Localized radial roll patterns in higher space dimensions

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Abstract

Localized roll patterns are structures that exhibit a spatially periodic profile in their center. When following such patterns in a systems parameter in one space dimension, the length of the spatial interval over which these patterns resemble a periodic profile stays either bounded, in which case branches form closed bounded curves ("isolas"), or the length increases to infinity so that branches are unbounded in function space ("snaking"). In two space dimensions, numerical computations show that branches of localized rolls exhibit a more complicated structure in which both isolas and snaking occur. In this paper, we analyse the structure of branches of localized radial roll solutions in dimension $1+\varepsilon$, with $\varepsilon > 0$ small, through a perturbation analysis. Our analysis provides an explanation for some of the features visible in the planar case.

1 Introduction

Spatially localized patterns can be observed in the natural world in a variety of places, such as vegetation patterns [14, 18], crime hotspots [9], and ferrofluids [6]. We are particularly interested in localized roll solutions: when the spatial variable x is in $\mathbb{R}$, these structures are spatially periodic for x in a bounded region, and they decay exponentially fast to zero as $x \to \pm\infty$; see Figure 1 for sample profiles. The bifurcation structure associated with localized rolls consists typically of curves that turn back and forth as they extend vertically upward when they are plotted as a function of a bifurcation parameter against the length of the roll plateau; see again Figure 1 for an illustration. These diagrams are often referred to as snaking [3, 4, 15, 16, 19].

A specific partial differential equation that is well known to exhibit snaking is the Swift–Hohenberg equation

$$U_t = -(1+\Delta)^2 U - \mu U + \nu U^2 - U^3, \qquad x \in \mathbb{R}^n, \tag{1.1}$$

where $U \in \mathbb{R}$, Δ denotes the usual Laplacian operator, the parameter $\nu > 0$ is typically held fixed, and $\mu > 0$ is taken to be the bifurcation parameter. As shown in Figure 1, this system exhibits snaking of even localized rolls when posed in one space dimension [2–4, 19].

In planar systems where $x \in \mathbb{R}^2$, localized ring and hexagon solutions take the place of the localized rolls we discussed above [11, 12]. We are particularly interested in localized radially-symmetric ring solutions, which are shown in Figure 2. The existence of planar radial localized rings for the system (1.1) with μ near zero was shown in [10, 13]. The numerical computations published in [12] indicate that the global bifurcation structure for these planar patterns differs from the one-dimensional case. Instead of exhibiting two unbounded snaking branches, the planar diagram consists of three distinct regions that are shown in Figure 2: there are a bounded lower and

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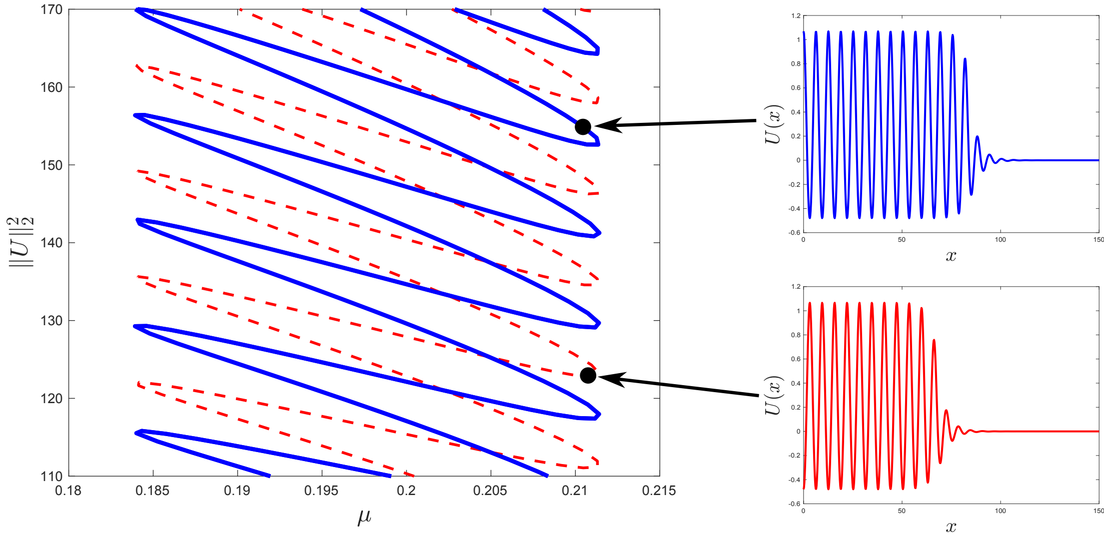

Figure 1: *The bifurcation diagram of localized rolls to equation (1.1) in dimension $n = 1$ for $\nu = 1.6$. Two branches of solutions move back and forth in μ whilst the L^2-norm of the associated localized solutions increases. Sample profiles are given on the right for each of the branches, where we restrict the domain to $x \geq 0$ and note that the full solutions on $\mathbb{R}$ are symmetric over $x = 0$. Solutions are obtained at the parameter values $\mu = 0.2105$ (top) and $\mu = 0.2110$ (bottom). Notice that the profiles along one branch have a maximum at $x = 0$, while the profiles along the other branch have a minimum at $x = 0$.*

an unbounded upper snaking branch, which are separated by a region of stacked isolas. Furthermore, the upper snaking branch seems to collapse onto a vertical line located approximately at $\mu = 0.204$. The spatial profiles retain the same basic shape through these different regions. However, along the lower branch, rolls are added in the far field, while rolls are added near the center of the pattern along the upper snaking branch.

The goal of this paper is to shed light on the differences between the bifurcation diagrams in one and two space dimensions. To illustrate the scope of our paper, we first outline how radial solutions can be found for the Swift–Hohenberg equation. Writing $U(x) = U(r)$ with $r := |x|$, we see that radially-symmetric steady-states of the Swift–Hohenberg equation (1.1) satisfy the ordinary differential equation (ODE)

$$0 = -\left(1 + \frac{n-1}{r}\partial_r + \partial_{rr}\right)^2 U - \mu U + \nu U^2 - U^3, \qquad r > 0. \tag{1.2}$$

Note that the space dimension n enters explicitly into the second term on the right-hand side. As suggested in [12], we will carry out a perturbation analysis near $n = 1$ by setting $n = 1 + \varepsilon$ with $0 < \varepsilon \ll 1$. While there is no meaningful connection between n and the underlying dimension unless n is an integer, we can study (1.2) in its own right for real n. Rewriting (1.2) as a first-order system, we obtain the nonautonomous ODE

$$\begin{aligned}
u_1' &= u_2, \\
u_2' &= u_3 - u_1 - \frac{\varepsilon}{r}u_2, \\
u_3' &= u_4, \\
u_4' &= -u_3 - \mu u_1 + \nu u_1^2 - u_1^3 - \frac{\varepsilon}{r}u_4,
\end{aligned} \tag{1.3}$$

where $' = \frac{d}{dr}$,

$$u_1 = U,\ u_2 = \partial_r U,\ u_3 = (1 + \frac{\varepsilon}{r}\partial_r + \partial_{rr})U,\ u_4 = \partial_r(1 + \frac{\varepsilon}{r}\partial_r + \partial_{rr})U,$$

and $0 \leq \varepsilon \ll 1$ is now a small parameter that connects the one-dimensional system ($\varepsilon = 0$, corresponding to $n = 1$) to the $1 + \varepsilon$-dimensional system. Since the one-dimensional case is well understood, we can leverage the

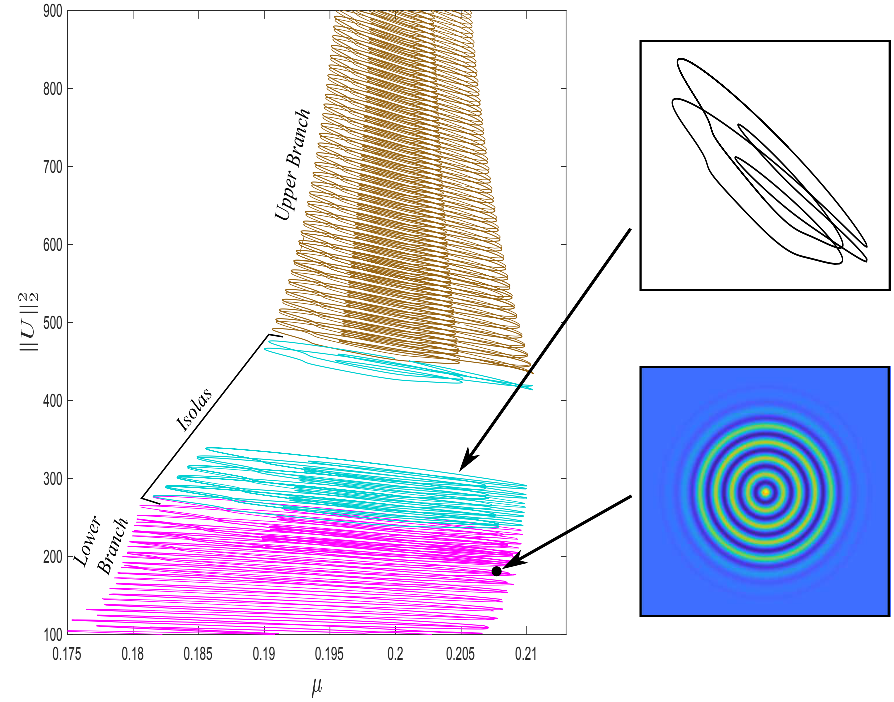

Figure 2: *Snaking in the planar Swift–Hohenberg equation with $\nu = 1.6$. The bifurcation diagram is composed of three distinct regions: the lower snaking branch (magenta), stacked isolas (turquoise), and the upper snaking branch (brown). The isolas are a collection of closed curves, as is demonstrated in the upper right inset. The bottom right inset provides a contour plot of the bifurcating localized spot solutions of the Swift–Hohenberg equation.*

existing theoretical approaches to understand what structural changes the nonautonomous perturbation term induces as ε changes near zero. Figure 3 illustrates the connection between even localized roll patterns of (1.1) and heteroclinic solutions that connect periodic solutions to the equilibrium $u = 0$ of the ODE (1.3) in the one-dimensional situation.

When $\varepsilon = 0$, equation (1.3) is autonomous, reversible under $r \mapsto -r$, and conservative. Reversibility and the existence of a conserved quantity $\mathcal{H} : \mathbb{R}^4 \to \mathbb{R}$ allows us to assume that (1.3) admits a cylinder of reversible periodic orbits that is parametrized by the phase of the periodic patterns along its circumference and the value

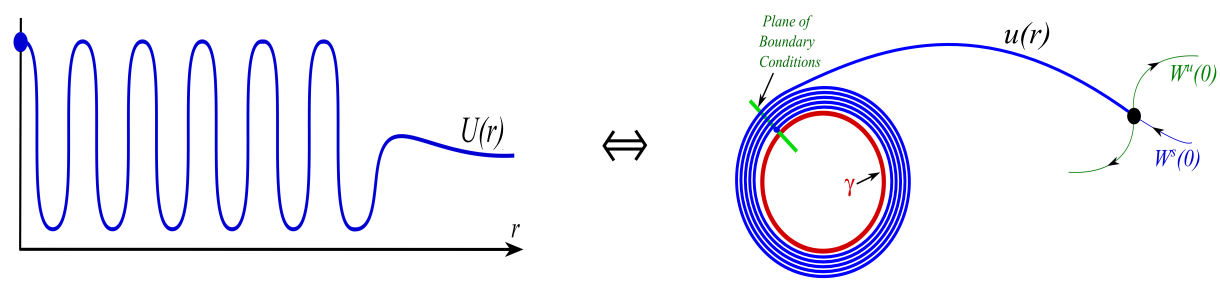

Figure 3: *The left panel illustrates an even localized roll solution $U(r)$ of the Swift–Hohenberg equation (1.2). The right panel shows the same profile now viewed as a solution $u(r)$ of the first-order dynamical system (1.3): the associated trajectory of (1.3) starts sufficiently close to the periodic orbit γ, follows this orbit for some amount of time r, and converges to the trivial equilibrium $u = 0$ along its stable manifold.*

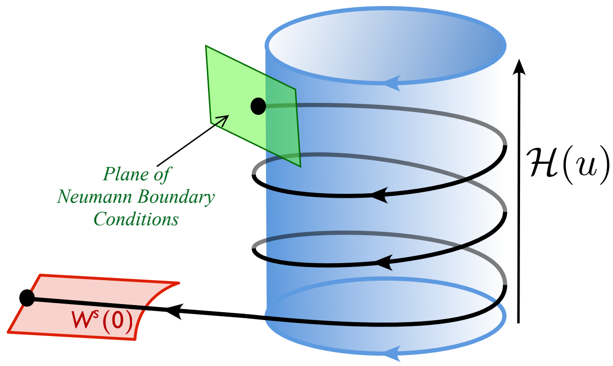

Figure 4: *A localized solution of (1.3) for $\varepsilon > 0$ starts near the cylinder and follows solutions on the cylinder before converging to the equilibrium $U = 0$.*

of the conserved quantity along its vertical direction; see Figure 4 for an illustration. The conserved quantity allows us to restrict the system to the three-dimensional invariant level set $\mathcal{H}^{-1}(0)$ that contains $u = 0$. The approach taken in [2] was to assume the existence of a heteroclinic orbit of (1.3) inside $\mathcal{H}^{-1}(0)$ that connects the periodic orbit in $\mathcal{H}^{-1}(0)$ to the rest state $u = 0$. The analysis in [2] then focused on constructing solutions that satisfy Neumann boundary conditions $u_2 = u_4 = 0$ at $r = 0$ and follow the periodic orbit for large times r inside the invariant three-dimensional level set before converging to $u = 0$ as $r \to \infty$: the resulting orbits are even in r and can be parametrized by the length L of the interval in r for which they stay close to the periodic orbit.

For $\varepsilon \neq 0$, we again seek solutions of (1.3) that start near a periodic orbit and satisfy the Neumann boundary conditions $u_2 = u_4 = 0$ at $r = 0$, resemble periodic profiles for r in an interval of length L, and converge to $u = 0$ as $r \to \infty$. As we shall see below, the Neumann boundary conditions will regularize the terms containing the factor $1/r$ which are singular at $r = 0$. In contrast to the case $\varepsilon = 0$, however, the conservative quantity $\mathcal{H}(u)$ that we exploited above to reduce the dimension from four to three is no longer conserved when $\varepsilon \neq 0$. We will show that the cylinder of periodic orbits will persist, though the dynamics on the cylinder will no longer be strictly periodic: in particular, the value of the conserved quantity will be a dynamic variable, and solutions can leave the cylinder through its top or bottom. We use averaging of the nonautonomous system (1.3) to transform the perturbed vector field on the cylinder into a form that is more tractable: this allows us to estimate the length L_* of the interval $[0, L_*]$ over which solutions will stay on the perturbed cylinder. We will show that we can track solutions near the perturbed cylinder using Shilnikov-type variables, similar to the approach taken in [2], which allows us to match solutions with the nonautonomous stable manifold of the rest state $u = 0$. Figure 4 provides a visualization of such a trajectory in the phase space of the four-dimensional system (1.3). Our main finding is that we can construct spatial radial profiles for any plateau length L less than L_*, and we will also state conditions on the averaged vector field on the persisting cylinder that allow us to construct profiles with arbitrarily long roll plateaus L. Instead of working with the specific system (1.3), we will consider a general system of differential equations and start with the assumptions from [2] that guarantee snaking. We will then add nonautonomous perturbations of a form that includes the terms appearing in (1.3) and carry out a general analysis of the persistence of snaking in such systems.

The remainder of this paper is organized as follows. In §2, we formulate our set of hypotheses and state our main result. In §3, we investigate our system near $r = 0$ to handle the singularity introduced into the system when $\varepsilon > 0$. §4 introduces the Shilnikov variables that allow us to describe trajectories in a neighbourhood of the invariant cylinder. In §5, we briefly review the persistence properties of the stable manifold of $u = 0$ under the nonautonomous perturbations, which are then used in §6 to construct localized roll solutions. In §7, we show that the averaged vector field on the persisting cylinder allows us to identify scenarios in which profiles with unbounded roll plateaus exist. Finally, in §8, we apply our theoretical results to the Swift–Hohenberg equation.

2 Main results

Consider the ordinary differential equation

$$u_x = f(u, \mu), \tag{2.1}$$

where $u \in \mathbb{R}^4$, $\mu \in \mathbb{R}$, and $f : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ is smooth. Our first assumption concerns reversibility.

Hypothesis 1. *There exists a linear map $\mathcal{R} : \mathbb{R}^4 \to \mathbb{R}^4$ with $\mathcal{R}^2 = 1$ and $\dim \mathrm{Fix}(\mathcal{R}) = 2$ so that $f(\mathcal{R}u, \mu) = -\mathcal{R}f(u, \mu)$ for all (u, μ).*

Hypothesis 1 implies that if $u(x)$ is a solution to (2.1), then so is $\mathcal{R}u(-x)$. Furthermore, if $u(0) \in \mathrm{Fix}(\mathcal{R})$ we have that $u(x) = \mathcal{R}u(-x)$ for all $x \in \mathbb{R}$, and hence we refer to such solutions as *symmetric*. Finally, we remark that $\mathbb{R}^4 = \mathrm{Fix}(\mathcal{R}) \oplus \mathrm{Fix}(-\mathcal{R})$. Next, we assume the existence of a conserved quantity.

Hypothesis 2. *There exists a smooth function $\mathcal{H} : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}$ with $\mathcal{H}(\mathcal{R}u, \mu) = \mathcal{H}(u, \mu)$ and $\langle \nabla_u \mathcal{H}(u, \mu), f(u, \mu) \rangle = 0$ for all (u, μ). We normalize $\mathcal{H}$ so that $\mathcal{H}(0, \mu) = 0$ for all μ.*

Our next hypothesis states that the origin is a hyperbolic saddle.

Hypothesis 3. *We have that $f(0, \mu) = 0$ for all μ and that $f_u(0, \mu)$ has exactly two eigenvalues with strictly negative real part and two eigenvalues with strictly positive real part.*

Next, we formalize the existence of hyperbolic periodic orbits that are parametrized by the value of the conserved quantity $\mathcal{H}(\cdot, \mu)$.

Hypothesis 4. *There exist compact intervals $J, K \subset \mathbb{R}$ with nonempty interior such that the differential equation (2.1) has, for each $(\mu, h) \in J \times K$, a periodic orbit $\gamma(x, \mu, h)$ with minimal period $2\pi T(\mu, h) > 0$ such that the following holds for each $(\mu, h) \in J \times K$:*

(i) *$\gamma(x, \mu, h)$ and $T(\mu, h)$ depend smoothly on (μ, h).*
(ii) *$\gamma(x, \mu, h)$ is symmetric: $\gamma(0, \mu, h) \in \mathrm{Fix}(\mathcal{R})$.*
(iii) *$\mathcal{H}(\gamma(x, \mu, h), \mu) = h$ and $\mathcal{H}_u(\gamma(x, \mu, h), \mu) \neq 0$ for one, and hence all, x.*
(iv) *Each $\gamma(x, \mu, h)$ has two positive nontrivial Floquet multipliers $e^{\pm 2\pi T(\mu,h)\alpha(\mu,h)}$ that depend smoothly on (μ, h) so that $\inf_{(\mu,h)\in J\times K} \alpha(\mu, h) > 0$.*

We remark that reversibility implies that the set of Floquet exponents of a symmetric periodic orbit is invariant under multiplication by -1. As shown in [1], the case where the two hyperbolic Floquet multipliers are negative may not lead to snaking. Hypothesis 4 implies that the union $\mathcal{C}(\mu) := \{\gamma(x, \mu, h) : x \in \mathbb{R}, h \in K\}$ of the periodic orbits forms a cylinder that is parametrized by its "height" $h \in K$.

