

Vortices and the Navier-Stokes equation:
understanding solutions of equations that we can't
actually solve

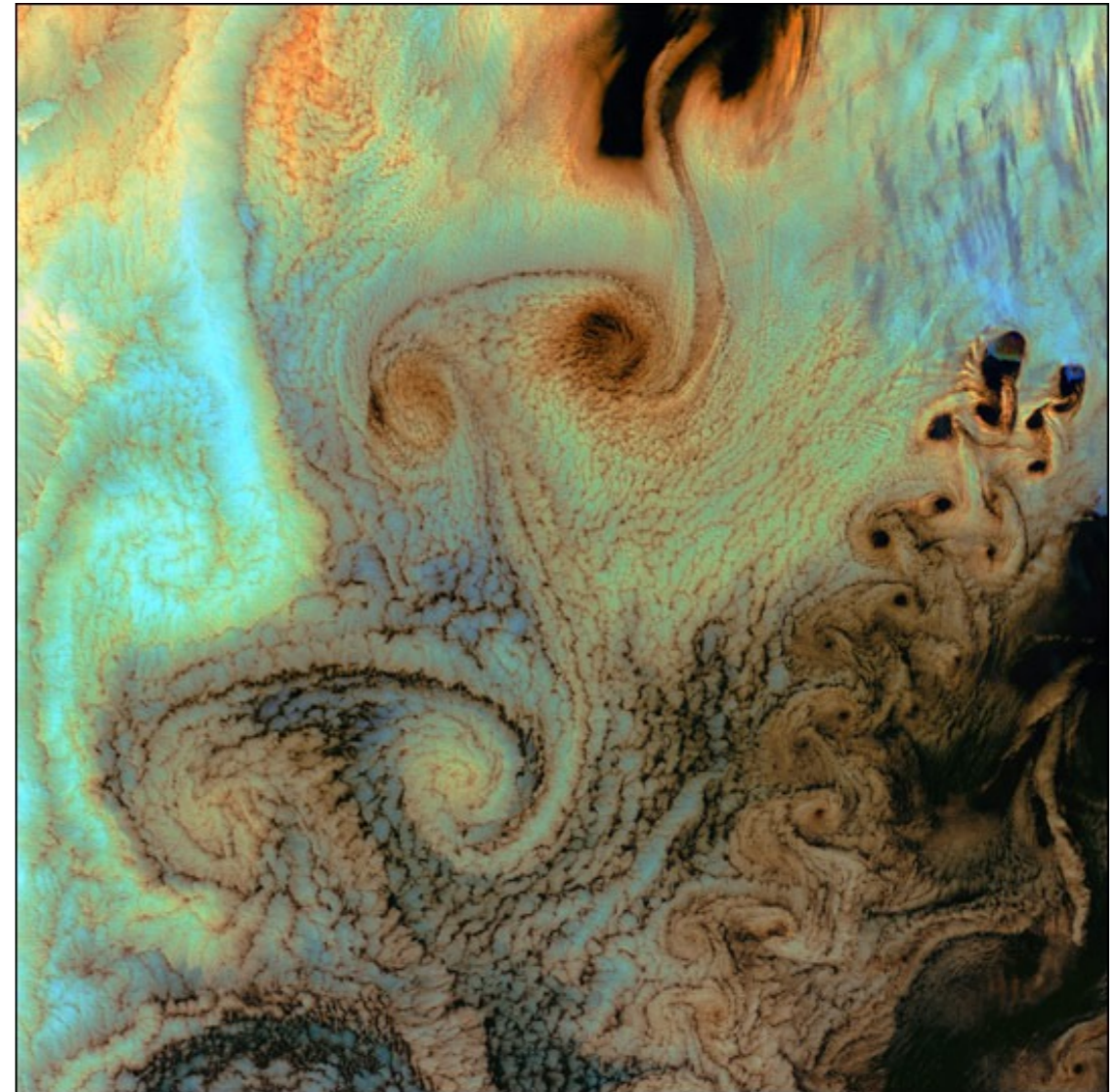
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Boston University

Haus der Wissenschaft
Bremen, 27 March 2015

What are vortices?



