

[RW95a] Luis G. Reyna and Michael J. Ward. On exponential ill-conditioning and internal layer behavior. *Numer. Funct. Anal. Optim.*, 16(3-4):475–500, 1995.

[RW95b] Luis G. Reyna and Michael J. Ward. On the exponentially slow motion of a viscous shock. *Comm. Pure Appl. Math.*, 48(2):79–120, 1995.

[Sac87] P. L. Sachdev. *Nonlinear diffusive waves.* Cambridge University Press, Cambridge, 1987.

[SW99] Xiaodi Sun and Michael J. Ward. Metastability and pinning for convection-diffusion-reaction equations in thin domains. *Methods Appl. Anal.*, 6(4):451–475, 1999.

[TE05] Lloyd N. Trefethen and Mark Embree. *Spectra and pseudospectra.* Princeton University Press, Princeton, NJ, 2005. The behavior of nonnormal matrices and operators.

[Whi99] G. B. Whitham. *Linear and nonlinear waves.* Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1999. Reprint of the 1974 original, A Wiley-Interscience Publication.

[WR95] Michael J. Ward and Luis G. Reyna. Internal layers, small eigenvalues, and the sensitivity of metastable motion. *SIAM J. Appl. Math.*, 55(2):425–445, 1995. Perturbation methods in physical mathematics (Troy, NY, 1993).

[YMC03] Z. Yin, D. C. Montgomery, and H. J. H. Clercx. Alternative statistical-mechanical descriptions of decaying two-dimensional turbulence in terms of "patches" and "points". *Phys. Fluids*, 15:1937–1953, 2003.