As in [2], we restrict the system (2.1) to the three-dimensional level set $\mathcal{H}^{-1}(0)$ and parametrize a neighborhood of the periodic orbit $\gamma(\cdot, \mu, 0)$ using the variables (φ, v^s, v^u), where $(\varphi, 0, 0)$ corresponds to $\gamma(\varphi T(\mu, 0), \mu, 0)$, and $(\varphi, v^s, 0)$ and $(\varphi, 0, v^u)$ parametrize the strong stable and strong unstable fibers $W^{ss}(\gamma(\varphi T(\mu, 0), \mu, 0), \mu)$ and $W^{uu}(\gamma(\varphi T(\mu, 0), \mu, 0), \mu)$, respectively, of $\gamma(\varphi T(\mu, 0), \mu, 0)$. Using the coordinates (φ, v^s, v^u), we then define the section

$$\Sigma_{\mathrm{out}} := \{(\varphi, v^s, v^u, h) \in S^1 \times [-\delta, \delta] \times [-\delta, \delta] \times K : v^u = \delta\}, \tag{2.2}$$

where $\delta > 0$ is a small positive constant. We can now formulate our assumptions on the existence of heteroclinic orbits that connect the periodic orbits γ to the rest state $u = 0$.

Hypothesis 5. *There exists a smooth function $G_0 : S^1 \times \mathcal{I} \times J \to \mathbb{R}$ such that $G_0(\varphi, v^s, \mu) = 0$ if, and only if, $(\varphi, v^s, \delta, 0) \in W^s(0, \mu) \cap \Sigma_{\mathrm{out}}$. In particular,*

$$\Gamma := \{(\varphi, \mu) \in S^1 \times J : G_0(\varphi, 0, \mu) = 0\} = \{(\varphi, \mu) \in S^1 \times J : W^s(0, \mu) \cap W^{uu}(\gamma(\varphi T(\mu, 0), \mu, 0), \mu) \cap \Sigma_{\mathrm{out}} \neq \emptyset\}, \tag{2.3}$$

and we assume that $\Gamma \subset S^1 \times \mathring{J}$ is nonempty with $\nabla_{(\varphi,\mu)} G_0(\varphi, 0, \mu) \neq 0$ for each $(\varphi, \mu) \in \Gamma$.

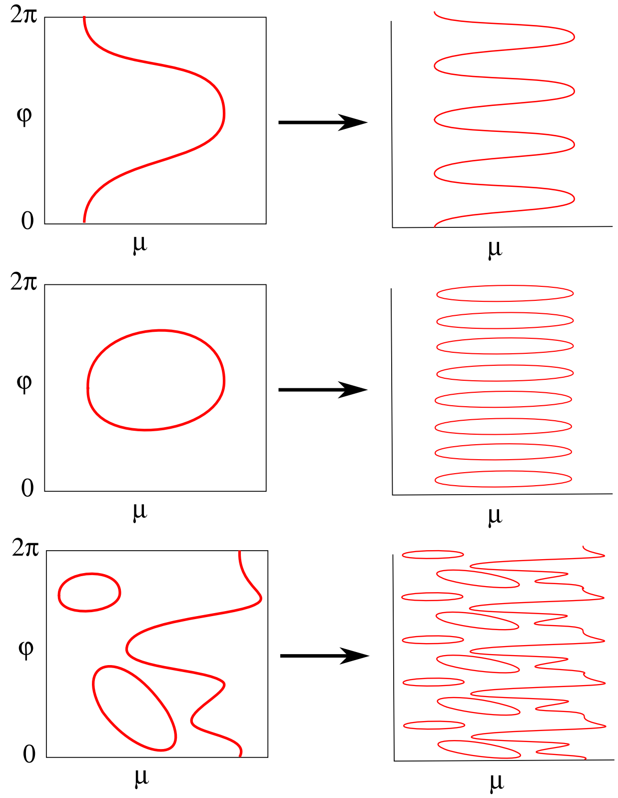

Figure 5: *The panels on the left contain, from top to bottom, a 1-loop, a 0-loop, and the union of 0- and 1-loops. The panels on the right illustrate the corresponding existence branches of even localized roll profiles consisting of, from top to bottom, snaking branches, isolas, and the union of isolas and snaking branches.*

As shown in [1, 2], Hypothesis 5 implies that Γ is the union of finitely many disjoint closed loops. Parametrizing one such loop by a function $(\varphi(s), \mu(s))$ with $s \in [0,1]$ and $\varphi(s)$ in the universal cover $\mathbb{R}$ of S^1, we must have either (i) $\varphi(0) = \varphi(1)$ or (ii) $\varphi(0) \neq \varphi(1)$. Following [1], we will refer to the case (i) as a 0-loop and case (ii) as a 1-loop. As illustrated in Figure 5 and proved in [1, 2], 0-loops lead to isolas and 1-loop to snaking branches.

Motivated by the structure of (1.3), our goal is to extend the results in [2] to systems of the form

$$u_x = f(u, \mu) + \frac{\varepsilon}{x} g(u, \mu, \varepsilon), \tag{2.4}$$

where $0 \leq \varepsilon \ll 1$ is a small perturbation parameter. For simplicity, we will assume that g is defined for all $\varepsilon \geq 0$.

Hypothesis 6. *The function $g : \mathbb{R}^4 \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^4$ is smooth in all its arguments, and $g(u, \mu, \varepsilon) = 0$ for all $(u, \mu, \varepsilon) \in \mathrm{Fix}(\mathcal{R}) \times J \times \mathbb{R}^+$.*

Hypothesis 6 implies in particular that $u = 0$ is a solution of (2.1) for all values of ε. We are interested in constructing solutions to (2.4) for $0 < \varepsilon \ll 1$ that remain close to the cylinder $\mathcal{C}(\mu)$ of periodic orbits for $x \in [0, L]$ for appropriate large values of $L \gg 1$ and converge to $u = 0$ as $x \to \infty$. More precisely, we denote the δ-neighbourhood of the manifold $\mathcal{C}(\mu)$ by $U_\delta(\mathcal{C}(\mu))$ and furthermore denote by $W^s_L(0, \mu, \varepsilon) \subset \mathbb{R}^4$ the slice of the stable manifold of the rest state $u = 0$ of (2.4) for $x = L$. We then say that $u(x)$ is a *radial pulse* if $u(x)$ is defined for $x \geq 0$, is a solution of (2.4) for $x > 0$ with (μ, ε) fixed, and satisfies the conditions

$$u(0) \in \mathrm{Fix}(\mathcal{R}), \qquad u(x) \in U_\delta(\mathcal{C}(\mu)) \text{ for } x \in [0, L], \qquad u(L) \in \partial U_\delta(\mathcal{C}(\mu)) \cap W^s_L(0, \mu, \varepsilon) \tag{2.5}$$

for some $L \gg 1$; see Figure 4 for an illustration. We denote by $\Gamma_{\text{lift}} \subset \mathbb{R} \times \mathring{J}$ the preimage of Γ under the natural covering projection from $\mathbb{R} \times \mathring{J}$ to $S^1 \times \mathring{J}$ so that 0-loops in Γ are lifted to an infinite number of disjoint copies of

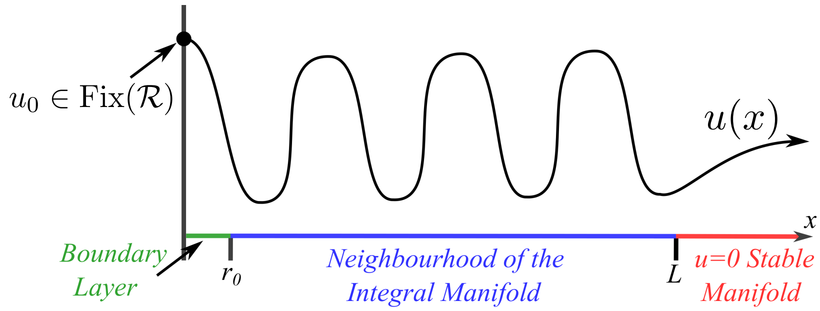

Figure 6: *To prove Theorem 2.1, we consider solutions of (2.4) in the separate regions of the independent variable x given by $[0, r_0]$, $[r_0, L]$, and $[L, \infty)$ for some small $0 < r_0 \ll 1$ and a large $L \gg 1$.*

the 0-loop, whereas 1-loops lift to an unbounded connected curve. The next theorem, which is our main result, relates the structure of Γ_{lift} to the the bifurcation structure of radial-pulses.

Theorem 2.1. *Assume that Hypotheses 1-6 are met, then there are constants $L_*, \varepsilon_0, \eta > 0$ and $b > 1$, a function $L_{\max}(\varepsilon)$ such that $L_{\max}(\varepsilon) \geq b^{\frac{1}{\varepsilon}}$, and sets $\Gamma^{\varphi_0,\varepsilon}_{\text{pulse}} \subset (L_*, L_{\max}(\varepsilon)) \times J$ defined for $\varphi_0 \in \{0, \pi\}$ and $\varepsilon \in [0, \varepsilon_0]$ so that the following is true:*

(i) Equation (2.4) admits a radial pulse if, and only if, $(L, \mu) \in \Gamma^{\varphi_0,\varepsilon}_{\text{pulse}}$ for $\varphi_0 = 0$ or $\varphi_0 = \pi$.

(ii) There exists a smooth function $g^c(L, \varepsilon) = \mathcal{O}(\varepsilon \ln(L))$ such that the one-dimensional manifolds

$$\tilde{\Gamma}^{\varphi_0}_{\text{lift}} := \{(L - g^c(L, \varepsilon) - \varphi_0, \mu) : \ (L, \mu) \in \Gamma_{\text{lift}} \cap ((L_*, L_{\max}(\varepsilon)) \times J)\},$$

and $\Gamma^{\varphi_0,\varepsilon}_{\text{pulse}}$ are $\mathcal{O}(\mathrm{e}^{-\eta L})$-close to each other in the C^0-sense near each point $(L, \mu) \in \tilde{\Gamma}^{\varphi_0}_{\text{lift}}$.

Remark 1. *We note that this theorem captures not only solutions that stay close to the level set $\mathcal{H} = 0$ for all values of x but also solutions whose energy varies in the interval K away from $h = 0$. In particular, the size of b is restricted only by the possibility that solutions can leave a neighborhood of the cylinders $\mathcal{C}(\mu)$ when their energy leaves the interval K.*

Theorem 2.1 will be proved in the remainder of this paper. To find radial pulses, we will construct solutions separately on the interval $[0, r_0]$ with $0 < r_0 \ll 1$ to account for the singularity of (2.4) at $x = 0$, the interval $[r_0, L]$ where solutions are close to the cylinder $\mathcal{C}(\mu)$, and the interval $[L, \infty)$ where solutions lie on the stable manifold of $u = 0$. We then match these solutions at $r = r_0$ and $r = L$: see Figures 4 and 6 for illustrations.

3 The boundary layer

In this section we will prove the existence of a solution to (2.4) on the interval $[0, r_0]$ for a small, positive r_0.

Lemma 3.1. *Assume Hypotheses 1, 3 and 6. Then, for each compact set B in $\mathrm{Fix}(\mathcal{R})$, there exists $C, R_0, \varepsilon_1 > 0$ such that, for all $\varepsilon \in [0, \varepsilon_1)$, $\mu \in J$, and $u_0 \in B$, there exists a unique solution $u = u_{\text{bdy}}(\cdot; u_0, \mu, \varepsilon) \in C^0([0, 2R_0], \mathbb{R}^4) \cap C^1((0, 2R_0), \mathbb{R}^4)$ of the differential equation (2.4) that satisfies the initial condition $u(0) = u_0$. Furthermore, this solution is of the form*

$$u_{\text{bdy}}(x) = u^0(x) + \bar{u}(x, u_0, \mu, \varepsilon),$$

where $u^0(x)$ satisfies the unperturbed system (2.1) with $u^0(0) = u_0$, and $\bar{u}$ depends smoothly on (u_0, μ, ε) with

$$|\bar{u}(x, u_0, \mu, \varepsilon)|, \ |\bar{u}_{u_0}(x, u_0, \mu, \varepsilon)|, \ |\bar{u}_\mu(x, u_0, \mu, \varepsilon)|, \ |\bar{u}_\varepsilon(x, u_0, \mu, \varepsilon)| \leq C\varepsilon x$$

uniformly in $0 < x \leq R_0$.

Proof. Let $u^0(x)$ be the solution of $u_x = f(u,\mu)$ with $u^0(0) = u_0 \in \mathrm{Fix}(\mathcal{R})$. Writing $u(x) = u^0(x) + v(x)$, we see that $u(x)$ is a solution to (2.4) of the form desired by the lemma if, and only if, $v(x)$ satisfies the nonautonomous initial-value problem

$$v_x = f(u^0(x)+v,\mu) - f(u^0(x),\mu) + \frac{\varepsilon}{x} g(u^0(x)+v,\mu,\varepsilon), \qquad v(0)=0. \tag{3.1}$$

We can write $u^0(x) = u_0 + x\tilde{u}(x)$ and

$$f(u^0(x)+v,\mu) - f(u^0(x),\mu) = \tilde{f}(x,v,\mu)v.$$

Furthermore, denoting by $\mathcal{P}_\mathcal{R}$ the projection onto $\mathrm{Fix}(-\mathcal{R})$ along $\mathrm{Fix}(\mathcal{R})$, we can use Hypothesis 6 to write

$$\frac{1}{x} g(u,\mu,\varepsilon) = \frac{1}{x}\tilde{g}(u,\mu,\varepsilon)\mathcal{P}_\mathcal{R} u.$$

Using the Banach space

$$X = \left\{ v \in C^0([0,2R_0],\mathbb{R}^4) : \; v(0)=0 \text{ and } \|v\| := \sup_{x\in[0,R_0]} \frac{|v(x)|}{x} < \infty \right\},$$

we then define, for $v \in X$, the new function

$$\begin{aligned}
[Tv](x) \quad &:= \quad \int_0^x \left[f(u^0(s)+v(s)) - f(u^0(s),\mu) + \frac{\varepsilon}{s} g(u^0(s)+v(s),\mu,\varepsilon)\right] \mathrm{d}s \\
&= \quad \int_0^x \left[\tilde{f}(s,v(s),\mu)v(s) + \frac{\varepsilon}{s}\tilde{g}(u^0(s)+v(s),\mu,\varepsilon)\mathcal{P}_\mathcal{R}(u_0 + s\tilde{u}(s)+v(s))\right] \mathrm{d}s \\
&=: \quad \int_0^x \left[\tilde{f}(s,v(s),\mu)v(s) + \varepsilon\tilde{g}_1(s,v(s),\mu,\varepsilon) + \frac{\varepsilon}{s}\tilde{g}_2(s,v(s),\mu,\varepsilon)v(s)\right] \mathrm{d}s
\end{aligned}$$

where we used that $\mathcal{P}_\mathcal{R} u_0 = 0$. Since fixed points of T are in one-to-one correspondence with solutions of (3.1), it suffices to show that, for sufficiently small $R_0, \delta > 0$, T maps the ball of radius δ centered at the origin in X into itself and is a uniform contraction on this ball: these properties are straightforward to verify using the uniform bounds on the smooth functions $\tilde{f}$ and $\tilde{g}_{1,2}$ and their Lipschitz constants in v. We omit the details. □

4 Shilnikov variables

The results of the preceding section allow us to restrict the analysis of (2.4) to the region $x \geq r_0$ for each fixed, but arbitrary, value of r_0. For each such fixed r_0, we will construct a local coordinate system akin to Shilnikov variables for the nonautonomous system (2.4) near the cylinder $\mathcal{C}(\mu)$ of periodic orbits.

Choose a closed nonempty interval $K_0 \subset \mathring{K}$. For each $\delta > 0$, we define $\mathcal{I} := [-\delta,\delta]$ and let $\mathcal{V} := \mathbb{R}\times\mathcal{I}\times\mathcal{I}\times K$, $\mathcal{V}_0 := \mathbb{R}\times\mathcal{I}\times\mathcal{I}\times K_0$, and $\mathcal{C}_0(\mu) := \{\gamma(x,\mu,h) : \; x\in\mathbb{R}, h \in K_0\}$.