Homemade vortex in colored fluid,
found online at
[http://www.flickr.com/photos/
bagrat/collections/72157626374676307/](http://www.flickr.com/photos/bagrat/collections/72157626374676307/)



Atmospheric vortices, visualized via
clouds above Alaska, acquired by
Landsat 7 (NASA, USGS)

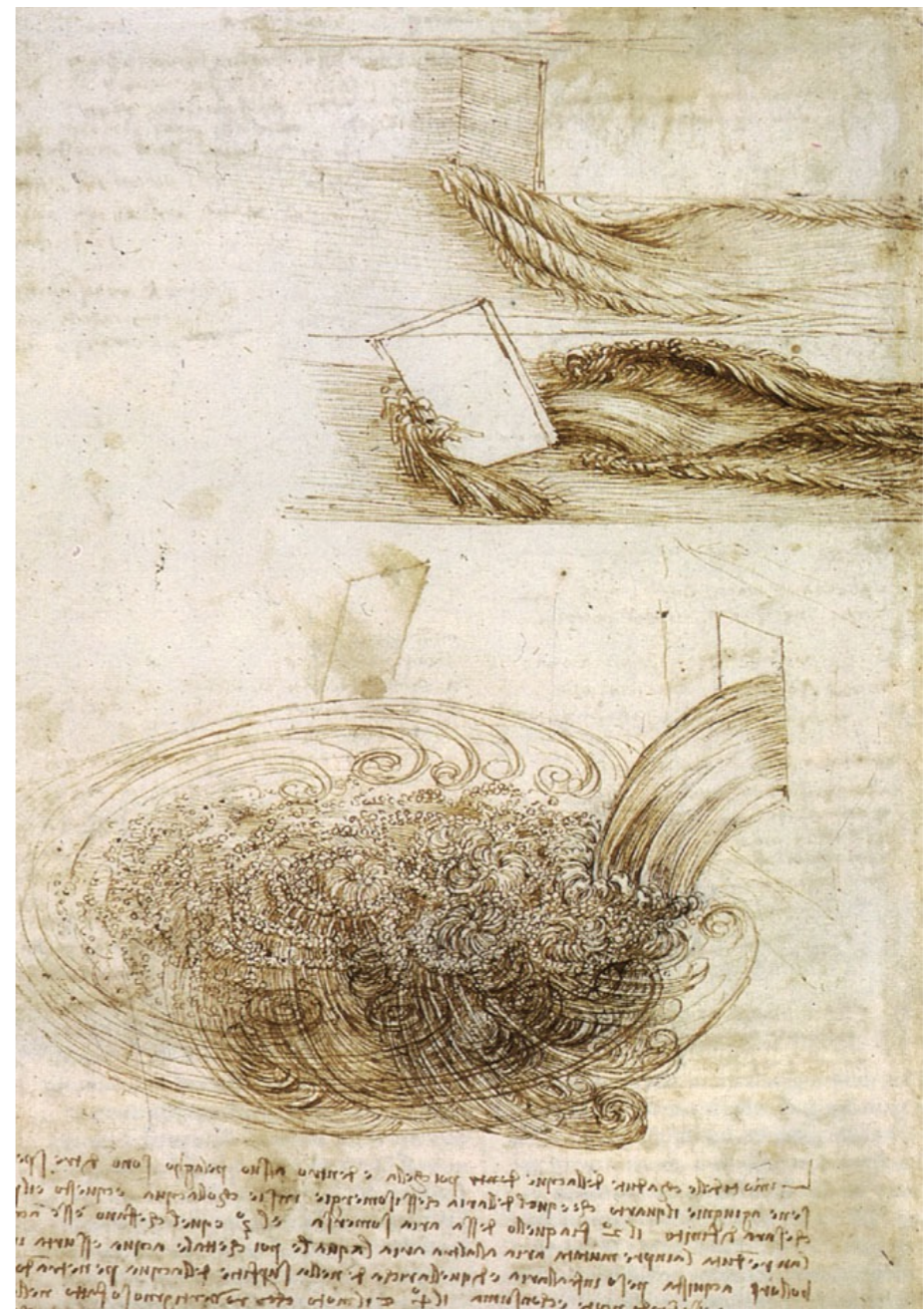
What are vortices?

Leonardo da Vinci, in what is believed to be the first study of turbulence (“turbolenza”) more than 500 years ago, made a sketch of the vortices he saw in his experiments.

“...moving water strives to maintain the course pursuant to the power which occasions it and, if it finds an obstacle in its path, completes the span of the course it has commenced by a circular and revolving movement.”

“...the smallest eddies are almost numberless, and large things are rotated only by large eddies and not by small ones, and small things are turned by small eddies and large...”

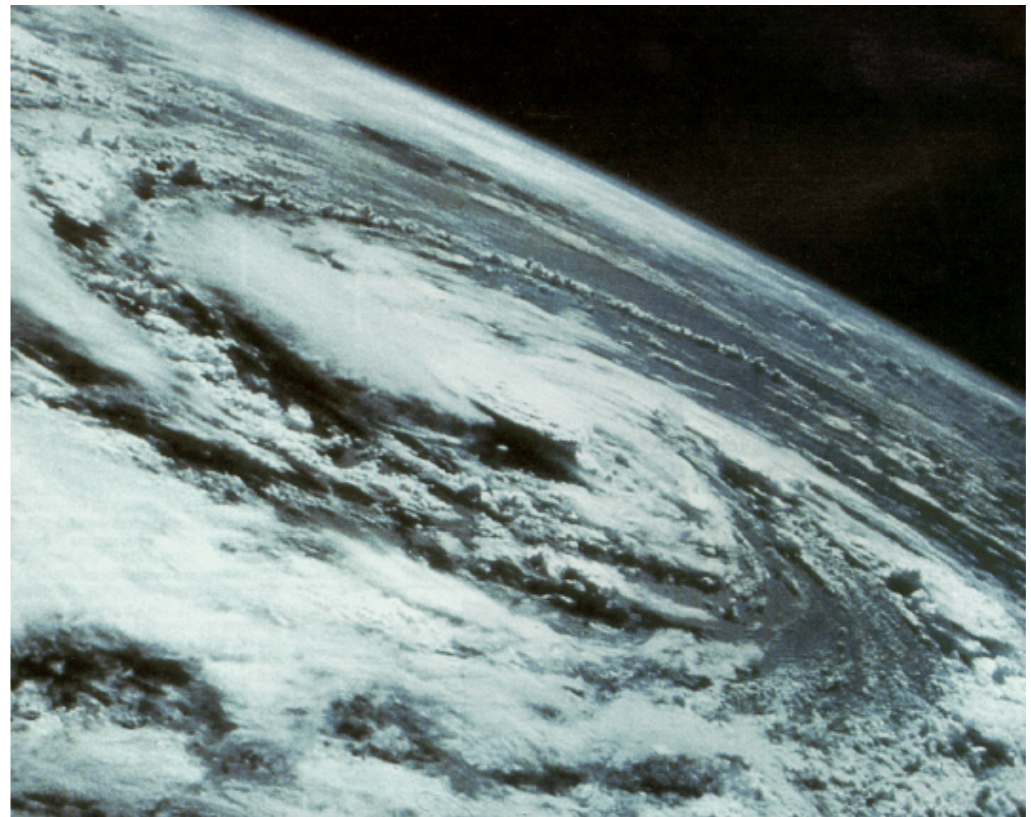
(Translated by Ugo Piomelli, University of Maryland.)



What is the Navier-Stokes equation?

- Models fluid dynamics in the ocean, atmosphere, climate, etc
- Formulated by Navier and Stokes in the 1800's.
- Partial differential equation: describes changes in both space and time

Vortices: key feature of both the model and real life!



Why is the Navier-Stokes equation famous?

- Fluid dynamics are important for many applications, and fluids move in very complicated ways.
- It is very difficult to solve!

Reasonable requirements for any mathematical model:

- (1) Solutions should exist (because reality exists).
- (2) Solutions should be unique (only one version of reality).
- (3) Related physical quantities - velocity, acceleration, etc - should remain finite (not “blow-up”).

We do not completely understand 1, 2, or 3!

- \$1 million Clay Millennium Prize: (dis)prove 1, 2, or 3.
- What use is a model we don't know how to solve?