Lemma 4.1. *Assume that Hypotheses 1-6 are met, then there is a $\delta > 0$ such that the following is true for each fixed $r_0 \in (0,R_0]$: there is an $\varepsilon_2 > 0$ such that there are smooth real-valued functions $(h^c_{1,2,3}, h^s, h^u, h^e_{1,2,3})$ and a smooth change of coordinates that puts (2.4) restricted to a uniform neighborhood of $\mathcal{C}_0(\mu)$ into the form*

$$\begin{aligned}
v^c_x &= 1 + \frac{\varepsilon}{x}h^c_1(v^c,v^h,\mu,\varepsilon) + \frac{\varepsilon^2}{x^2}h^c_2(x,v^c,v^h,\mu,\varepsilon) + \frac{\varepsilon}{x}h^c_3(x,v,\mu,\varepsilon)v^s v^u \\
v^s_x &= -[\alpha(\mu,v^h) + h^s(x,v,\mu,\varepsilon)]v^s \\
v^s_u &= [\alpha(\mu,v^h) + h^u(x,v,\mu,\varepsilon)]v^u \\
v^h_x &= \frac{\varepsilon}{x}h^e_1(v^c,v^h,\mu,\varepsilon) + \frac{\varepsilon^2}{x^2}h^e_2(x,v^c,v^h,\mu,\varepsilon) + \frac{\varepsilon}{x}h^e_3(x,v,\mu,\varepsilon)v^s v^u,
\end{aligned} \tag{4.1}$$

where $(x,\mu,\varepsilon) \in [r_0,\infty) \times J \times [0,\varepsilon_2]$ and $v = (v^c,v^s,v^u,v^h) \in \mathcal{V}$. The functions $(h^c_{1,2,3},h^s,h^u,h^e_{1,2,3})$ are $2\pi T(\mu,v^h)$-periodic in v^c, uniformly bounded for $x \geq r_0$, and globally Lipschitz in $v \in \mathcal{V}$. Moreover, h^s and h^u vanish identically when $(v^s,v^u,\varepsilon) = (0,0,0)$, and $h^e_{1,2}$ vanish identically for $v^h \in \partial K$. The reverser $\mathcal{R}$ acts via

$$\mathcal{R}(v^c,v^s,v^u,v^h) = (-v^c,v^u,v^s,v^h).$$

Proof. Let $v^h := \mathcal{H}(u,\mu)$, then, for each solution $u(x)$ of (2.4), we have

$$v^h_x = \nabla_u\mathcal{H}(u,\mu)\cdot u_x = \nabla_u\mathcal{H}(u,\mu)\cdot\Big(f(u,\mu)+\frac{\varepsilon}{x}g(u,\mu,\varepsilon)\Big) = \frac{\varepsilon}{x}\nabla_u\mathcal{H}(u,\mu)\cdot g(u,\mu,\varepsilon) \tag{4.2}$$

upon using Hypothesis 2. Note that the identity $\mathcal{H}(\mathcal{R}u,\mu) = \mathcal{H}(u,\mu)$ implies that v^h remains unchanged under the action of $\mathcal{R}$. As in [2, 5], we can now use Hypothesis 4(iv) to introduce the invertible coordinate transformation $u = Q(v,\mu)$ defined for $v = (v^c,v^s,v^u,v^h) \in \mathbb{R}\times\mathcal{I}\times\mathcal{I}\times K$ so that, for $\varepsilon = 0$, $Q(v^c,0,0,v^h,\mu) = \gamma(v^c,\mu,v^h)$ parametrizes the periodic orbits, the sets $\{v^u = 0\}$ and $\{v^s = 0\}$ parametrize, respectively, the strong stable and strong unstable fibers of $\gamma(v^c,\mu,v^h)$, and $\{v^h = h\}$ gives $\mathcal{H}^{-1}(h)$. Note that the action of the reverser follows from Hypothesis 4(ii). Referring again to [2, 5], the vector field in the new coordinates v is then given by

$$\begin{aligned}
v^c_x &= 1 + f^c(v,\mu)v^sv^u + \frac{\varepsilon}{x}\tilde{f}^c(x,v,\mu,\varepsilon)\\
v^s_x &= -[\alpha(\mu,v^h) + f^s_1(v,\mu)v^s + f^s_2(v,\mu)v^u]v^s + \frac{\varepsilon}{x}\tilde{f}^s(x,v,\mu,\varepsilon)\\
v^s_u &= [\alpha(\mu,v^h) + f^u_1(v,\mu)v^s + f^u_2(v,\mu)v^u]v^u + \frac{\varepsilon}{x}\tilde{f}^u(x,v,\mu,\varepsilon)\\
v^h_x &= \frac{\varepsilon}{x}\tilde{f}^h(x,v,\mu,\varepsilon)
\end{aligned} \tag{4.3}$$

for each $\mu \in J$ and $\varepsilon \geq 0$, where the functions $\tilde{f}^j$ with $j = c,s,u,h$ represent the terms coming from the perturbation $g(Q(v),\mu,\varepsilon)$, with $\tilde{f}^h$ given by (4.2).

Note that the set $\{v^s = v^u = 0\}$, which corresponds to the cylinder $\mathcal{C}(\mu)$ in the original variables, forms an invariant, normally hyperbolic manifold for (4.3) when $\varepsilon = 0$. We claim that this invariant manifold persists as an integral manifold for (4.3) for all sufficiently small $\varepsilon > 0$. Indeed, it is straightforward to check that our system satisfies the hypotheses stated in [7, Theorem 2.2], and this theorem then guarantees the existence of a smooth function $\Theta(x,v^c,v^h,\mu,\varepsilon)$ that is defined for all $0 < \varepsilon \ll 1$ such that

$$\Big\{(v^s,v^u) = \frac{\varepsilon}{x}\Theta(x,v^c,v^h,\mu,\varepsilon); (v^c,v^h) \in \mathbb{R}\times\mathcal{I}\Big\}$$

is a smooth integral manifold for (4.3). Furthermore, the transformation Θ is $2\pi T(\mu,v^h)$-periodic in v^c, uniformly bounded in $x \geq r_0$, and does not depend on the choice of r_0. Defining the new variables

$$(\tilde{v}^s,\tilde{v}^u) := (v^s,v^u) - \frac{\varepsilon}{x}\Theta(x,v^c,v^h,\mu,\varepsilon)$$

and noting that $\{(\tilde{v}^s,\tilde{v}^u) = 0\}$ corresponds to the integral manifold we just constructed, we see that, upon dropping the tildes, the system (4.3) becomes

$$\begin{aligned}
v^c_x &= 1 + \frac{\varepsilon}{x}H^c_1(v^c,v^h,\mu,\varepsilon) + \frac{\varepsilon^2}{x^2}H^c_2(x,v^c,v^h,\mu,\varepsilon) + H^c_3(x,v,\mu,\varepsilon)v^s + H^c_4(x,v,\mu,\varepsilon)v^u\\
v^s_x &= -[\alpha(\mu,v^h) + H^s_1(x,v,\mu,\varepsilon)]v^s + H^s_2(x,v,\mu,\varepsilon)v^u\\
v^s_u &= [\alpha(\mu,v^h) + H^u_1(x,v,\mu,\varepsilon)]v^u + H^u_2(x,v,\mu,\varepsilon)v^s\\
v^h_x &= \frac{\varepsilon}{x}H^h_1(v^c,v^h,\mu,\varepsilon) + \frac{\varepsilon^2}{x^2}H^h_2(x,v^c,v^h,\mu,\varepsilon) + \frac{\varepsilon}{x}H^h_3(x,v,\mu,\varepsilon)v^s + \frac{\varepsilon}{x}H^h_4(x,v,\mu,\varepsilon)v^u,
\end{aligned} \tag{4.4}$$

where $H^{c,h}_1(v^c,v^h,\mu,\varepsilon) = \tilde{f}^{c,h}(v^c,0,0,v^h,\mu,\varepsilon)$. The functions appearing in (4.4) are $2\pi T(\mu,v^h)$-periodic in v^c, uniformly bounded in their arguments, independent of r_0 for $x \geq r_0$, and globally Lipschitz in v. Furthermore, $H^{s,u}_j$ vanish when $(v^s,v^u,\varepsilon) = (0,0,0)$ for $j = 1,2$. By locally straightening the stable and unstable fibers of the integral manifold so that they correspond, respectively, to $v^u = 0$ and $v^s = 0$, we can bring (4.4) into the normal form (4.1); see [5] or [8, Chapter 4] for details. Finally, we can multiply the functions $h^e_{1,2}$ by appropriate cutoff functions so that the products coincide with the original functions for $v^h \in K_0$ and vanish identically when $v^h \in \partial K$. Throughout these transformations, the action of the reverser remains as stated in the lemma. □

Note that Lemma 4.1 shows that $\{(v^s, v^u) = 0\}$ is an integral manifold of (2.4) that corresponds to the perturbed invariant cylinder $\mathcal{C}(\mu)$ of periodic orbits of (2.1). The vector field on this integral manifold is given by

$$\begin{pmatrix} v^c \\ v^h \end{pmatrix}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\varepsilon}{x} h_1(v^c, v^h, \mu, \varepsilon) + \frac{\varepsilon^2}{x^2} h_2(x, v^c, v^h, \mu, \varepsilon) =: \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\varepsilon}{x} F^c(x, v^c, v^h, \mu, \varepsilon). \tag{4.5}$$

Due to the fact that when $\varepsilon > 0$ the flow of (2.4) can move between the level sets of the conserved quantity $\mathcal{H}$, there is the possibility that trajectories can leave $\mathcal{V}$ by having $v^h \notin K$, representing a trajectory that flows off the top or bottom of the integral manifold. To prevent this from happening we have applied cutoff functions to guarantee that $\mathcal{V}$ is invariant, but we note that the system (4.1) does not completely correspond to the original perturbed differential equation (2.4). This leads to the following lemma which precisely determines when a trajectory of (4.5) corresponds to a trajectory of (2.4), along with some important properties about solutions on the integral manifold.

Lemma 4.2. *For each fixed $r_0 \in (0, 1]$ and given $L \geq 1/r_0$, $\varphi_1 \in \mathbb{R}$, and $h_1 \in K$, there exists a unique solution $\Phi(x; r_0, L, \varphi_1, h_1, \mu, \varepsilon)$ to (4.5) that satisfies the boundary conditions*

$$v^c(L) = \varphi_1, \qquad v^h(L) = h_1 \tag{4.6}$$

and lies in $\mathcal{I} \times K$ for $x \in [r_0, L]$. This solution is smooth in $(x, r_0, L, \varphi_1, h_1, \mu, \varepsilon)$ and we have

$$v^c(r_0) = \varphi_1 - L + r_0 + g^c(L, \mu, \varepsilon), \tag{4.7}$$

with $|g^c(L, \mu, \varepsilon)|, |v^h_\varphi(r_0)|, |v^h_{h_1}(r_0)|, |v^h_\mu(r_0)| \leq C_g \varepsilon \ln(L)$, for some constant $C_g > 0$. This solution provides a genuine solution of the original system whenever $\Phi^h(x; r_0, L, \varphi_1, h_1, \mu, \varepsilon) \in K_0$ for $x \in [r_0, L]$.

Proof. Since $\mathcal{I} \times \partial K$ is invariant under (4.5) for all $\varepsilon > 0$, existence, uniqueness, and smoothness of the solution is immediate. Therefore, it remains to estimate $v^c(r_0)$. We write $v^c(x) = \varphi_1 + x - L + \tilde{v}^c(x)$ so that the boundary condition $v^c(L) = \varphi_1$ is equivalent to $\tilde{v}^c(L) = 0$. Thus, $\tilde{v}^c(r_0)$ is given implicitly by

$$\tilde{v}^c(r_0) = \varepsilon \int_L^{r_0} \frac{h^c_1(v^c(s), v^h(s), \mu, \varepsilon)}{s} \mathrm{d}s + \varepsilon^2 \int_L^{r_0} \frac{h^c_2(s, v^c(s), v^h(s), \mu, \varepsilon)}{s^2} \mathrm{d}s.$$

Bounding $h^c_{1,2}$ by a uniform constant $C_1 > 0$, we obtain

$$|\tilde{v}^c(r_0)| = |g^c(L, \mu, \varepsilon)| \leq \varepsilon C_1(\ln(L) + |\ln(r_0)|) \leq \varepsilon C_g \ln(L)$$

with $C_g := C_1(1 + |\ln(r_0)|/\ln(L)) \leq 2C_1$ for $L \geq 1/r_0$. The bounds on $|v^h(r_0)|$ and its derivatives are handled in an identical manner. □

Proposition 4.3. *There exist $\eta, L_0, M > 0$ such that, for each fixed $0 < r_0 \leq 1$, there exists an $\varepsilon_3 > 0$ such that the following holds: pick $0 \leq \varepsilon \leq \varepsilon_3$, $L \geq L_0$, and let $\Phi(x; r_0, L, \varphi_1, h_1, \mu, \varepsilon)$ be as above, then, for each $a^s \in \mathcal{I}$, there exists a unique solution $v(x) = v(x; r_0, L, a^s, \varphi_1, h_1, \mu, \varepsilon) \in \mathcal{V}$ to (4.1) defined for $x \in [r_0, L]$ so that*

$$v^c(L) = \varphi_1, \qquad v^s(r_0) = a^s, \qquad v^u(L) = \delta, \qquad v^h(L) = h_1.$$

Furthermore, this solution satisfies

$$|v^s(x)| \leq M\mathrm{e}^{-\eta x}, \quad |v^u(x)| \leq M\mathrm{e}^{\eta(x-L)}, \quad |(v^c(x), v^h(x))^T - \Phi(x; r_0, L, \varphi_1, h_1, \mu, \varepsilon)| \leq M\mathrm{e}^{-\eta L} \tag{4.8}$$

for all $x \in [r_0, L]$, $v(x)$ is smooth in $(x, r_0, L, a^s, \varphi_1, h_1, \mu, \varepsilon)$, and the bounds (4.8) also hold for these derivatives.

Proof. We will show that the assumptions of [17, Theorem 2.2] are satisfied: our statements then follow directly from this theorem. Note that restricting to $x \geq r_0$ and choosing $0 \leq \varepsilon_2 \leq r_0^2$ guarantees that the right-hand side of (4.1) is bounded uniformly in x. It remains to establish appropriate exponential bounds for solutions of the linearized dynamics of (4.1). Linearizing (4.1) along the solution $(v^c, v^s, v^u, v^h) = (\Phi^c(x), 0, 0, \Phi^h(x))$, where $(\Phi^c(x), \Phi^h(x))^T = \Phi(x; r_0, L, \mu, \varepsilon)$ satisfies (4.5)–(4.6) on $[r_0, L]$, we arrive at the linear system

$$v_x^s = -[\alpha(\Phi^h(x)) + h^s(x, \Phi^c(x), 0, 0, \Phi^h(x), \varepsilon)]v^s \tag{4.9a}$$

$$v_x^u = [\alpha(\Phi^h(x)) + h^u(x, \Phi^c(x), 0, 0, \Phi^h(x), \varepsilon)]v^u \tag{4.9b}$$

$$w_x = \frac{\varepsilon}{x} D_{(v^c, v^h)} F^c(x, \Phi^c(x), \Phi^h(x), \varepsilon)c, \tag{4.9c}$$

where $w = (v^c, v^h)^T$; note that we have suppressed the dependence on $\mu \in J$ for notational convenience. Lemma 4.1 implies that (h^s, h^u) vanish uniformly when $\varepsilon = 0$, and Hypothesis 4 implies that $\alpha(\mu, v^h)$ is bounded away from zero uniformly in $(\mu, v^h) \in J \times K$. Hence, for $\varepsilon > 0$ taken sufficiently small, the right-hand sides of (4.9a) and (4.9b) are uniformly bounded away from zero,and we conclude that there are constants $\eta^s, \eta^u > 0$ and $M_0 > 0$ such that the solution operators $\Psi^s(x, s), \Psi^u(x, s)$ of (4.9a) and (4.9b), respectively, satisfy

$$|\Psi^s(x, y)| \leq M_0 \mathrm{e}^{-\eta^s(x-y)} \text{ and } |\Psi^u(y, x)| \leq M_0 \mathrm{e}^{\eta^u(y-x)}$$

for $r_0 \leq y \leq x \leq L$ and $\varepsilon > 0$ sufficiently small. Furthermore, we may take $\eta^u > 0$ so that $-\eta^s + \eta^u < 0$.

We now turn to (4.9c). Lemma 4.1 guarantees that there is a constant $C > 0$ with $|D_{(v^c, v^h)} F^c(x, \Phi^c(x), \Phi^h(x), \varepsilon)| \leq C$ uniformly in all arguments, and we conclude that

$$|w_x| \leq \frac{\varepsilon C}{x}|w| \leq \varepsilon^{\frac{1}{2}} C|w|$$

provided we pick $0 \leq \varepsilon_2 \leq r_0^2$. Denoting the solution operator to (4.9c) by $\Psi^c(x, y)$, we have

$$|\Psi^c(x, y)| \leq \mathrm{e}^{\sqrt{\varepsilon} C(x-y)}$$

for all $x, y \in [r_0, L]$, independently of r_0 and L, which verifies [17, Hypothesis (E1) in Theorem 2.2]. Finally, taking $\varepsilon > 0$ and sufficiently small, we can guaranteed that

$$\eta^s + \eta^u + \sqrt{\varepsilon} K_2 < 0 < \eta^u - \sqrt{\varepsilon} C,$$

which verifies [17, Hypothesis (D2) in Theorem 2.2]. We have now verified all hypotheses required to obtain the result in the statement of Proposition 4.3. ☐

Proposition 4.3 shows that we can understand the dynamics of (4.1) via solutions $\Phi(x; r_0, L, \varphi_1, h_1, \mu, \varepsilon)$ to (4.5). In §7, we will explore conditions on the vector field on the integral manifold that imply that solutions to (4.5)–(4.6) stay in K_0 for all x, and therefore by Lemma 4.2 correspond to genuine solution to (2.4). Prior to doing so though, we continue our exploration of the full augmented vector field (4.5) by obtaining radial pulse solutions in this situation and work to understand their bifurcation characteristics.

5 The stable manifold

We now describe the set of solutions $u(x)$ that converge to the trivial equilibrium as $x \to \infty$. Recall that Hypothesis 6 implies that $u = 0$ persists as an equilibrium of the nonautonomous system (2.4) for all $\varepsilon \geq 0$. Hence, for any $L \geq 1$, we define the section of the stable manifold of the trivial solution at $x = L$ to be

$$W_L^s(0, \mu, \varepsilon) := \{u_0 \in \mathbb{R}^4 : u(x) \text{ satisfies (2.4) with } u(L) = u_0, \text{ and } u(x) \to 0 \text{ as } x \to \infty\}.$$

Using the constant $\delta > 0$ introduced in Lemma 4.1, we define the section $\Sigma_{\rm out}$ to be

$$\Sigma_{\rm out} = \mathbb{R} \times \mathcal{I} \times \{v^u = \delta\} \times K.$$

First, we show that $W^s_L(0,\mu,\varepsilon)$ is a regular perturbation of $W^s(0,\mu)$ in ε.

Lemma 5.1. *For each $L \geq 1$, the set $W^s_L(0,\mu,\varepsilon)$ is a two-dimensional manifold that is $\mathcal{O}(\varepsilon L^{-1})$-close in the C^1 sense to $W^s(0,\mu)$, uniformly in μ.*

Proof. The lemma follows from the uniform contraction mapping principle. □

Next, we use the preceding lemma to provide a parametrization of the stable manifold in the Shilnikov variables.