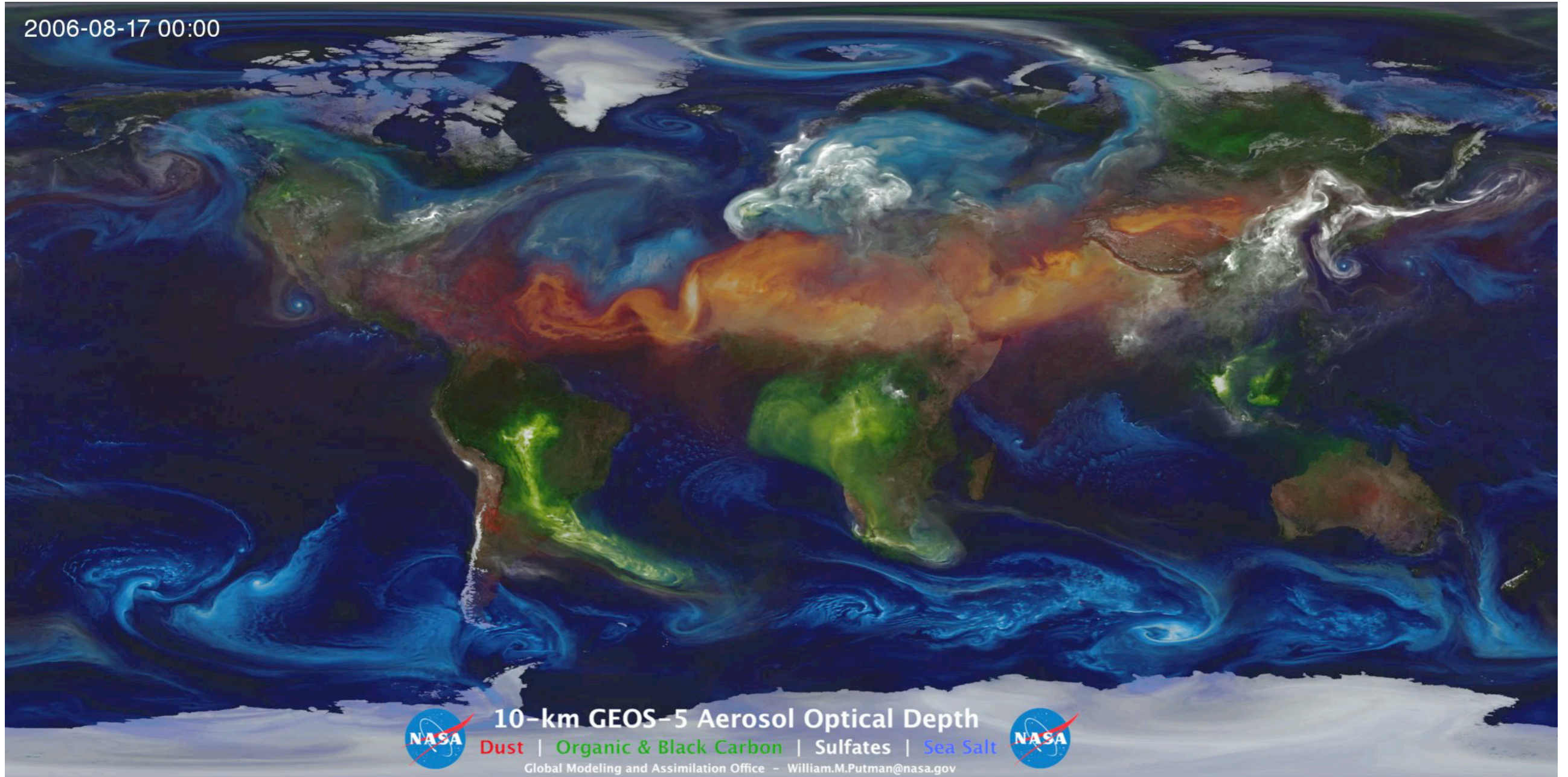
Why use mathematical models?

- Incorporate physical principles
- Provide insight without experiments or field studies
- Yield predictions to help focus future research

How do we analyze the models?

- Identify mathematical structures within the model that control the behavior of solutions
- Determine corresponding physical properties that produce these structures, and hence the behavior
- Results can suggest how to produce/prevent these behaviors in the real world systems

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Goal of Talk: use mathematics and the Navier-Stokes equation to understand why vortices play a key role in the behavior of fluids