Lemma 5.2. *There exists $\varepsilon_4 > 0$ and smooth real-valued functions z^Γ, z^h so that the following is true for each $\varepsilon \in [0,\varepsilon_4]$ and $L \geq 1$: we have that $(\varphi, v^s, \delta, v^h)$ with $|v^s|, |v^h| < \varepsilon_4$ lies in $W^s_L(0,\mu,\varepsilon) \cap \Sigma_{\rm out}$ if, and only if, there exists $(\varphi^*, \mu^*) \in \Gamma$ such that*

$$\begin{aligned}
\varphi &= \varphi^* + z^\Gamma(L,\varphi^*,v^s,\mu^*,\varepsilon)\frac{\partial G_0}{\partial \varphi}(\varphi^*,0,\mu^*) \\
\mu &= \mu^* + z^\Gamma(L,\varphi^*,v^s,\mu^*,\varepsilon)\frac{\partial G_0}{\partial \mu}(\varphi^*,0,\mu^*) \\
v^h &= z^h(L,\varphi^*,v^s,\mu^*,\varepsilon),
\end{aligned}$$

where the function G_0 was defined in Hypothesis 5, and the functions z^Γ and z^h are bounded uniformly in $L \geq 1$, independent of L when $\varepsilon = 0$, and satisfy

$$z^\Gamma(\cdot,\varphi^*,0,\mu^*,0) = 0, \qquad z^h(\cdot,\varphi^*,v^s,\mu^*,0) = 0.$$

Proof. Define the subset of $\Sigma_{\rm out} \times J \cong \mathbb{R} \times \mathcal{I} \times K \times J$ given by

$$\tilde{W}^s_L(0,\varepsilon) := \bigcup_{\mu \in J} (W^s_L(0,\mu,\varepsilon) \cap \{v^u = \delta\}) \times \{\mu\}.$$

First, note that $\tilde{W}^s_L(0,0)$ is independent of L since (2.4) is autonomous when $\varepsilon = 0$. Hypothesis 5 implies that

$$\Gamma = \tilde{W}^s_L(0,0) \cap (\mathbb{R} \times \{0\} \times \{0\} \times J),$$

that the function $G : \mathbb{R} \times \mathcal{I} \times K \times J \to \mathbb{R}^2$ given by

$$G(\varphi, v^s, v^h, \mu) = (G_0(\varphi, v^s, \mu), v^h)$$

satisfies $G^{-1}(0) = \tilde{W}^s_L(0,0)$, and that $\nabla_{(\varphi,v^h,\mu)}G(\varphi,0,0,\mu) \in \mathbb{R}^{2\times 3}$ has full rank for all $(\varphi,\mu) \in \Gamma$. Therefore, we may use the columns of $\nabla_{(\varphi,v^h,\mu)}G(\varphi,0,0,\mu)$ to define normal vectors to Γ as a subset of $\tilde{W}^s_L(0,\varepsilon)$ in the space $\Sigma_{\rm out} \times J$ to obtain the existence of the functions z^Γ and z^h, which represent deviations from Γ along these normal vectors with respect to small perturbations in v^s and ε. Smoothness properties and boundedness with respect to $L \geq 1$ follow from Lemma 5.1. The property $z^h(\varphi^*, v^s, \mu^*, 0) = 0$ follows from the fact that $W^s(0,\mu) \subset \mathcal{H}^{-1}(0)$ for all $\mu \in J$. This completes the proof. □

6 Matching conditions

We can now match the solution segments we obtained in the preceding sections for $0 \leq x \leq r_0$, $r_0 \leq x \leq L$, and $L \geq x$. Before doing so, we restate our definition (2.5) of radial pulses. We say that $u(x)$ is a radial-pulse solution to (2.4) if

$$u(0) \in \mathrm{Fix}(\mathcal{R}), \tag{6.1a}$$
$$u(x) \in \mathcal{V}, \quad x \in [0, L), \tag{6.1b}$$
$$u(L) \in W_L^s(0, \mu, \varepsilon) \cap \Sigma_{\mathrm{out}}. \tag{6.1c}$$

Furthermore, recall that $\Gamma_{\mathrm{lift}} \subset \mathbb{R} \times \mathring{J}$ is defined to be the preimage of Γ from (2.3) under the natural covering map from $\mathbb{R} \times \mathring{J}$ to $S^1 \times \mathring{J}$. Due to the cutoff functions applied to the vector field (4.1) that guarantee that $\mathcal{V}$ is an invariant region for all $x \geq r_0$, we remind the reader that only those trajectories $(v^c(x), v^s(x), v^u(x), v^h(x))$ of (4.1) for which $v^h(x) \in K_0$ correspond to genuine trajectories to (2.4). It should therefore be noted that the following result is for the augmented system (4.1), which eases the analysis, and in Section 7 we describe how these solutions can be related to the original differential equation (2.4). Throughout the proof of Theorem 2.1, we will make use of the following theorem that we state without proof.

Theorem 6.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function and assume there exists an invertible matrix $A \in \mathbb{R}^{n\times n}$, $w_0 \in \mathbb{R}^n$, $0 < \kappa < 1$, and $\rho > 0$ such that*

(i) $\|1 - A^{-1}DF(w)\| \leq \kappa$ *for all* $w \in B_\rho(w_0)$,
(ii) $\|A^{-1}F(w_0)\| \leq (1-\kappa)\rho$.

Then F has a unique root w_* in $B_\rho(w_0)$, and this root satisfies $|w_* - w_0| \leq \frac{1}{1-\kappa}\|A^{-1}F(w_0)\|$.

Theorem 6.2. *Assume that Hypotheses 1-6 are met. There exist constants $L_*, \varepsilon_0, \eta > 0$ and sets $\Gamma_{\mathrm{pulse}}^{\varphi_0,\varepsilon} \subset (L_*, \infty) \times J$ defined for $\varphi_0 \in \{0, \pi\}$ and $\varepsilon \in [0, \varepsilon_0]$ so that the following is true:*

(i) The modified equation (2.4) using the flow (4.1) admits a radial-pulse solution if, and only if, $(L, \mu) \in \Gamma_{\mathrm{pulse}}^{\varphi_0,\varepsilon}$ for $\varphi_0 = 0$ or $\varphi_0 = \pi$.

(ii) There exists a smooth function $g_{\mathrm{pulse}}(L, \mu, \varepsilon) = \mathcal{O}(\varepsilon \ln(L))$ such that the one-dimensional manifolds

$$\tilde{\Gamma}_{\mathrm{lift}}^{\varphi_0} := \{(L - g_{\mathrm{pulse}}(L, \mu, \varepsilon) - \varphi_0, \mu) : \ (L, \mu) \in \Gamma_{\mathrm{lift}} \cap ((L_* \infty) \times J)\},$$

and $\Gamma_{\mathrm{pulse}}^{\varphi_0,\varepsilon}$ are $\mathcal{O}(\mathrm{e}^{-\eta L})$-close to each other in the C^0-sense near any point $(L, \mu) \in \tilde{\Gamma}_{\mathrm{lift}}^{\varphi_0}$. In particular, the function g_{pulse} is such that

$$g_{\mathrm{pulse}}(L, \mu, \varepsilon) = g^c(L, \mu, \varepsilon) + \mathcal{O}(\varepsilon^2 \ln(L)),$$

where $g^c(L, \mu, \varepsilon)$ is defined in (4.7).

Proof. Lemma 3.1, Proposition 4.3, and Lemma 5.2 show that it suffices to match the solution segments we constructed there at $x = r_0$ and $x = L$. More precisely, we need to match

(i) the boundary-layer solution of Lemma 3.1 with the Fenichel solution obtained in Proposition 4.3 at $x = r_0$ to satisfy (6.4);
(ii) the Fenichel solution and the stable manifold $W_L^s(0, \mu, \varepsilon)$ characterized in Lemma 5.2 at $x = L$ in Σ_{out} to satisfy (6.1c).

We begin by precisely denoting each of the relevant solutions used to perform the matchings. Take some $u_0 \in \mathrm{Fix}(\mathcal{R})$ convert the solution guaranteed by Lemma 3.1 associated to this u_0 into the coordinates of Lemma 4.1. That is, for $\varphi_0 \in \{0, \pi\}$ and arbitrary $a_0 \in \mathcal{I}$, $h_0 \in K$ we can write $u_0 = (\varphi_0, v_0, v_0, h_0) \in \mathrm{Fix}(\mathcal{R})$, and consider $r_0 > 0$ so that we write this solution as

$$v_{\mathrm{b}}(r_0, u_0, \mu, \varepsilon) = (\varphi_0, a_0, a_0, h_0) + (v_{\mathrm{b}}^c(r_0, u_0, \mu, \varepsilon), v_{\mathrm{b}}^s(r_0, u_0, \mu, \varepsilon), v_{\mathrm{b}}^u(r_0, u_0, \mu, \varepsilon), v_{\mathrm{b}}^h(r_0, u_0, \mu, \varepsilon)),$$

where we $v_{\mathrm{b}}^j(0, u_0, \mu, \varepsilon) = 0$ for all $j = c, u, s, h$ and $v_{\mathrm{b}}(0, r_0, u_0, \mu, \varepsilon) = u_0 \in \mathrm{Fix}(\mathcal{R})$. Furthermore, at $\varepsilon = 0$ we have

$$v_{\mathrm{b}}(r_0, u_0, \mu, \varepsilon) = (\varphi_0 + r_0, a_0, v_0, h_0) + \mathcal{O}(r_0)a_0,$$

since $v_{\rm b}$ depends smoothly on x and taking $a_0 = 0$ results in the flow restricted to the invariant periodic orbits governed by $v^s = v^u = 0$ and $v^h = h_0$. Then, from Lemma 3.1 for sufficiently small $\varepsilon > 0$ we get

$$v_{\rm b}(r_0, u_0, \mu, \varepsilon) = (\varphi_0 + r_0, a_0, v_0, h_0) + \mathcal{O}(r_0 a_0) + \mathcal{O}(\varepsilon). \tag{6.2}$$

We further note that the dependence of $v_{\rm b}(r_0, u_0, \mu, \varepsilon)$ on h_0 is absent from the $\mathcal{O}(r_0 a_0)$ term since when $\varepsilon = 0$ the solution $v_{\rm b}(x, u_0, \mu, 0)$ belongs to the h_0 level set of the conserved quantity $\mathcal{H}$.

Continuing, we now assume that $(\Phi^c, \Phi^h)(x; r_0, L, \varphi_1, h_1, \mu, \varepsilon)$ is a solution on the integral manifold satisfying the conclusions of Lemma 4.2 in that

$$\Phi^c(L; r_0, L, \varphi_1, h_1, \mu, \varepsilon) = \varphi_1, \qquad \Phi^h(L; r_0, L, \varphi_1, h_1, \mu, \varepsilon) = h_1,$$

for some $\varphi_1 \in \mathbb{R}$, $h_1 \in K$. Then, from Proposition 4.3 we have the existence of a Fenichel solution, denoted $v_{\rm f}(x; r_0, L, b_0, \varphi_1, h_1, \mu, \varepsilon)$, smooth in all parameters, and satisfying

$$\begin{aligned} v_{\rm f}(r_0; r_0, L, b_0, \varphi_1, h_1, \mu, \varepsilon) &= (\Phi^c(r_0) + \mathcal{O}(\mathrm{e}^{-\eta L}), b_0, \mathcal{O}(\mathrm{e}^{-\eta L}), \Phi^h(r_0) + \mathcal{O}(\mathrm{e}^{-\eta L})), \\ v_{\rm f}(L; r_0, L, b_0, \varphi_1, h_1, \mu, \varepsilon) &= (\varphi_1, \mathcal{O}(\mathrm{e}^{-\eta L}), \delta, h_1). \end{aligned} \tag{6.3}$$

where $(\Phi^c, \Phi^h)(x) = (\Phi^c, \Phi^h)(x; r_0, L, \varphi_1, h_1, \mu, \varepsilon)$ for the ease of notation. Moreover, the error estimates in (6.3) can be differentiated and hold for all derivatives with respect to $(x, r_0, L, b_0, \varphi_1, h_1, \mu, \varepsilon)$.

We now state the first matching equation, corresponding to 1. above. This requires matching $v_{\rm b}(r_0) = v_{\rm f}(r_0)$, which becomes

$$\varphi_0 + r_0 + \mathcal{O}(r_0 a_0 + \varepsilon) = \Phi^c(r_0; r_0, L, b_0, \varphi_1, h_1, \mu, \varepsilon) + \mathcal{O}(\mathrm{e}^{-\eta L}) \tag{6.4a}$$
$$a_0 + \mathcal{O}(r_0 a_0 + \varepsilon) = b_0 + \mathcal{O}(\mathrm{e}^{-\eta L}) \tag{6.4b}$$
$$a_0 + \mathcal{O}(r_0 a_0 + \varepsilon) = \mathcal{O}(\mathrm{e}^{-\eta L}) \tag{6.4c}$$
$$h_0 + \mathcal{O}(r_0 a_0 + \varepsilon) = \Phi^h(r_0; r_0, L, b_0, \varphi_1, h_1, \mu, \varepsilon) + \mathcal{O}(\mathrm{e}^{-\eta L}) \tag{6.4d}$$

using the expansion for $v_{\rm b}(r_0)$ stated in (6.2). We begin by focussing on (6.4b)-(6.4d) and then return to (6.4a) after performing all other matching. Hence, let us define the smooth function $F_0 : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F_0(a_0, b_0, h_0) = \begin{pmatrix} a_0 - b_0 \\ a_0 \\ h_0 - \Phi^h(r_0; r_0, L, \varphi_1, h_1, \mu, \varepsilon) \end{pmatrix} + \mathcal{O}(r_0 a_0 + \mathrm{e}^{-\eta L} + \varepsilon),$$

so that roots of F_0 are exactly the solutions of (6.4b)–(6.4d) with the variables $(r_0, L, \varphi_0, \varphi_1, h_1, \mu, \varepsilon)$ treated as parameters in the system. Using the notation of Theorem 6.1 we take $w_0 = (0, 0, \Phi^h(r_0; r_0, L, b_0, \varphi_1, h_1, \mu, \varepsilon))$ and A_0 to be the invertible matrix

$$A_0 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We then have

$$F_0(w_0) = \mathcal{O}(\mathrm{e}^{-\eta L} + \varepsilon) \implies \|A_0^{-1} F_0(w_0)\| = \mathcal{O}(\mathrm{e}^{-\eta L} + \varepsilon)$$

since $a_0 = 0$, and

$$\|1 - A_0^{-1} DF_0(w)\| = \mathcal{O}(r_0 + \mathrm{e}^{-\eta L} + \varepsilon),$$

since $\Phi^h(r_0; r_0, L, \varphi_1, h_1, \mu, \varepsilon)$ is independent of (a_0, b_0, h_0). Hence, taking $\kappa = \frac{1}{2}$ and $\rho = 1$ allows for the application of Theorem 6.1 for fixed $r_0 > 0$ sufficiently small, which gives the existence a unique function $(a_0^*, b_0^*, h_0^*)(L, \varphi_0, \varphi_1, h_1, \mu, \varepsilon)$ defined for all sufficiently small $\varepsilon > 0$, large $L > 0$, $\varphi_0 \in \{0, \pi\}$ and arbitrary (φ_1, h_1, μ), smooth in the arguments $(L, \varphi_1, h_1, \mu, \varepsilon)$, and given by

$$(a_0^*, b_0^*, h_0^*)(L, \varphi_0, \varphi_1, h_1, \mu, \varepsilon) = (0, 0, \Phi^h(r_0; r_0, L, \varphi_1, h_1, \mu, \varepsilon)) + \mathcal{O}(\mathrm{e}^{-\eta L} + \varepsilon).$$

Moreover, recalling from Lemma 4.2 that

$$\Phi^h_{\varphi_1}(r_0;r_0,L,\varphi_1,h_1,\mu,\varepsilon),\Phi^h_{h_1}(r_0;r_0,L,\varphi_1,h_1,\mu,\varepsilon),\Phi^h_{\mu}(r_0;r_0,L,\varphi_1,h_1,\mu,\varepsilon)=\mathcal{O}(\varepsilon\ln(L)),\tag{6.5}$$

we therefore find that

$$\begin{aligned}\partial_{\varphi_1}(a_0^*,b_0^*,h_0^*)(L,\varphi_0,\varphi_1,h_1,\mu,\varepsilon)&=\mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon+\varepsilon\ln(L)),\\ \partial_{h_1}(a_0^*,b_0^*,h_0^*)(L,\varphi_0,\varphi_1,h_1,\mu,\varepsilon)&=\mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon+\varepsilon\ln(L)),\\ \partial_{\mu}(a_0^*,b_0^*,h_0^*)(L,\varphi_0,\varphi_1,h_1,\mu,\varepsilon)&=\mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon+\varepsilon\ln(L)).\end{aligned}\tag{6.6}$$

We now move to the second matching condition, stated in 2. above, which requires obtaining $v_\mathrm{f}(L)\in W^s_L(0,\mu,\varepsilon)$. To use $v_\mathrm{f}(L;r_0,L,b_0,\varphi_1,h_1,\mu,\varepsilon)$ to match with the stable manifold $W^s_L(0,\mu,\varepsilon)$ we start by following [1] to parameterize Γ. For each 0-loop in Γ, we parametrize the loop by 2π-periodic functions $(\tilde\varphi(s),\tilde\mu(s))$ with $0\leq s\leq 2\pi$ so that

$$\Gamma_\mathrm{lift}=\{(\tilde\varphi(s)+2\pi j,\tilde\mu(s))\ :\ 0\leq s\leq 2\pi, j\in\mathbb{N}\}.$$

If Γ is a 1-loop, we can parametrize Γ_lift by a curve $(\tilde\varphi(s),\tilde\mu(s))$ with $s\geq 0$, where $(\tilde\varphi(s+2\pi),\tilde\mu(s+2\pi))=(\tilde\varphi(s)+2\pi,\tilde\mu(s))$. Hence, from Lemma 5.2 this matching condition is equivalent to solving

$$\begin{aligned}\varphi_1&=\tilde\varphi(s)+z^\Gamma(L,\tilde\varphi(s),v^s_\mathrm{f}(L;r_0,L,b_0,\varphi_1,h_1,\mu,\varepsilon),\tilde\mu(s),\varepsilon)\frac{\partial G_0}{\partial\varphi}(\tilde\varphi(s),0,\tilde\mu(s)),\\ \mu&=\tilde\mu(s)+z^\Gamma(L,\tilde\varphi(s),v^s_\mathrm{f}(L;r_0,L,b_0,\varphi_1,h_1,\mu,\varepsilon),\tilde\mu(s),\varepsilon)\frac{\partial G_0}{\partial\mu}(\tilde\varphi(s),0,\tilde\mu(s)),\\ h_1&=z^h(L,\tilde\varphi(s),v^s_\mathrm{f}(L;r_0,L,b_0,\varphi_1,h_1,\mu,\varepsilon),\tilde\mu(s),\varepsilon),\end{aligned}\tag{6.7}$$

since $v^c(L;r_0,L,\varphi_1,h_1,\mu,\varepsilon)=\varphi_1$ and $v^h(L;r_0,L,\varphi_1,h_1,\mu,\varepsilon)=h_1$.

We now use the function $b_0^*(L,\varphi_0,\varphi_1,h_1,\mu,\varepsilon)$, determined above, to solve (6.7). From Proposition 4.3 we have the estimate

$$|v^s_\mathrm{f}(L;r_0,L,b_0,\varphi_1,h_1,\mu,\varepsilon)|\leq M\mathrm{e}^{-\eta L}\tag{6.8}$$

for some $M>0$, and furthermore, the bound (6.8) holds for all derivatives of $v^s_\mathrm{f}(L;r_0,L,b_0,\varphi_1,h_1,\mu,\varepsilon)$ with respect to $(x,r_0,L,b_0,\varphi_1,h_1,\mu,\varepsilon)$ evaluated at $x=L$. Then, evaluating v^s_f at $b_0=b_0^*(L,\varphi_0,\varphi_1,h_1,\mu,\varepsilon)$, we take partial derivatives and using (6.6) we find that (upon suppressing parameter dependence for notational convenience)

$$\partial_{\varphi_1}v^s_\mathrm{f}(L;r_0,L,b_0^*,\varphi_1,h_1,\mu,\varepsilon)=\partial_{\varphi_1}v^s_\mathrm{f}+\partial_{b_0}v^s_\mathrm{f}\partial_{\varphi_1}b_0^*=\mathcal{O}(\mathrm{e}^{-\eta L}+\mathrm{e}^{-2\eta L}+\varepsilon\mathrm{e}^{-\eta L}+\varepsilon\ln(L)\mathrm{e}^{-\eta L}),$$

and the same result further holds for both $\partial_{h_1}v^s_\mathrm{f}(L;r_0,L,\varphi_1,h_1,\mu,\varepsilon)$ and $\partial_{\mu}v^s_\mathrm{f}(L;r_0,L,\varphi_1,h_1,\mu,\varepsilon)$. Then, since $\mathrm{e}^{-\eta L}$ and $\ln(L)\mathrm{e}^{-\eta L}$ are bounded uniformly and $\mathrm{e}^{-\eta L}\geq\mathrm{e}^{-2\eta L}$ for all $L\geq 1$, we find that

$$\mathcal{O}(\mathrm{e}^{-\eta L}+\mathrm{e}^{-2\eta L}+\varepsilon\mathrm{e}^{-\eta L}+\varepsilon\ln(L)\mathrm{e}^{-\eta L})=\mathcal{O}(^{-\eta L}+\varepsilon),$$

simplifying the estimates on $\partial_{\varphi_1}v^s_\mathrm{f}(L;r_0,L,b_0^*,\varphi_1,h_1,\mu,\varepsilon)$, $\partial_{h_1}v^s_\mathrm{f}(L;r_0,L,b_0^*,\varphi_1,h_1,\mu,\varepsilon)$ and $\partial_{\mu}v^s_\mathrm{f}(L;r_0,L,b_0^*,\varphi_1,h_1,\mu,\varepsilon)$. Using the fact that z^Γ, z^h vanish when $v^s_\mathrm{f}=\varepsilon=0$ we find that (6.7) can be written

$$\begin{aligned}\varphi_1&=\tilde\varphi(s)+\mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon),\\ \mu&=\tilde\mu(s)+\mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon),\\ h_1&=\mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon).\end{aligned}\tag{6.9}$$

We introduce the function $F_1:\mathbb{R}^3\to\mathbb{R}^3$ given by

$$F_1(\varphi_1,\mu,h_1)=\begin{pmatrix}\varphi_1-\tilde\varphi(s)\\ \mu-\tilde\mu(s)\\ h_1\end{pmatrix}+\mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon),$$

so that roots of F_1 are exactly the solutions to (6.9) evaluated at $b_0^*(L,\varphi_0,\varphi_1,h_1,\mu,\varepsilon,s)$, where we consider $(L,\varphi_0,\varepsilon)$ to be parameters. Proceeding as before, we take $w_1 = (\tilde{\varphi}(s),\tilde{\mu}(s),0)$ and A_1 to be the invertible matrix

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This then gives

$$F_1(w_1) = \mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon) \implies \|A_1^{-1}F_1(w_1)\| = \mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon),$$

and using the above error bounds for the partial derivatives on $v_\mathrm{f}^s(L;r_0,L,b_0^*,\varphi_1,h_1,\mu,\varepsilon)$ we further have

$$\|1 - A_1^{-1}DF_1(w)\| = \mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon).$$

Taking again $\kappa = \frac{1}{2}$ and $\rho = 1$ allows for the application of Theorem 6.1 which gives the existence a unique function $(\varphi_1^*,\mu^*,h_1^*)(L,\varphi_0,\varepsilon,s)$ defined for all sufficiently small $\varepsilon > 0$, large $L > 0$, $\varphi_0 \in \{0,\pi\}$ and arbitrary s, smooth in the arguments (L,ε,s), and given by

$$(\varphi_1^*,\mu^*,h_1^*)(L,\varphi_0,\varepsilon,s) = (\tilde{\varphi}(s),\tilde{\mu}(s),0) + \mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon).$$

Evaluating (a_0^*,b_0^*,h_0^*) at $(\varphi_1^*,\mu^*,h_1^*)(L,\varphi_0,\varepsilon,s)$ allows one to write the unique solution to equations (6.4b)–(6.4d) and (6.7) as functions of only $(L,\varphi_0,\varepsilon,s)$ so that

$$\begin{aligned}
a_0^*(L,\varphi_0,\varepsilon,s) &= \mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon), \\
b_0^*(L,\varphi_0,\varepsilon,s) &= \mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon), \\
h_0^*(L,\varphi_0,\varepsilon,s) &= \Phi^h(r_0;r_0,L,\varphi_1(s),0,\mu(s),\varepsilon) + \mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon+\varepsilon^2\ln(L)), \\
\varphi_1^*(L,\varphi_0,\varepsilon,s) &= \tilde{\varphi}(s) + \mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon), \\
\mu^*(L,\varphi_0,\varepsilon,s) &= \tilde{\mu}(s) + \mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon), \\
h_1^*(L,\varphi_0,\varepsilon,s) &= \mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon),
\end{aligned} \tag{6.10}$$

upon evaluating $\Phi^h(r_0;r_0,L,\varphi_1,h_1,\mu,\varepsilon)$ at $(\varphi_1^*,\mu^*,h_1^*)(L,\varphi_0,\varepsilon,s)$ and expanding using (6.5).

We now turn to equation (6.4a), which is the only remaining matching equation from (6.4) and (6.7) to be solved. Evaluating (6.4a) at the previously obtained solution (6.10), the unique solution of (6.4b)–(6.4d) and (6.7), we are required to solve

$$\Phi^c(r_0;r_0,L,\varphi_1^*(L,\varphi_0,\varepsilon,s),h_1^*(L,\varphi_0,\varepsilon,s),\mu^*(L,\varphi_0,\varepsilon,s),\varepsilon) + r_0 - \varphi_0 + \mathcal{O}(r_0 a_0^*(L,\varphi_0,\varepsilon,s) + \mathrm{e}^{-\eta L}+\varepsilon) = 0. \tag{6.11}$$

We note that the $\mathcal{O}(r_0 a_0^*(L,\varphi_0,\varepsilon,s))$ term in (6.11) depends on $h_0^*(L,\varphi_0,\varepsilon,s)$, which belongs to K for all $(L,\varphi_0,\varepsilon,s)$. Since K is compact, we use the fact that $a_0^*(L,\varphi_0,\varepsilon,s) = \mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon)$ to find that

$$\mathcal{O}(r_0 a_0^*(L,\varphi_0,\varepsilon,s)) = \mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon),$$

for $r_0 > 0$ fixed. Using (6.10) we can further expand

$$\begin{aligned}
\Phi^c(r_0;&r_0,L,\varphi_1^*(L,\varphi_0,\varepsilon,s),h_1^*(L,\varphi_0,\varepsilon,s),\mu^*(L,\varphi_0,\varepsilon,s),\varepsilon) \\
&= \Phi^c(r_0;r_0,L,\tilde{\varphi}(s),0,\tilde{\mu}(s),\varepsilon) + \mathcal{O}(\varepsilon\ln(L)(\mathrm{e}^{-\eta L}+\varepsilon)) \\
&= \tilde{\varphi}(s) + r_0 - L + g^c(L,\tilde{\mu}(s),\varepsilon) + \mathcal{O}(\varepsilon+\varepsilon^2\ln(L))),
\end{aligned} \tag{6.12}$$

where we have applied (4.7), the form of $\varphi_1^*(L,\varphi_0,\varepsilon,s)$, and used the previously stated fact that $\mathcal{O}(\varepsilon\ln(L)\mathrm{e}^{-\eta L}) = \mathcal{O}(\varepsilon)$. Recall that $g^c(L,\mu,\varepsilon) = \mathcal{O}(\varepsilon\ln(L))$ for all $\mu \in J$.

Substitution of (6.12) into (6.11) gives

$$\tilde{\varphi}(s) - \varphi_0 - L + g^c(L,\tilde{\mu}(s),\varepsilon) + \mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon+\varepsilon^2\ln(L)) = 0. \tag{6.13}$$

This expression can be rearranged to read

$$\tilde{\varphi}(s) - \varphi_0 + L\Big(-1 + \frac{g^c(L, \tilde{\mu}(s), \varepsilon)}{L} + \mathcal{O}(\mathrm{e}^{-\eta L} + \frac{\varepsilon}{L} + \frac{\varepsilon^2 \ln(L)}{L})\Big) = 0,$$

which is unbounded in L and monotonically decreasing when L is large since $g^c(L, \mu, \varepsilon) = \mathcal{O}(\varepsilon \ln(L))$. Therefore, since $\tilde{\varphi}(s) \to \infty$ as $s \to \infty$, we may use the Intermediate Value Theorem to infer that for all s sufficiently large there exists a unique solution $L^*(\varphi_0, \varepsilon, s)$ solving (6.11) with the property that $L^*(\varphi_0, \varepsilon, s) \to \infty$ as $s \to \infty$. Moreover, we define $g_{\text{pulse}}(L, \mu, \varepsilon)$ to be so that (6.13) is written

$$\tilde{\varphi}(s) - \varphi_0 - L + g_{\text{pulse}}(L, \tilde{\mu}(s), \varepsilon) + \mathcal{O}(\mathrm{e}^{-\eta L} + \varepsilon),$$

so that $g_{\text{pulse}}(L, \mu, \varepsilon) = g^c(L, \mu, \varepsilon) + \mathcal{O}(\varepsilon^2 \ln(L))$. Note that $g_{\text{pulse}}(L, \mu, \varepsilon) = \mathcal{O}(\varepsilon \ln(L))$, as claimed, since its leading order term is the function g^c. Hence, the functions $(a_0^*, b_0^*, h_0^*, \varphi_1^*, \mu^*, h_1^*)(L, \varphi_0, \varepsilon, s)$ evaluated at $L = L^*(\varphi_0, \varepsilon, s)$ solve the matching equations (6.4b) and (6.9), and the claims of the theorem now follow. □

Remark 2. *Since we solved (6.11) using the Intermediate Value Theorem, it is not clear whether $\Gamma_{\text{pulse}}^{\varphi_0,\varepsilon}$ is a smooth manifold. We were not able to gain sufficient control over the flow on the integral manifold to prove that $\Gamma_{\text{pulse}}^{\varphi_0,\varepsilon}$ is indeed a smooth manifold and therefore state only that $\Gamma_{\text{pulse}}^{\varphi_0,\varepsilon}$ is a set for $\varepsilon > 0$. We note, however, that the bifurcation curves $\Gamma_{\text{pulse}}^{\varphi_0,\varepsilon}$ are indeed smooth and unique whenever $\varepsilon \ln(L)$ is sufficiently small, as we can then differentiate and bound the derivative of the left-hand side of (6.11) with respect to L.*

7 Dynamics on the integral manifold

Recall that

$$v_x^c = 1 + \frac{\varepsilon}{x} h_1^c(v^c, v^h, \mu, \varepsilon) + \frac{\varepsilon^2}{x^2} h_2^c(x, v^c, v^h, \mu, \varepsilon), \tag{7.1a}$$

$$v_x^h = \frac{\varepsilon}{x} h_1^e(v^c, v^h, \mu, \varepsilon) + \frac{\varepsilon^2}{x^2} h_2^e(x, v^c, v^h, \mu, \varepsilon), \tag{7.1b}$$

governs the dynamics on the two-dimensional invariant integral manifold that continues the cylinder, $\mathcal{C}(\mu)$, of wave trains that exist at $\varepsilon = 0$ to positive values of ε. We proved in Theorem 6.2 that radial pulses persist for a given sufficiently small value of $\varepsilon > 0$ based upon the results of Lemma 4.2 and Proposition 4.3, which rely on the fact that the cutoff functions guarantee that there is a solution $(v^c(x), v^h(x)) \in \mathbb{R} \times K$ for all $x \in [0, L]$. Here we will extend the results of Lemma 4.2 in an effort to better describe bifurcating radial pulse solutions to (2.4) using the following averaging result, whose proof is left to §7.2.

Theorem 7.1. *For arbitrary $r_0 > 0$, the transformation*

$$v^c = x + w^c + \varepsilon W^c(x, w^c, w^h, \mu), \qquad v^h = w^h + \varepsilon W^h(x, w^c, w^h, \mu), \tag{7.2}$$

where W^c and W^h are defined in (7.14), transforms (7.1) to the form

$$\begin{aligned} w_x^c &= \frac{\varepsilon}{x}\Big[F_0^c(w^h, \mu) + \varepsilon F_1^c(x, w^c, w^h, \mu, \varepsilon)\Big], \\ w_x^h &= \frac{\varepsilon}{x}\Big[F_0^h(w^h, \mu) + \varepsilon F_1^h(x, w^c, w^h, \mu, \varepsilon)\Big]. \end{aligned} \tag{7.3}$$

The functions $F_1^{c,h}$ are smooth, $2\pi T(\mu, h)$-periodic with respect to w^c, and uniformly bounded in $x \geq r_0$, $w^h \in (-k, k)$, $\mu \in J$ and $\varepsilon \geq 0$ sufficiently small and satisfies $\varepsilon \leq r_0^2$.

Along with the proof of Theorem 7.1, we also show that one may obtain the exact form for the leading order functions F_0^c and F_0^h as they relate to the perturbed function g in (2.4). In particular, our results show that

$$F_0^h(h,\mu) = \frac{1}{2\pi T(\mu,h)} \int_0^{2\pi T(\mu,h)} \langle \nabla H(\gamma(x,\mu,h)), g(\gamma(x,\mu,h),\mu,0)\rangle \, \mathrm{d}x, \tag{7.4}$$

so that F_0^h can be thought of as the average of g evaluated on the invariant manifold $\mathcal{C}(\mu)$ of the unperturbed system (2.1). Hence, given a function g, F_0^h can be computed numerically using the periodic orbits of the unperturbed system (2.1). Moreover, in the following section we present a case study of the Swift-Hohenberg equation and show that in the case of the Swift-Hohenberg equation the function F_0^c identically vanishes, which therefore gives that the leading order dynamics of w^c are at $\mathcal{O}(\varepsilon^2)$.

The averaged system (7.3) of Theorem 7.1 allows one to better understand the dynamics of the invariant integral manifold. That is, in §7.1 we provide a series of results that connect the bifurcation curves obtained in Theorem 6.2 to the original differential equation (2.4). Our first result, Lemma 7.2 begins with system (7.1) to provide a lower bound on how long one can ensure trajectories on the invariant integral manifold can be guaranteed to stay in K_0, which from Lemma 4.2 guarantees that these solutions correspond to genuine trajectories associated to (2.4). Following this result we focus on system (7.3) and formulate hypotheses which can be used to extrapolate valuable insight into the nature of the existence and bifurcation structure of radial pulse solutions to (2.4). One such result provides a 'best-case scenario' for the function F_0^h which guarantees that appropriate trajectories on the invariant integral manifold remain in K_0 for arbitrarily long amounts of time, whereas the second major result is based upon our numerical investigation of the Swift-Hohenberg equation using the identity (7.4). In Section 8 we make clear the connection between the results in §7.1 and the motivating Swift-Hohenberg equation.

7.1 Bifurcation diagrams

Prior to working with (7.3), we provide the most general result which gives the lower bound for the height of the bifurcation curves stated in Theorem 2.1.

Lemma 7.2. *For each fixed $r_0 \in (0,1]$ and each closed nonempty subset $K_1 \subset \mathring{K}_0$, there exist constants $\varepsilon_5 > 0$ and $C_2 > 0$ such that, for each $(\varphi_1, h_1, \varepsilon) \in \mathbb{R} \times K_1 \times [0, \varepsilon_5]$ and $0 < L \leq \mathrm{e}^{C_2/\varepsilon}$, there exists a unique solution $(v^c(x), v^h(x))$ of (4.5)–(4.6) so that $(v^c, v^h) \in \mathbb{R} \times K_0$ for all $x \in [r_0, L]$.*

Proof. Based upon the results of Lemma 4.2, the global solution to (4.5) provides a genuine solution of the original system whenever $v^h(x; r_0, L, \varphi_1, h_1, \mu, \varepsilon) \in K_0$ for $x \in [r_0, L]$. Hence, it suffices to find conditions that guarantee that $v^h(x) \in K_0$ for all $x \in [r_0, L]$ whenever $v^h(L) = h_1 \in K_1$. To simplify notation, we assume that $K_j = [-k_j, k_j]$ with $0 < k_1 < k_0$. If $(v^c(x), v^h(x))$ is a solution to (4.5) with $|h_1| \leq k_1$, then v^h satisfies

$$v^h(x) = h_1 + \varepsilon \int_L^x \frac{h_1^e(v^c(s), v^h(s), \mu, \varepsilon)}{s} \, \mathrm{d}s + \varepsilon^2 \int_L^x \frac{h_2^e(s, v^c(s), v^h(s), \mu, \varepsilon)}{s^2} \, \mathrm{d}s.$$

As in the proof of Lemma 4.2, we bounding $h_{1,2}^e$ by a uniform constant $C_3 > 0$ to obtain

$$|v^h(x)| \leq k_1 + \varepsilon C_3 (\ln(L) + |\ln(r_0)|)$$

uniformly on $r_0 \leq x \leq L$. Hence, setting $\varepsilon_3 \leq \frac{k_0 - k_1}{2C_3 |\ln(r_0)|}$, $C_2 := \frac{k_0 - k_1}{2C_3}$ and restricting $L \leq \mathrm{e}^{C_2/\varepsilon}$ guarantees that $|v^h(x)| \leq k_0$ for $r_0 \leq x \leq L$, proving the lemma. □

Of course, one can see from Lemma 7.2 that the major hindrance on the height of the bifurcation diagrams is attributed to the possibility that trajectories on the integral manifold can leave in finite time. In Section 4 this was overcome by introducing the cutoff functions which guarantee that trajectories are confined to the integral

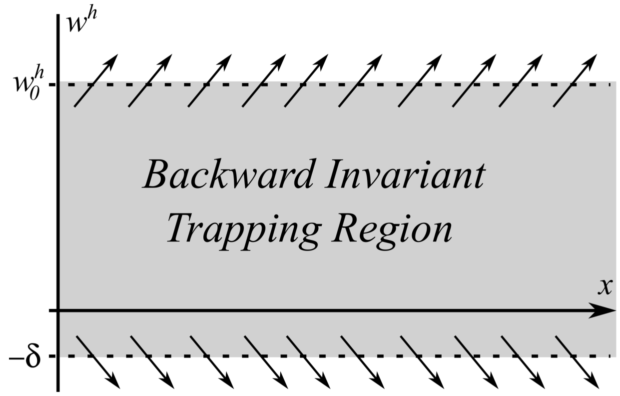

Figure 7: *The hypotheses of Proposition 7.3 imply the existence of a backward invariant trapping region to the differential equation governing w^h in (7.3). Here we illustrate the second of the two hypotheses.*

manifold for all time, but in the most general case it is only the results of Lemma 7.2 which can be guaranteed to provide the bifurcation curves for radial pulse solutions to (2.4). The following result uses the averaged integral manifold equations (7.3) to infer conditions which guarantee that the bifurcation diagram is unbounded in the vertical direction. That is, we show that if the backwards flow of solutions to the averaged vector field

$$w_x^h = F_0(w^h, \mu), \qquad w^h \in K = [-k, k],$$

can be trapped inside K_0 for all $x \leq L$, then the results of Theorem 6.2 are valid for system (2.4). We now present the following proposition.

Proposition 7.3. *Suppose that for all $\mu \in J$ one of the following cases is true:*

(i) $F_0(0, \mu) > 0$ and there exists negative $w_0^h \in \mathring{K}_0$ such that $F_0(w_0^h, \mu) < 0$.
(ii) $F_0(0, \mu) < 0$ and there exists positive $w_0^h \in \mathring{K}_0$ such that $F_0(w_0^h, \mu) > 0$.