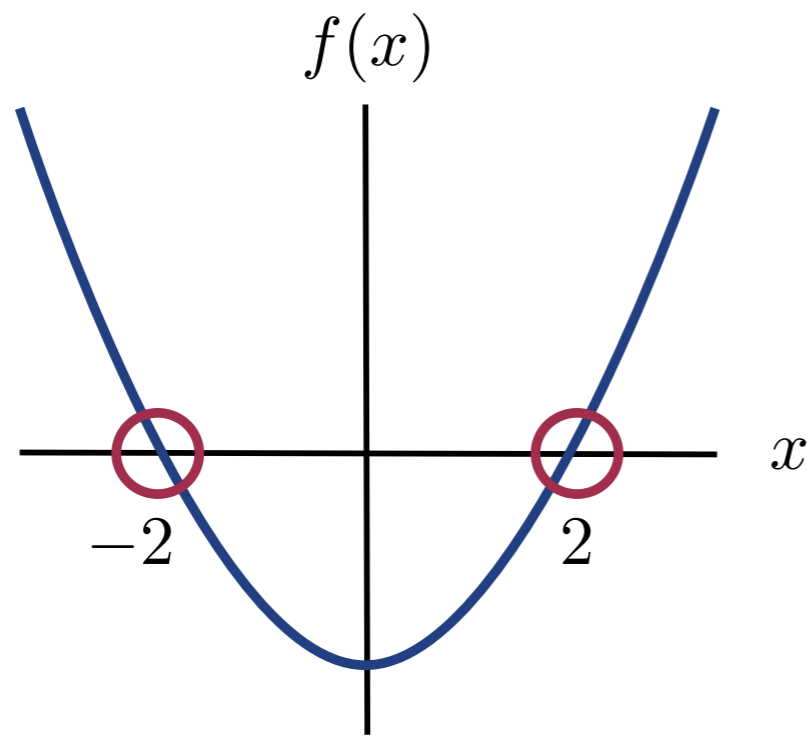
What do you mean by “Understanding solutions of equations that we can’t actually solve”?

Example: Some equations we can solve.

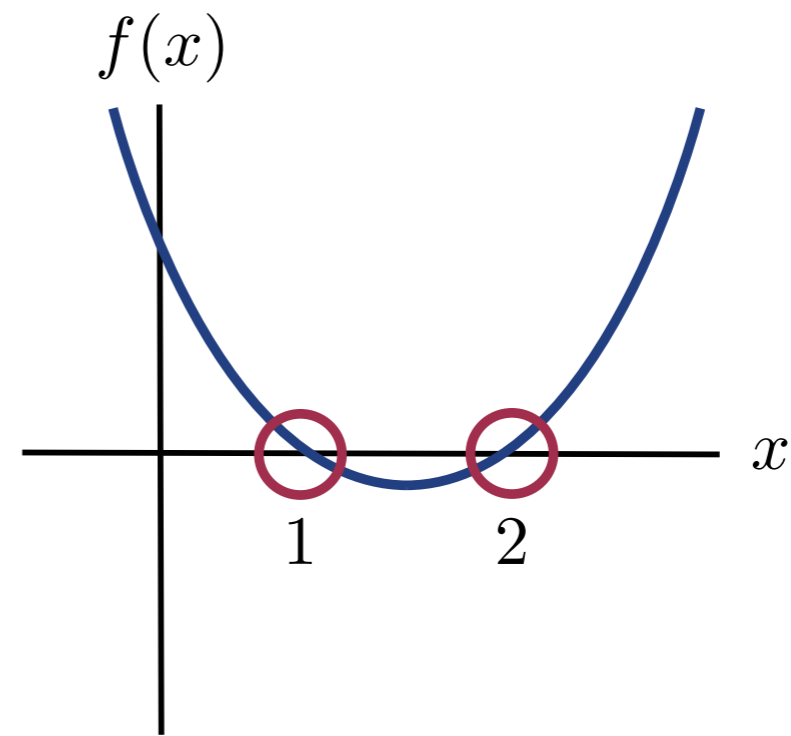
$$x^2 - 4 = 0 \quad \Rightarrow \quad x = 2 \text{ or } -2$$

$$x^2 - 3x + 2 = 0 \quad \Rightarrow \quad x = 1 \text{ or } 2$$

Graphically visualize solutions:



$$f(x) = x^2 - 4$$



$$f(x) = x^2 - 3x + 2$$

What do you mean by “Understanding solutions of equations that we can’t actually solve”?

Example: We can solve any quadratic equation.

$$ax^2 + bx + c = 0$$

Formula for solutions:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$$

Example: An equation we can’t solve.

$$x^5 - x + 1 = 0$$

- No (simple) formula for solutions
- Can we obtain any information about solutions?

What do you mean by “Understanding solutions of equations that we can’t actually solve”?

Example: An equation we can’t solve.

$$f(x) = x^5 - x + 1 = 0$$

What can we do?