Then the results of Theorem 6.2 are valid for system (2.4).

Proof. In Lemma 7.2 we saw that our limitation on L comes from guaranteeing that $v^h(x)$ remain in K_0 for all x. Using the averaging results of Theorem 7.1 we see that for $\varepsilon > 0$ sufficiently small our hypotheses guarantee that the system

$$w_x^h = \frac{\varepsilon}{x}\left[F_0^h(w^h, \mu) + \varepsilon F_1^h(x, w^c, w^h, \mu, \varepsilon)\right],$$

has an invariant trapping region when flowing backwards from $x = L$. This follows simply from the hypotheses of the proposition and the fact that all functions are continuous with respect to all arguments and uniformly bounded in $x \geq r_0$. Furthermore, this trapping region is contained in the interior of K_0, and therefore using the coordinate transformation (7.2) we find that upon returning to the original variable $v^h(x)$ we can guarantee that $v^h(x) \in K_0$ for all $r_0 \leq x \leq L$, for arbitrary $0 < r_0 < L$ and $\varepsilon > 0$. The results now follow from Lemma 4.2 and Theorem 6.2. $\square$

In Section 8 we will show that in the motivating example of the Swift-Hohenberg equation the hypotheses of Proposition 7.3 are not met. Hence, motivated by our numerical computations for the Swift–Hohenberg equation (8.1), we describe a different mechanism that leads to persistence of localized patterns of arbitrary large width L. We make the following hypothesis.

Hypothesis 7. *There exists $\mu_0 \in \mathring{J}$ such that $F_0(0, \mu_0) = 0$, $\partial_{w^h} F_0(0, \mu_0) < 0$ and $\partial_\mu F_0(0, \mu_0) \neq 0$.*

Hypothesis 7 says that at $\mu = \mu_0$ the trivial equilibrium is a stable steady-state of the averaged energy vector field $w_x^h = F_0(v^h, \mu)$. Moreover, since $\mu_0 \in \mathring{J}$ we may apply the Implicit Function Theorem to $F(w^h, \mu) = 0$ to

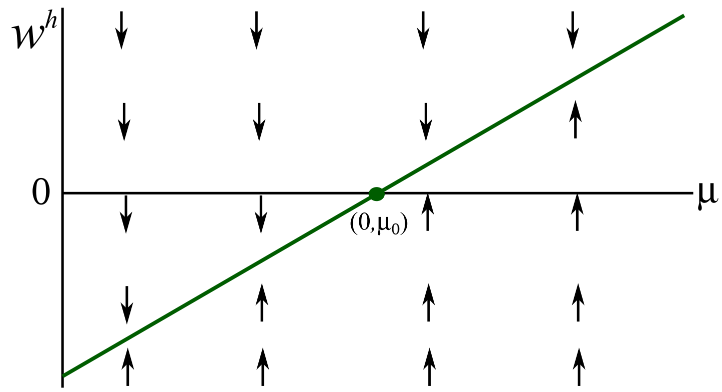

Figure 8: *Hypothesis 7 implies that in a neighbourhood of the point $(w^h, \mu) = (0, \mu)$ the averaged phase equation $w^h_x = F^h_0(w^h, \mu)$ has a curve of unique stable steady-states parametrized by μ.*

find that there exists $\mu^* > 0$ and a function $w^* : [\mu_0 - \mu^*, \mu_0 + \mu^*] \to K$ such that

$$F(w^*(\mu), \mu) = 0 \qquad \forall \mu \in [\mu_0 - \mu^*, \mu_0 + \mu^*],$$

with $w^*(\mu_0) = 0$ and $\frac{d}{d\mu} w^*(\mu_0) \neq 0$. We may restrict the value μ^* so that $w^*(\mu) \in \mathring{K}_0$ and

$$a(\mu) := \partial_{w^h} F_0(w^*(\mu), \mu) < 0$$

for all $\mu \in [\mu_0 - \mu^*, \mu_0 + \mu^*]$. Figure 8 provides a phase diagram of the dynamics of the averaged phase equation in a neighbourhood of $(w^h, \mu) = (0, \mu)$. We now present the following results.

Lemma 7.4. *Fix $r_0 \in (0, 1]$ and assume Hypothesis 7. Then, for all $\varepsilon > 0$ sufficiently small, there exists a smooth function $\Pi(x, w^c, \mu, \varepsilon)$ such that*

$$w^h = w^*(\mu) + \varepsilon \Pi(x, w^c, \mu, \varepsilon) \tag{7.5}$$

is an integral manifold of system (7.3) for all $x \geq r_0$, $w^c \in \mathbb{R}$, $\mu \in [\mu_0 - \mu^*, \mu_0 + \mu^*]$.

Proof. We introduce the change of variable $w^h(x) = w^*(\mu) + \tilde{w}^h(x)$, so that (7.3) becomes

$$\begin{aligned} w^c_x &= \frac{\varepsilon}{x}\Big[F^c_0(w^*(\mu) + \tilde{w}^h, \mu) + \varepsilon F^c_1(x, w^c, w^*(\mu) + \tilde{w}^h, \mu, \varepsilon)\Big], \\ \tilde{w}^h_x &= \frac{\varepsilon}{x}\Big[F^h_0(w^*(\mu) + \tilde{w}^h, \mu) + \varepsilon F^h_1(x, w^c, w^*(\mu) + \tilde{w}^h, \mu, \varepsilon)\Big]. \end{aligned} \tag{7.6}$$

Now, write

$$F_0(w^*(\mu) + \tilde{w}^h, \mu) = a(\mu)\tilde{w}^h + (F_0(w^*(\mu) + \tilde{w}^h, \mu) - a(\mu)\tilde{w}^h),$$

so that $(F_0(w^*(\mu) + \varepsilon\tilde{w}^h, \mu) - \varepsilon a(\mu)\tilde{w}^h) = \mathcal{O}(|w^h|^2)$ and (7.6) becomes

$$\begin{aligned} w^c_x &= \frac{\varepsilon}{x}\Big[F^c_0(w^*(\mu) + \tilde{w}^h, \mu) + \varepsilon F^c_1(x, w^c, w^*(\mu) + \tilde{w}^h, \mu, \varepsilon)\Big], \\ \tilde{w}^h_x &= \frac{\varepsilon a(\mu)}{x}\tilde{w}^h + \frac{\varepsilon}{x}\Big[F^h_0(w^*(\mu) + \varepsilon\tilde{w}^h, \mu) - a(\mu)\tilde{w}^h + \varepsilon F^h_1(x, w^c, w^*(\mu) + \tilde{w}^h, \mu, \varepsilon)\Big]. \end{aligned}$$

Then, for $\varepsilon > 0$ and $x \geq r_0$, we introduce the new independent variable $y = \varepsilon \ln(x)$ so that our differential equation is now cast as

$$\begin{aligned} w^c_y &= F^c_0(w^*(\mu) + \tilde{w}^h, \mu) + \varepsilon F^c_1(\mathrm{e}^{\frac{y}{\varepsilon}}, w^c, w^*(\mu) + \tilde{w}^h, \mu, \varepsilon), \\ \tilde{w}^h_y &= a(\mu)\tilde{w}^h + [F_0(w^*(\mu) + \tilde{w}^h, \mu) - a(\mu)\tilde{w}^h] + \varepsilon F_1(\mathrm{e}^{\frac{y}{\varepsilon}}, w^c, w^*(\mu) + \tilde{w}^h, \mu, \varepsilon)\Big]. \end{aligned} \tag{7.7}$$

Hence, we are now in the appropriate form to apply [7, Theorem 2.2] to see that there exists a smooth, uniformly bounded function $\tilde{\Pi}(y, w^c, \mu, \varepsilon)$ such that

$$\tilde{w}^h = \tilde{\Pi}(y, w^c, \mu, \varepsilon)$$

is an integral manifold for (7.7). Moreover, from [7, Theorem 2.2] we have that $\tilde{\Pi}(y, w^c, \mu, 0) = 0$ for all (y, w^c, μ). Therefore, tracing back all of our coordinate transformations we arrive at the integral manifold for (7.3) in the form (7.5) given by

$$\Pi(x, w^c, \mu, \varepsilon) := \varepsilon^{-1}\tilde{\Pi}(\varepsilon \ln(x), w^c, \mu, \varepsilon),$$

which is smooth and uniformly bounded. This completes the proof. □

Corollary 7.5. *Fix $r_0 \in (0, 1]$ and assume Hypothesis 7. There exists $\varepsilon_6 > 0$ such that for each $L > 1$, $\mu \in [\mu_0 - \mu^*, \mu_0 + \mu^*]$, $\varphi_1 \in \mathbb{R}$, and $\varepsilon \in [0, \varepsilon_6]$, then the solution $(v^c(x), v^h(x)) \in \mathbb{R} \times K$ satisfying*

$$v^c(L) = \varphi_1, \qquad v^h(L) = w^*(\mu) + \varepsilon\Pi(L, \varphi_1, \mu, \varepsilon) + \varepsilon W^h(L, \varphi_1, w^*(\mu) + \varepsilon\Pi(L, \varphi_1, \mu, \varepsilon)\mu).$$

where W^h is defined in (7.14), is such that $v^h(x) \in K_0$ for all $x \in [r_0, L]$.

Proof. From Lemma 7.5 we may obtain a solution to (7.3) by restricting $w^h = w^*(\mu) + \varepsilon\Pi(x, w^c, \mu, \varepsilon)$ and inspecting the resulting one-dimensional non-autonomous differential equation

$$w^c_x = \frac{\varepsilon}{x}\bigg[F_0^c(w^*(\mu) + \varepsilon\Pi(x, w^c, \mu, \varepsilon), \mu) + F_1^c(x, w^c, w^*(\mu) + \varepsilon\Pi(x, w^c, \mu, \varepsilon), \mu, \varepsilon)\bigg].$$

Smoothness of the differential equation, and uniform boundedness with respect to w^c imply the existence of a solution for all $x \in [r_0, L]$ with the boundary condition $w^c(L) = \varphi_1' \in \mathbb{R}$, for each $L > 1$. Furthermore, using the invertible change of variables introduced in Theorem 7.1, we can trace back these changes of variable to the original variables (v^c, v^h) so that

$$\begin{aligned} v^c(x) &= x + w^c(x) + \varepsilon W^c(x, w^c(x), w^*(\mu) + \varepsilon\Pi(x, w^c(x), \mu, \varepsilon), \mu), \\ v^h(x) &= w^*(\mu) + \varepsilon\Pi(x, w^c(x), \mu, \varepsilon) + \varepsilon W^s(x, w^c(x), w^*(\mu) + \varepsilon\Pi(x, w^c(x), \mu, \varepsilon), \mu). \end{aligned}$$

Then, at $x = L$ we have

$$v^c(L) = L + \varphi_1' + \varepsilon W^c(L, \varphi_1', w^*(\mu) + \varepsilon\Pi(L, \varphi_1', \mu, \varepsilon), \mu).$$

For all $L > 1$ and $\varepsilon \geq 0$ sufficiently small, the function

$$\varphi_1' \mapsto L + \varphi_1' + \varepsilon W^c(L, \varphi_1', w^*(\mu) + \varepsilon\Pi(L, \varphi_1', \mu, \varepsilon), \mu),$$

is surjective from $\mathbb{R}$ to $\mathbb{R}$, and therefore, for all $\varphi_1 \in \mathbb{R}$, there exists a choice of $\varphi_1' \in \mathbb{R}$ such that $v^c(L) = \varphi_1$ when $w^c(L) = \varphi_1'$. This confirms the boundary conditions at $x = L$ and concludes the proof. □

Lemma 7.6. *Assume Hypotheses 1-7 are met and we further have the following:*

(i) There exists $\varphi \in S^1$ such that $(\varphi, \mu_0) \in \Gamma$.

(ii) $\partial_\varphi G_0(\varphi, 0, \mu_0) \neq 0$ for some $(\varphi, \mu_0) \in \Gamma$ (i.e. the point (φ, μ_0) is not a saddle-node unfolded in φ).

Then, for $\varepsilon \geq 0$ sufficiently small and each $\varphi_0 \in \{0, \pi\}$, the following holds: there exists a sequence $\{(L_n^{\varphi_0}, \mu_n^{\varphi_0})\}_{n=1}^\infty$ with $L_n^{\varphi_0} \to \infty$ monotonically so that at $\mu = \mu_n^{\varphi_0}$ the differential equation (2.4) admits radial pulse solutions, $u_n(x; \varphi_0)$, with the property that $|u_n(x; \varphi_0)| \to \infty$ monotonically.

Proof. This proof proceeds in a similar way to that of the proof of Theorem 6.2 in that we must match solutions at $x = r_0$ and $x = L$, and therefore we will primarily focus on what is different in our current situation. From our assumptions, we can locally parametrize a neighbourhood of $(\varphi, \mu_0) \in \Gamma$ in Γ as $(\tilde{\varphi}(s), \tilde{\mu}(s)) \in \Gamma$, for a small parameter s so that $\tilde{\mu}(0) = 0$ and $\tilde{\mu}'(0) \neq 0$. Hence, the function $s \mapsto \tilde{\mu}(s)$ is locally surjective onto an open, connected, neighbourhood of $\mu_0 \in \mathring{J}$.

The major difference from the proof of Theorem 6.2 is that we now assume that $(\Phi^c, \Phi^h)(x; r_0, L, \varphi_1, \mu, \varepsilon)$ is a solution on the integral manifold satisfying the conclusions of Corollary 7.5 satisfying the boundary conditions

$$\Phi^c(L; r_0, L, \varphi_1, \mu, \varepsilon) = \varphi_1, \qquad \Phi^h(L; r_0, L, \varphi_1, \mu, \varepsilon) = w^*(\mu) + \mathcal{O}(\varepsilon),$$

for some $\varphi_1 \in \mathbb{R}$ and $\Phi^c(r_0; r_0, L, \varphi_1, \mu, \varepsilon)$ again satisfying (4.7). Thus, the matching condition (6.4) at $x = r_0$ remains unchanged, and can again be solved for fixed $r_0 > 0$ sufficiently small to obtain the existence a unique function $(a_0^*, b_0^*, h_0^*)(L, \varphi_0, \varphi_1, \mu, \varepsilon)$ defined for all sufficiently small $\varepsilon > 0$, large $L > 0$, $\varphi_0 \in \{0, \pi\}$ and arbitrary (φ_1, μ), smooth in the arguments $(L, \varphi_1, \mu, \varepsilon)$, and given by

$$(a_0^*, b_0^*, h_0^*)(L, \varphi_0, \varphi_1, \mu, \varepsilon, s) = (0, 0, \Phi^h(r_0; r_0, L, \varphi_1, \mu, \varepsilon)) + \mathcal{O}(\mathrm{e}^{-\eta L} + \varepsilon),$$

and again satisfying (6.6).

Now, (6.9) becomes

$$\begin{aligned}
\varphi_1 &= \tilde{\varphi}(s) + 2\pi n \\
&\quad + z^\Gamma(L, \tilde{\varphi}(s), v_{\mathrm{f}}^s(L; r_0, L, b_0^*(L, \varphi_0, \varphi_1, \mu, \varepsilon), \varphi_1, \mu, \varepsilon), \tilde{\mu}(s), \varepsilon) \frac{\partial G_0}{\partial \varphi}(\tilde{\varphi}(s), 0, \tilde{\mu}(s)), \\
\mu &= \tilde{\mu}(s) + z^\Gamma(L, \tilde{\varphi}(s), b_0^*(L, \varphi_0, \varphi_1, \mu, \varepsilon), \varphi_1, \mu, \varepsilon), \tilde{\mu}(s), \varepsilon) \frac{\partial G_0}{\partial \mu}(\tilde{\varphi}(s), 0, \tilde{\mu}(s)), \\
w^*(\mu) + \mathcal{O}(\varepsilon) &= z^h(L, \tilde{\varphi}(s), b_0^*(L, \varphi_0, \varphi_1, \mu, \varepsilon), \varphi_1, \mu, \varepsilon), \tilde{\mu}(s), \varepsilon),
\end{aligned} \tag{7.8}$$

where $(\tilde{\varphi}(s) + 2\pi n, \tilde{\mu}(s)) \in \Gamma_{\mathrm{lift}}$ for every $n \in \mathbb{N}$. All error bounds remain identical to those in the proof of Theorem 6.2, and therefore we define the function $F_2 : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F_2(\varphi_1, \mu, s) = \begin{pmatrix} \varphi_1 - \tilde{\varphi}(s) - 2\pi n \\ \mu - \tilde{\mu}(s) \\ w^*(\mu) \end{pmatrix} + \mathcal{O}(\mathrm{e}^{-\eta L} + \varepsilon),$$

so that roots of F_2 are exactly the solutions to (7.8) evaluated at $b_0^*(L, \varphi_0, \varphi_1, \mu, \varepsilon)$, where we consider $(L, \varphi_0, \varepsilon, n)$ to be parameters. Proceeding as before, we take $w_2 = (\tilde{\varphi}(0) + 2\pi n, \mu_0, 0)$ and A_2 to be the matrix

$$A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\tilde{\mu}'(0) \\ 0 & 0 & (w^*)'(\mu_0) \end{bmatrix},$$

which is invertible since $(w^*)'(\mu_0) \neq 0$. Since $\tilde{\mu}(0) = 0$ and $w^*(\mu_0) = 0$ this then gives

$$F_2(w_2) = \mathcal{O}(\mathrm{e}^{-\eta L} + \varepsilon) \implies \|A_2^{-1} F_2(w_2)\| = \mathcal{O}(\mathrm{e}^{-\eta L} + \varepsilon),$$

and as before we further have

$$\|1 - A_2^{-1} DF_2(w)\| = \mathcal{O}(\mathrm{e}^{-\eta L} + \varepsilon + s + \mu), \tag{7.9}$$

where the $\mathcal{O}(s + \mu)$ term comes from the smooth properties of $(\tilde{\varphi}(s), \tilde{\mu(s)})$ and $w^*(\mu)$ with respect to s and μ, respectively. Hence, taking $\kappa = \frac{1}{2}$ and $\rho > 0$ sufficiently small to restrict the size of the $\mathcal{O}(s + \mu)$ in (7.9), we can apply Theorem 6.1 to obtain the existence a unique function $(\varphi_1^*, \mu^*, s^*)(L, \varphi_0, \varepsilon, n)$ defined for all sufficiently small $\varepsilon > 0$, large $L > 0$, $\varphi_0 \in \{0, \pi\}$ and arbitrary $n \in \mathbb{N}$, smooth in the arguments (L, ε), and given by

$$(\varphi_1^*, \mu^*, s^*)(L, \varphi_0, \varepsilon, n) = (\tilde{\varphi}(0) + 2\pi n, \mu_0, 0) + \mathcal{O}(\mathrm{e}^{-\eta L} + \varepsilon).$$

Now, we again follow as in the proof of Theorem 6.2 to solve (6.4a), which requires solving

$$\Phi^c(r_0; r_0, L, \varphi_1^*(L, \varphi_0, \varepsilon, n), \mu^*(L, \varphi_0, \varepsilon, n), \varepsilon) + r_0 - \varphi_0 + \mathcal{O}(r_0 a_0^*(L, \varphi_0, \varepsilon, s) + \mathrm{e}^{-\eta L} + \varepsilon) = 0.$$

Similar manipulations to that of Theorem 6.2 give

$$\tilde{\varphi}(0)+2\pi n-\varphi_0-L+g^c(L,\mu,\varepsilon)+\mathcal{O}(\mathrm{e}^{-\eta L}+\varepsilon)=0, \tag{7.10}$$

where we recall that $g^c(L,\mu,\varepsilon)=\mathcal{O}(\varepsilon\ln(L))$. Then, as in the proof of Theorem 2.1, the left-hand-side of (7.10) is unbounded in L and is monotonically decreasing for all L sufficiently large. Hence, for each n sufficiently large, we can apply the Intermediate Value Theorem to obtain a function $L(r_0,n,\varepsilon)$, such that $L(\varphi_0,\varepsilon,n)\to\infty$ as $n\to\infty$, which satisfies (7.10). We note that for each sufficiently small $\varepsilon>0$ we have obtained a discrete sequence, $L_n^{\varphi_0}=L(\varphi_0,\varepsilon,n)$, of radial pulse solutions to (2.4) at $\mu=\mu^*(L(\varphi_0,\varepsilon,n),\varphi_0,\varepsilon,n)$. Moreover, as $n\to\infty$ we also have $L_n^{\varphi_0}\to\infty$ monotonically, thus giving that $|u_n(x;\varphi_0)|\to\infty$ monotonically. This completes the proof. $\square$

7.2 Proof of Theorem 7.1

The proof of Theorem 7.1 will be broken down throughout this subsection in an effort to clarify the proof by presenting some necessary preliminary results. To begin, let $v^c(x)=x+\tilde{v}^c(x)$, so that we obtain

$$v_x^c=1+\tilde{v}_x^c,$$

and therefore (7.1) becomes

$$\begin{aligned}\tilde{v}_x^c&=\frac{\varepsilon}{x}h_1^c(x+\tilde{v}^c,v^h,\mu,\varepsilon)+\frac{\varepsilon^2}{x^2}h_2^c(x,x+\tilde{v}^c,v^h,\mu,\varepsilon),\\ v_x^h&=\frac{\varepsilon}{x}h_1^e(x+\tilde{v}^c,v^h,\mu,\varepsilon)+\frac{\varepsilon^2}{x^2}h_2^e(x,x+\tilde{v}^c,v^h,\mu,\varepsilon).\end{aligned} \tag{7.11}$$

Note that from Lemma 4.1, all functions are $2\pi T(\mu,v^h)$-periodic with respect to $\tilde{v}^c$ and now $h_1^{c,e}$ are also $2\pi T(\mu,v^h)$-periodic with respect the independent variable x. This leads to the first result.

Lemma 7.7. *The function F_0^h defined in (7.4) is such that*

$$F_0^h(v^h,\mu)=\frac{1}{2\pi T(\mu,v^h)}\int_0^{2\pi T(\mu,v^h)}h_1^e(x+\tilde{v}^c,v^h,\mu,0)\,\mathrm{d}x.$$

Proof. Let us define

$$I(\tilde{v}^c,v^h,\mu)=\frac{1}{2\pi T(\mu,v^h)}\int_0^{2\pi T(\mu,v^h)}h_1^e(x+\tilde{v}^c,v^h,\mu,0)\,\mathrm{d}x.$$

Following the coordinate transformations of Lemma 4.1, it is clear that

$$h_1^e(x+\tilde{v}^c,v^h,\mu,0)=\langle\nabla H(\gamma(x+\tilde{v}^c,\mu,v^h)),g(\gamma(x+\tilde{v}^c,\mu,h),\mu,0)\rangle,$$

which we recall from Hypothesis 4 is $2\pi T(\mu,v^h)$-periodic in x and $\tilde{v}^c$. Hence, $I(v^h,\mu)$ becomes

$$I(\tilde{v}^c,v^h,\mu)=\frac{1}{2\pi T(\mu,v^h)}\int_0^{2\pi T(\mu,v^h)}\langle\nabla H(\gamma(x+\tilde{v}^c,\mu,v^h)),g(\gamma(x+\tilde{v}^c,\mu,v^h),\mu,0)\rangle\,\mathrm{d}x.$$

Then, $\tilde{v}^c$ acts as a phase-advance and hence it is clear that $I(\tilde{v}^c,v^h,\mu)$ is independent of $\tilde{v}^c$. Therefore, it follows that $I(\tilde{v}^c,v^h,\mu)=F_0^h(v^h,\mu)$ for all (v^h,μ), thus completing the proof. $\square$

Along with F_0^h, we will also define

$$F_0^c(v^h,\mu)=\frac{1}{2\pi T(\mu,v^h)}\int_0^{2\pi T(\mu,v^h)}h_1^c(x+\tilde{v}_x^c,v^h,\mu,0)\,\mathrm{d}x. \tag{7.12}$$

Then, we consider the functions

$$\tilde{F}^{c,h}(x,\tilde{v}^c,v^h,\mu) := h_1^{c,e}(x+\tilde{v}^c,v^h,\mu,0) - F_0^{c,h}(v^h,\mu), \tag{7.13}$$

which are $2\pi T(\mu,v^h)$-periodic in x and average to zero. Following the proof of Theorem 7.1, we will return to F_0^c to provide an explicit form in terms of the original perturbed vector field (2.4) as was done for F_0^h in Lemma 7.7. We now state the following lemma which details the boundedness of the functions $W^{c,h}$ used to perform the averaging of the vector field (7.1).

Lemma 7.8. *There exists a $C>0$ such that for every $r_0>0$ the functions*

$$\begin{aligned} W^c(x,\tilde{v}^c,v^h,\mu) &:= \int_{r_0}^x \frac{\tilde{F}^c(s,\tilde{v}^c,v^h,\mu)}{s}\mathrm{d}s, \\ W^h(x,\tilde{v}^c,v^h,\mu) &:= \int_{r_0}^x \frac{\tilde{F}^h(s,\tilde{v}^c,v^h,\mu)}{s}\mathrm{d}s, \end{aligned} \tag{7.14}$$

and their derivatives with respect to $\tilde{v}^c$ and v^h are uniformly bounded by Cr_0^{-1} for all $x\geq r_0$, $\tilde{v}^c\in\mathbb{R}$, $v^h\in K$, and $\mu\in J$.

Proof. Integrating (7.14) by parts gives

$$W^{c,h}(x,\tilde{v}^c,v^h,\mu) = \frac{1}{x}\int_{r_0}^x \tilde{F}^{c,h}(s,\tilde{v}^c,v^h,\mu)\,\mathrm{d}s + \int_{r_0}^x \frac{1}{s^2}\int_{r_0}^s \tilde{F}^{c,h}(t,\tilde{v}^c,v^h,\mu)\,\mathrm{d}t\,\mathrm{d}s.$$

Recall that the functions $\tilde{F}^{c,h}$ defined in (7.13) have zero average for each fixed v^h. Hence, there exists a constant $C_W>0$ such that the functions

$$s\mapsto \int_{r_0}^s \tilde{F}^c(t,\tilde{v}^c,v^h,\mu)\,\mathrm{d}t, \qquad s\mapsto \int_{r_0}^s \tilde{F}^c(t,\tilde{v}^c,v^h,\mu)\,\mathrm{d}t,$$

are uniformly bounded by C_W for all $s\geq r_0$, $\tilde{v}^c\in\mathbb{R}, v^h\in K$ and $\mu\in J$. Thus, for all $x\geq r_0$ we have that

$$|W^{c,h}(x,\tilde{v}^c,v^h,\mu)| \leq \frac{C_W}{r_0} + C_W\int_{r_0}^x \frac{1}{s^2}\,\mathrm{d}s = \frac{C_W}{r_0} + \frac{C_W}{r_0} - \frac{C_W}{x} \leq \frac{2C_W}{r_0}.$$

The estimates for the derivatives of $W^{c,h}$ with respect to $\tilde{v}^c$ and v^h follow in exactly the same way since the functions are again periodic in x for each fixed $v^h\in K$. ☐

Prior to presenting the following result, it should be noted that using the function definitions (7.13), differentiating $W^{c,h}$ with respect to x gives

$$W_x^{c,h}(x,\tilde{v}^c,v^h,\mu) = \frac{\tilde{F}^{c,h}(x,\tilde{v}^c,v^h,\mu)}{x} = \frac{1}{x}(h_1^{c,e}(x+\tilde{v}^c,v^h,\mu,0) - F_0^{c,h}(v^h,\mu)). \tag{7.15}$$

We now present the proof of Theorem 7.1.

Proof of Theorem 7.1. We will use the notation

$$W(x,w^c,w^h,\mu) = [W^c(x,w^c,w^h,\mu), W^h(x,w^c,w^h,\mu)]^T$$

for simplicity. Then, using the invertible, near-identity change of variable

$$\begin{pmatrix}\tilde{v}^c\\ v^h\end{pmatrix} = \begin{pmatrix}w^c\\ w^h\end{pmatrix} w^c + \varepsilon W(x,w^c,w^h,\mu),$$

we have that

$$\begin{pmatrix}\tilde{v}_x^c\\ v_x^h\end{pmatrix} = (I+\varepsilon D_{(w^c,w^h)}W(x,w^c,w^h,\mu))\begin{pmatrix}w_x^c\\ w_x^h\end{pmatrix} + \begin{pmatrix}\varepsilon W_x^c(x,w^c,w^h,\mu)\\ \varepsilon W_x^h(x,w^c,w^h,\mu)\end{pmatrix}, \tag{7.16}$$

where I denotes the 2×2 identity matrix. and .

Then, using (7.11) and (7.15) we can rearrange (7.16) to arrive at

$$\begin{aligned}(I + \varepsilon D_{(w^c,w^h)} W(x, w^c, w^h, \mu)) \begin{pmatrix} w^c_x \\ w^h_x \end{pmatrix} &= \frac{\varepsilon}{x} \begin{pmatrix} h^c_1(x + w^c + \varepsilon W^c(x, w^c, w^h, \mu), w^h + \varepsilon W^h(x, w^c, w^h, \mu), \mu, \varepsilon) \\ h^e_1(x + w^c + \varepsilon W^c(x, w^c, w^h, \mu), w^h + \varepsilon W^h(x, w^c, w^h, \mu), \mu, \varepsilon) \end{pmatrix} \\ &+ \frac{\varepsilon^2}{x^2} \begin{pmatrix} h^c_2(x, x + w^c + \varepsilon W^c(x, w^c, w^h, \mu), w^h + \varepsilon W^h(x, w^c, w^h, \mu), \mu, \varepsilon) \\ h^e_2(x, x + w^c + \varepsilon W^c(x, w^c, w^h, \mu), w^h + \varepsilon W^h(x, w^c, w^h, \mu), \mu, \varepsilon) \end{pmatrix} \\ &- \frac{\varepsilon}{x} \begin{pmatrix} h^c_1(x + w^c, w^h, \mu, 0) \\ h^e_1(x + w^c, w^h, \mu, 0) \end{pmatrix} \\ &+ \frac{\varepsilon}{x} \begin{pmatrix} F^c_0(w^h, \mu) \\ F^h_0(w^h, \mu) \end{pmatrix}. \end{aligned} \tag{7.17}$$

From Lemma 7.8 we have that $D_{(w^c,w^h)} W(x, w^c, w^h, \mu)$ is uniformly bounded for $x \geq r_0$, and therefore for $\varepsilon \geq 0$ sufficiently small we have that $(I + \varepsilon D_{(w^c,w^h)} W(x, w^c, w^h, \mu))$ is invertible for all (x, w^c, w^h, μ). Hence, we may apply $(I + \varepsilon D_{(w^c,w^h)} W(x, w^c, w^h, \mu))^{-1}$ to both sides of (7.17) to obtain

$$\begin{pmatrix} w^c_x \\ w^h_x \end{pmatrix} = \frac{\varepsilon}{x} (I + \varepsilon D_{(w^c,w^h)} W(x, w^c, w^h, \mu))^{-1} \begin{pmatrix} F^c_0(w^h, \mu) + \tilde{F}^c_1(x, w^c, w^h, \mu, \varepsilon) \\ F^h_0(w^h, \mu) + \tilde{F}^h_1(x, w^c, w^h, \mu, \varepsilon) \end{pmatrix},$$

where we have introduced the function $\tilde{F}^{c,h}_1$ to compactly write the right-hand-side of (7.17), and note that $\tilde{F}^{c,h}_1(x, w^c, w^h, \mu, \varepsilon) = \mathcal{O}(\varepsilon)$.

Finally, remarking that $[(I + \varepsilon D_{(w^c,w^h)} W(x, w^c, w^h, \mu))^{-1} - I] = \mathcal{O}(\varepsilon)$, we have

$$\begin{aligned}\begin{pmatrix} w^c_x \\ w^h_x \end{pmatrix} = \frac{\varepsilon}{x} \begin{pmatrix} F^c_0(w^h, \mu) \\ F^h_0(w^h, \mu) \end{pmatrix} &+ \underbrace{\frac{\varepsilon}{x} [(I + \varepsilon D_{(w^c,w^h)} W(x, w^c, w^h, \mu))^{-1} - I] \begin{pmatrix} F^c_0(w^h, \mu) \\ F^h_0(w^h, \mu) \end{pmatrix}}_{\mathcal{O}(\varepsilon^2)} \\ &+ \underbrace{\frac{\varepsilon}{x} (I + \varepsilon D_{(w^c,w^h)} W(x, w^c, w^h, \mu))^{-1} \begin{pmatrix} \tilde{F}^c_1(x, w^c, w^h, \mu, \varepsilon) \\ \tilde{F}^h_1(x, w^c, w^h, \mu, \varepsilon) \end{pmatrix}}_{\mathcal{O}(\varepsilon^2)}. \end{aligned}$$

Hence, defining the functions

$$\begin{aligned}\begin{pmatrix} F^c_1(x, w^c, w^h, \mu, \varepsilon) \\ F^h_1(x, w^c, w^h, \mu, \varepsilon) \end{pmatrix} &= \varepsilon^{-1} [(I + \varepsilon D_{(w^c,w^h)} W(x, w^c, w^h, \mu))^{-1} - I] \begin{pmatrix} F^c_0(w^h, \mu) \\ F^h_0(w^h, \mu) \end{pmatrix} \\ &+ (I + \varepsilon D_{(w^c,w^h)} W(x, w^c, w^h, \mu))^{-1} \begin{pmatrix} \tilde{F}^c_1(x, w^c, w^h, \mu, \varepsilon) \\ \tilde{F}^h_1(x, w^c, w^h, \mu, \varepsilon) \end{pmatrix}, \end{aligned}$$

shows that our differential equation is now in the form of (7.3). Note $F^{c,h}_1$ are well-defined, smooth and uniformly bounded when $\varepsilon \leq r_0^2$. This completes the proof. □

Having now proved Theorem 7.1, we now provide an explicit form for the function F^c_0 in terms of the original perturbed vector field (2.4).. Recall that in (7.12) we have that

$$F^c_0(v^h, \mu) = \frac{1}{2\pi T(\mu, v^h)} \int_0^{2\pi T(\mu, v^h)} h^c_1(x + \tilde{v}^c_x, v^h, \mu, 0)\, \mathrm{d}x.$$

Furthermore, since the periodic orbits $\gamma(x, \mu, h)$ of the unperturbed system (2.1) depend smoothly on h, we find that $\gamma_h(x, \mu, h)$ (the partial derivative of $\gamma(x, \mu, h)$ with respect to h) is a solution to the variational equation

$u_x = f_u(\gamma(x,\mu,h))u$. Furthermore, $\gamma_h(0,\mu,h) \in \mathrm{Fix}(\mathcal{R})$, or equivalently, $\gamma_h(x,\mu,h) = \mathcal{R}\gamma_h(-x,\mu,h)$ for all (x,μ,h).

From the discussion following Hypothesis 4, standard Floquet Theory tells us that for each (x,μ,h) the set

$$\{\gamma_x(x,\mu,h), p^s(x,\mu,h), p^u(x,\mu,h), \gamma_h(x,\mu,h)\}$$

is a linearly-independent, spanning set of $\mathbb{R}^4$. Then, we define a smooth function $\psi(x,\mu,h)$ so that ψ is orthogonal to $\mathrm{span}\{p^s, p^u, \gamma_h\}$ with $|\psi(x,\mu,h)| = 1$ for each (x,μ,h). This leads to the following result detailing the precise expression for F_0^c in terms of the perturbed vector field (2.4).

Lemma 7.9. *The function F_0^c defined in (7.12) is such that*

$$F_0^c(v^h,\mu) = \frac{1}{2\pi T(\mu,v^h)} \int_0^{2\pi T(\mu,v^h)} \frac{\langle \psi(x,\mu,v^h), g(\gamma(x,\mu,v^h))\rangle}{\langle \psi(x,\mu,v^h), \gamma_x(x,\mu,v^h)\rangle}\,\mathrm{d}x.$$

Proof. In Lemma 4.1 we introduced the change of variable

$$u = Q(v) = \gamma(v^c,\mu,v^h) + v^s p^s(v^c,\mu,v^h) + v^u p^u(v^c,\mu,v^h) + h.o.t., \tag{7.18}$$

so that when $\varepsilon = 0$ we have $v^u = 0$ and $v^s = 0$ parametrize the strong stable and strong unstable fibres of $\gamma(v^c,\mu,v^h)$, respectively. Differentiating (7.18) results in

$$f(Q(v)) + \frac{\varepsilon}{x} g(Q(v),\mu,\varepsilon) = \gamma_x(v^c,\mu,v^h)v_x^c + \gamma_h(v^c,\mu,v^h)v_x^h + p^s(v^c,\mu,v^h) + p^u(v^c,\mu,v^h) + h.o.t., \tag{7.19}$$

where we have used the fact that u is a solution of (2.4). Then, to obtain the form of the v_x^c equation in (4.3) we take the inner product with ψ of both sides of (7.19) and divide by $\langle \psi(x,\mu,v^h), \gamma_x(x,\mu,v^h)\rangle$. It should be noted that $\langle \psi(x,\mu,v^h), \gamma_x(x,\mu,v^h)\rangle$ never vanishes due to the linear independence of the set $\{\gamma_x, p^s, p^u, \gamma_h\}$ and the fact that ψ is orthogonal to $\mathrm{span}\{p^s, p^u, \gamma_h\}$. This process is exactly how the form for the v_x^c equation is derived to obtain it as stated in (4.3). Particularly, following the coordinate transformations of Lemma 4.1 we have that

$$h_1^c(v^c,v^h,\mu,\varepsilon) = \frac{\langle \psi(v^c,\mu,v^h), g(Q(v^c,0,0,v^h))\rangle}{\langle \psi(v^c,\mu,v^h), \gamma_x(v^c,\mu,v^h)\rangle}.$$

Now, using the fact that $Q(v^c,0,0,v^h) = \gamma(v^c,\mu,v^h)$, from (7.12) we have that $F_0^c(v^h,\mu)$ is in the form stated in the lemma. ☐

In Section 8 we will use Lemma 7.9 to show that when considering the motivating case of the Swift–Hohenberg equation we find that F_0^c vanishes for all $(v^h,\mu) \in K \times J$.

8 Application to the Swift-Hohenberg Equation

We now apply our results to the Swift–Hohenberg equation (1.1). Recall that radial steady-states of the Swift–Hohenberg equation posed on $\mathbb{R}^n$ satisfy the system

$$0 = -\left(1 + \frac{n-1}{x}\partial_x + \partial_{xx}\right)^2 U - \mu U + \nu U^2 - U^3, \tag{8.1}$$

where we denote the radial variable by x. Setting $\varepsilon := n-1$, $u_1 = U$, $u_2 = \partial_x u_1$, $u_3 = (1 + \frac{\varepsilon}{x}\partial_x + \partial_{xx})u_1$ and $u_4 = \partial_x u_3$, equation (8.1) is equivalent to

$$\begin{aligned}
u_1' &= u_2, \\
u_2' &= u_3 - u_1 - \frac{\varepsilon}{x}u_2, \\
u_3' &= u_4, \\
u_4' &= -u_3 - \mu u_1 + \nu u_1^2 - u_1^3 - \frac{\varepsilon}{x}u_4.
\end{aligned} \tag{8.2}$$

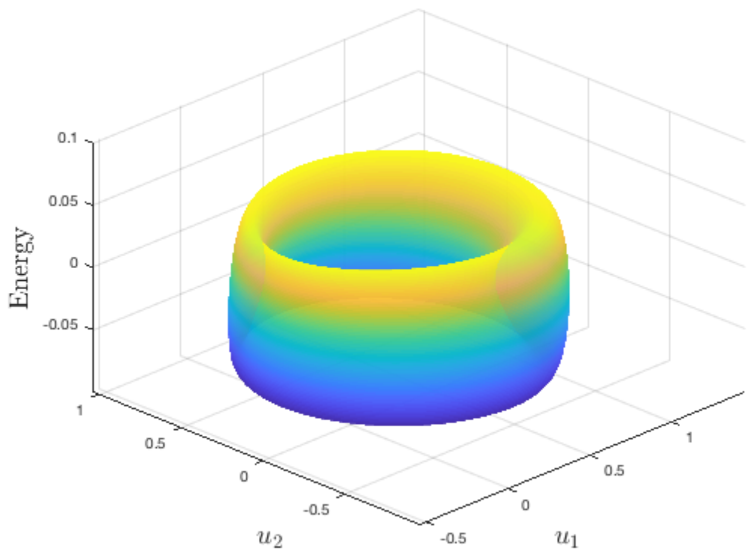

Figure 9: *Shown is the torus of periodic orbits of (8.2) with* $(\mu, \nu, \varepsilon) = (0.2, 1.6, 0)$. *The solutions on the inside of the torus are hyperbolic, while those on the outside are elliptic.*

Comparing with (2.4), we see that g is given by

$$g(u, \mu, \varepsilon) = -\begin{pmatrix} 0 \\ u_2 \\ 0 \\ u_4 \end{pmatrix}.$$

Furthermore, taking $\varepsilon = 0$ in (8.2), we find that the reverser $\mathcal{R}$ and the conserved quantity $\mathcal{H}$ are given by

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$\mathcal{H}(u, \mu) = u_2 u_4 + u_1 u_3 - \frac{u_3^2}{2} + \frac{\mu u_1^2}{2} - \frac{\nu u_1^3}{3} + \frac{u_1^4}{4}, \tag{8.3}$$

respectively. It is now straightforward to verify that Hypotheses 1–3 hold and that $g(u, \mu, \varepsilon)$ vanishes precisely when $u \in \mathrm{Fix}(\mathcal{R})$ as required in Hypothesis 6.

Figure 9 reflects the numerical evidence for the existence of a torus of periodic orbits to (8.2) when $\varepsilon = 0$. Numerically, the periodic orbits in the inside of the torus shown in Figure 9 are hyperbolic, thus indicating that Hypothesis 4 is indeed met. Furthermore, Figure 1 contains the numerical snaking diagram of localized rolls of (8.2) for $\varepsilon = 0$. As shown in [2], the structure of the branches visible in this figure is consistent with the assumption that the set Γ consists of a single 1-loop that satisfies Hypothesis 5.

With the caveat that Hypotheses 4–5 can be verified only numerically, Theorem 2.1 then implies that snaking persists for all values of L for which there is a solution on the perturbed cylinder that starts on the cylinder and reaches the zero-energy level set when $x = L$. This theorem also shows that such solutions exists at least for all L for which $\varepsilon \ln(L) \leq C$ for some constant C.

Next, we explore whether we can guarantee the existence of solutions for arbitrarily large values of L. As shown in §7, the vector field on the perturbed cylinder is of the form $v_x^h = \frac{\varepsilon}{x} F_0^h(v^h)$, where the averaged vector field

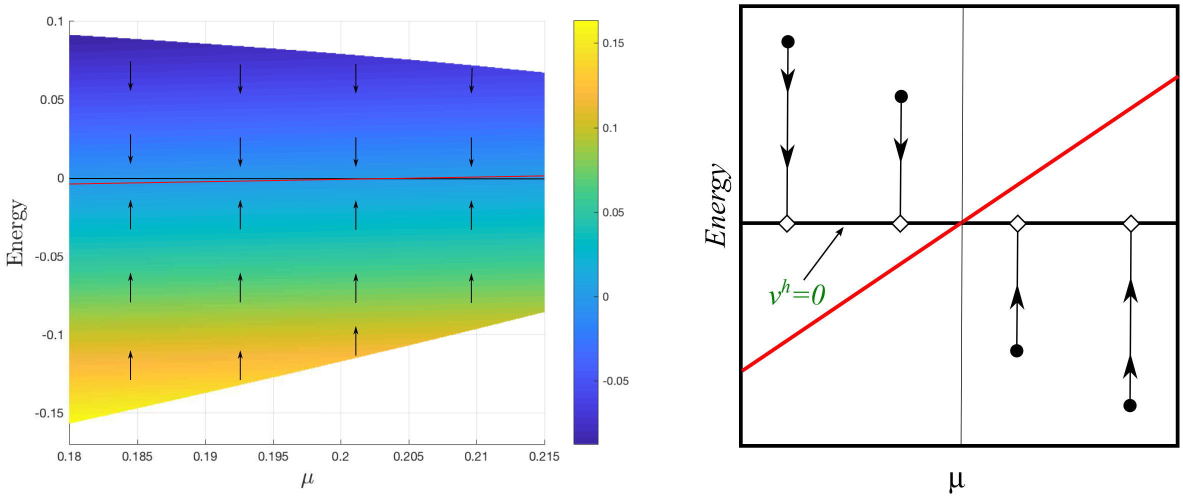

Figure 10: *Shown is the contour plot of the function* F_0^h *corresponding to the system (8.2) with* $\nu = 1.6$ *obtained using* AUTO09P *[left] and a schematic illustration of the result [right]. Note that the averaged vector field vanishes along a curve that intersects the* $\mathcal{H} = 0$ *axis transversely at approximately* $\mu = 0.204$.

F_0^h is given by the spatial average of the scalar product of the gradient of the energy with the perturbation g evaluated on the cylinder $\mathcal{C}(\mu)$ of the unperturbed system (8.2) with $\varepsilon = 0$. Using the energy functional (8.3), we find that

$$\nabla_u H(u, \mu) = \begin{pmatrix} u_3 + \mu u_1 - \nu u_1^2 + u_1^3 \\ u_4 \\ u_1 - u_3 \\ u_2 \end{pmatrix},$$

and Lemma 7.7 shows that the averaged vector field on the perturbed cylinder is given by

$$F_0^h(u, \mu) = \langle \nabla_u H(u, \mu), g(u, \mu, 0) \rangle = -u_2 u_4,$$

evaluated along the unperturbed cylinder $\mathcal{C}(\mu)$. We used AUTO09P to compute the averaged vector field F_0^h: the resulting contour plot is shown in Figure 10. As indicated there, the averaged equation has a unique equilibrium for each value of μ, and these equilibria are stable. Furthermore, the curve of equilibria intersects the $\mathcal{H} = 0$ axis precisely once, namely at $\mu = \mu_*$ with μ_* approximately equal to 0.204. This numerical result has two consequences. First, Lemma 7.6 therefore applies to guarantee the existence of a sequence of radial pulse solutions at $\mu = \mu_*$ whose associated values of L are not bounded; in particular, this confirms that the existence of localized radial spots with unbounded plateau lengths near $\mu = \mu_* \approx 0.204$ that is visible along the upper snaking branch in Figure 2. Second, as illustrated in the schematic picture in the left panel in Figure 11, the solutions $u(x)$ on the perturbed cylinder that reach the $\mathcal{H} = 0$ level set at $x = L$ necessarily have $\mathcal{H}(u(0), \mu) > 0$ for $\mu < \mu_*$ and $\mathcal{H}(u(0), \mu) < 0$ for $\mu > \mu_*$, provided L is sufficiently large. The right panel in Figure 11 confirms this prediction.

We conclude this section by proving that the averaged phase equation for the Swift–Hohenberg equation near $\varepsilon = 0$ vanishes identically.

Lemma 8.1. *For equation (8.2), we have* $F_0^c(v^h, \mu) = 0$ *for all* (v^h, μ).

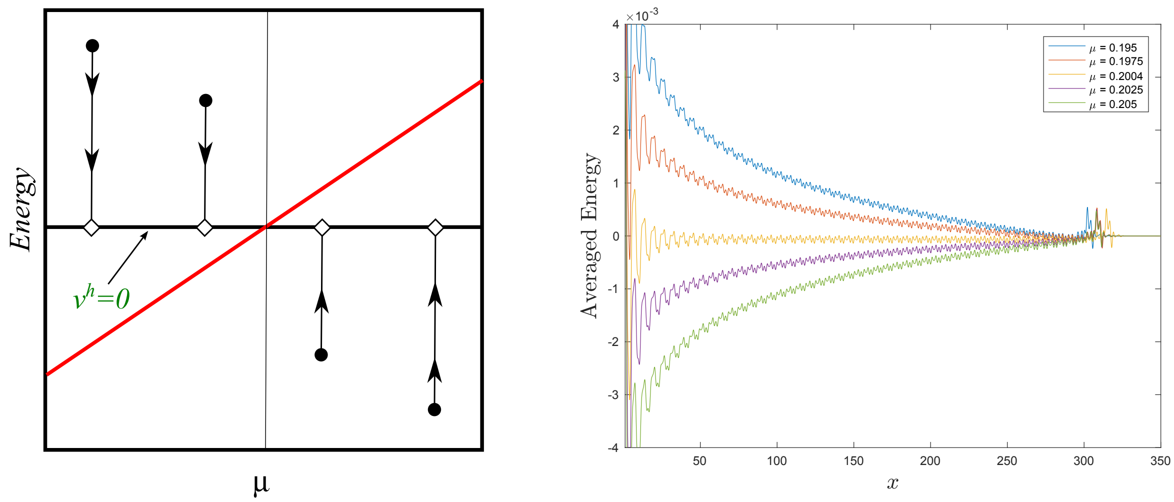

Figure 11: *The left panel contains a schematic picture of the averaged vector field on the perturbed cylinder and representative solutions $u(x)$ that satisfy $\mathcal{H}(u(L),\mu) = 0$ for large L: note that these trajectories necessarily satisfy $\mathcal{H}(u(0),\mu) > 0$ for $\mu < \mu_*$ and $\mathcal{H}(u(0),\mu) < 0$ for $\mu > \mu_*$. The right panel contains the graphs of $\mathcal{H}(u(x),\mu)$, averaged over one wave-train period, of radial pulses at five different values of μ, corroborating the theoretical predictions for the evolution of $\mathcal{H}(u(x),\mu)$ depending on whether μ is smaller or larger than $\mu_* \approx 0.204$.*

Proof. We proved in Lemma 7.9 that $F_0^c(v^h,\mu)$ is given by

$$\begin{aligned} F_0^c(v^h,\mu) &= \frac{1}{2\pi T(\mu,v^h)} \int_0^{2\pi T(\mu,v^h)} \frac{\langle \psi(x,\mu,v^h), g(\gamma(x,\mu,v^h))\rangle}{\langle \psi(x,\mu,v^h), \gamma_x(x,\mu,v^h)\rangle}\,\mathrm{d}s \\ &= \frac{1}{2\pi T(\mu,v^h)} \int_{-\pi T(\mu,v^h)}^{\pi T(\mu,v^h)} \frac{\langle \psi(x,\mu,v^h), g(\gamma(x,\mu,v^h))\rangle}{\langle \psi(x,\mu,v^h), \gamma_x(x,\mu,v^h)\rangle}\,\mathrm{d}s. \end{aligned}$$

Furthermore, using (8.2), we have

$$g(\gamma(x,\mu,h),\mu,\varepsilon) = -\begin{pmatrix} 0 \\ \gamma_2(x,\mu,h) \\ 0 \\ \gamma_4(x,\mu,h) \end{pmatrix},$$

and we conclude that the integrand in the equation above is given by

$$\frac{\langle \psi(x,\mu,h), g(\gamma(x,\mu,h),\mu,\varepsilon)\rangle}{\langle \psi(x,\mu,h), \gamma_x(x,\mu,h)\rangle} = \frac{\psi_2(x,\mu,h)\gamma_2(x,\mu,h) + \psi_4(x,\mu,h)\gamma_4(x,\mu,h)}{\langle \psi(x,\mu,h), \gamma_x(x,\mu,h)\rangle}. \tag{8.4}$$

It therefore suffices to prove that (8.4) is odd in x. Since $\gamma(x,\mu,h) = \mathcal{R}\gamma(-x,\mu,h)$, we see that both γ_2 and γ_4 are odd functions of x and that $\gamma_x(x,\mu,h) = -\mathcal{R}\gamma_x(-x,\mu,h)$ for all x. We claim that $\psi(x,\mu,h) = -\mathcal{R}\psi(-x,\mu,h)$ holds for all values of (x,μ,h). If true, this would imply that $\langle \psi(x,\mu,h), \gamma_x(x,\mu,h)\rangle$, ψ_2, and ψ_4 are all even in x, which implies that the expression in (8.4) is odd in x as claimed.

It therefore remains to prove that $\psi(x,\mu,h) = -\mathcal{R}\psi(-x,\mu,h)$ for all values of (x,μ,h). We recall that $p^u(x,\mu,h) = \mathcal{R}p^s(-x,\mu,h)$. Hence, using the definition of ψ, we see that, for all fixed constants $(c_1,c_2,c_3) \in \mathbb{R}^3$, we have

$$\begin{aligned} 0 &= \langle \psi(x), c_1 p^s(x) + c_2 p^u(x) + c_3\gamma_h(x)\rangle \\ &= \langle \psi(x), c_1\mathcal{R}p^u(-x) + c_2\mathcal{R}p^s(-x) + c_3\mathcal{R}\gamma_h(-x)\rangle \\ &= \langle \mathcal{R}\psi(x), c_1 p^u(-x) + c_2 p^s(-x) + c_3\gamma_h(-x)\rangle, \end{aligned}$$

where we have suppressed the dependence of all functions on (μ, h) for notational convenience. Since (c_1, c_2, c_3) were arbitrary and $\psi(x)$ spans the orthogonal compliment of $\text{span}\{p^s(x), p^u(x), \gamma_h(x)\}$ for all (x, μ, h), it follows that

$$\mathcal{R}\psi(x) = \pm\psi(-x) \implies \psi(x) = \pm\mathcal{R}\psi(-x),$$

for every x, since ψ is smooth and is such that $|\psi(x, \mu, h)| = 1$ for all (x, μ, h). Now, recall that $\gamma_h(0) \in \text{Fix}(\mathcal{R})$, and, furthermore, we have

$$p^s(0) + p^u(0) = p^s(0) + \mathcal{R}p^s(0) \in \text{Fix}(\mathcal{R}).$$

Linear independence of the set $\{p^s(x), p^u(x), \gamma_h(x)\}$ for all x coupled with the fact that $\text{Fix}(\mathcal{R})$ is two-dimensional then implies that

$$\text{Fix}(\mathcal{R}) = \text{span}\{p^s(0) + p^u(0), \gamma_h(0)\}.$$

Then, by definition of $\psi(x)$, for all $(c_1, c_2) \in \mathbb{R}^2$ we have

$$\langle \psi(0), c_1(p^s(0) + p^u(0)) + c_2\gamma_h(0)\rangle = 0,$$

and hence we must have that $\psi(0) \in \text{Fix}(-\mathcal{R})$. Therefore, $\psi(x) = -\mathcal{R}\psi(-x)$ and the lemma is proved. $\square$

9 Discussion

In this paper, we analyzed the existence and bifurcation structure of localized roll solutions to a class of singularly perturbed ODEs. The class of ODEs was motivated by the system (1.3), whose solutions correspond to stationary radial spots of the Swift–Hohenberg equation (1.2) posed on $\mathbb{R}^n$. The two key differences between $n = 1$ and $n > 1$ are that (1.3) no longer possesses a conserved quantity when $n > 1$ and that the term $1/x$ that becomes effective for $n > 1$ is not integrable on $[1, \infty)$. Through a perturbation analysis near $n = 1$, we were able to attribute the breakup of the snaking structure in dimensions $n > 1$ to these differences: the lack of a conservative quantity allows trajectories to track the entire invariant cylinder of periodic orbits, and the non-integrability of the perturbation allows solutions on the invariant cylinder to leave the cylinder through its top or bottom in finite time. Our analysis explains why the lower branch shown in Figure 2 for radial spots of the planar Swift–Hohenberg equation is bounded.

In addition, our description of the flow on the invariant perturbed cylinder of periodic orbits via averaging provided an explanation for the vertical asymptote of the upper snaking branch near $\mu = 0.204$. Specifically, we proved that, if the averaged vector field on the cylinder admits solutions that stay on the cylinder for all times and reach the periodic orbit that has zero energy, then the full system has radial pulses for each sufficiently large roll plateau length. We showed numerically that the averaged equation belonging to the Swift–Hohenberg equation has an equilibrium in the zero energy level set when μ is near 0.204.

Our analysis does not explain the precise structure of the branches shown in Figure 2 for the planar Swift–Hohenberg equation. In particular, we can explain neither the existence of an intermediate stack of isolas nor the fact the upper snaking branch forms a connected curve. We believe that the specific shape of the snaking diagram is determined by the behavior of solutions away from the invariant cylinder of periodic orbits and will therefore likely depend on the global dynamics rather than local properties near the cylinder of rolls and the stable manifold of the origin.

Finally, we observed numerically that the vertical asymptote $\mu = \mu_*$ of the upper snaking branch for the Swift–Hohenberg equation with $n > 1$ agrees, to numerical accuracy, with the Maxwell point. As pointed out in the last section, the conditions that give, respectively, the vertical asymptote and the Maxwell point differ, and it is not clear to us why their difference vanishes for all values of ν.

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