- Notice: $f(0) = 1$
- As x approaches $-\infty$, $f(x)$ also approaches $-\infty$

$$x = -10, -100, -1000, \dots$$

$$x^5 = -100000, -10000000000, -1000000000000000, \dots$$

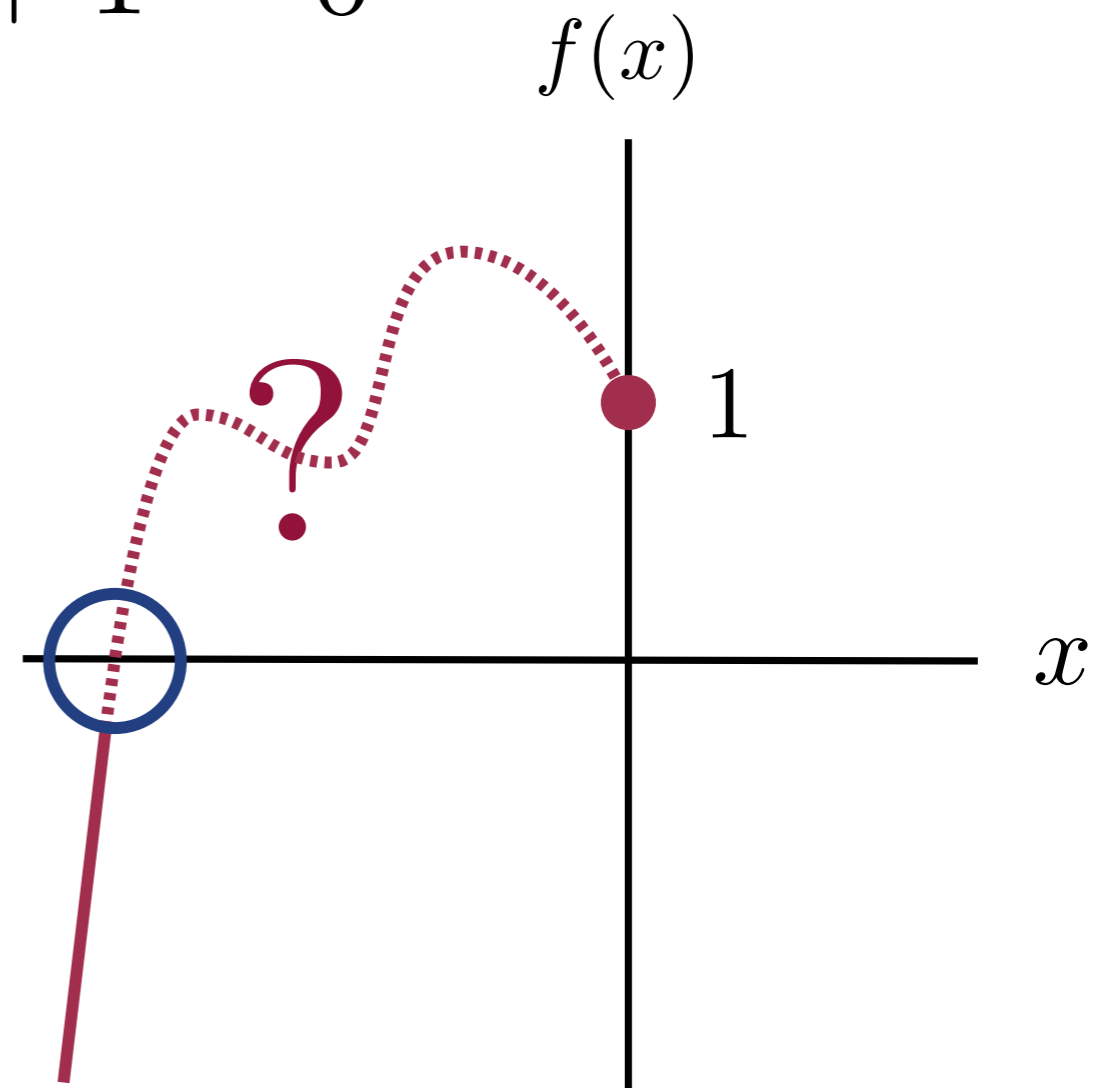
What do you mean by “Understanding solutions of equations that we can’t actually solve”?

Example: An equation we can’t solve.

$$f(x) = x^5 - x + 1 = 0$$

What can we do?

- Notice: $f(0) = 1$
- As x approaches $-\infty$
 $f(x)$ also approaches $-\infty$
- There must be at least one negative solution!



What do you mean by “Understanding solutions of equations that we can’t actually solve”?

Example: An equation we can’t solve.

$$f(x) = x^5 - x + 1 = 0$$

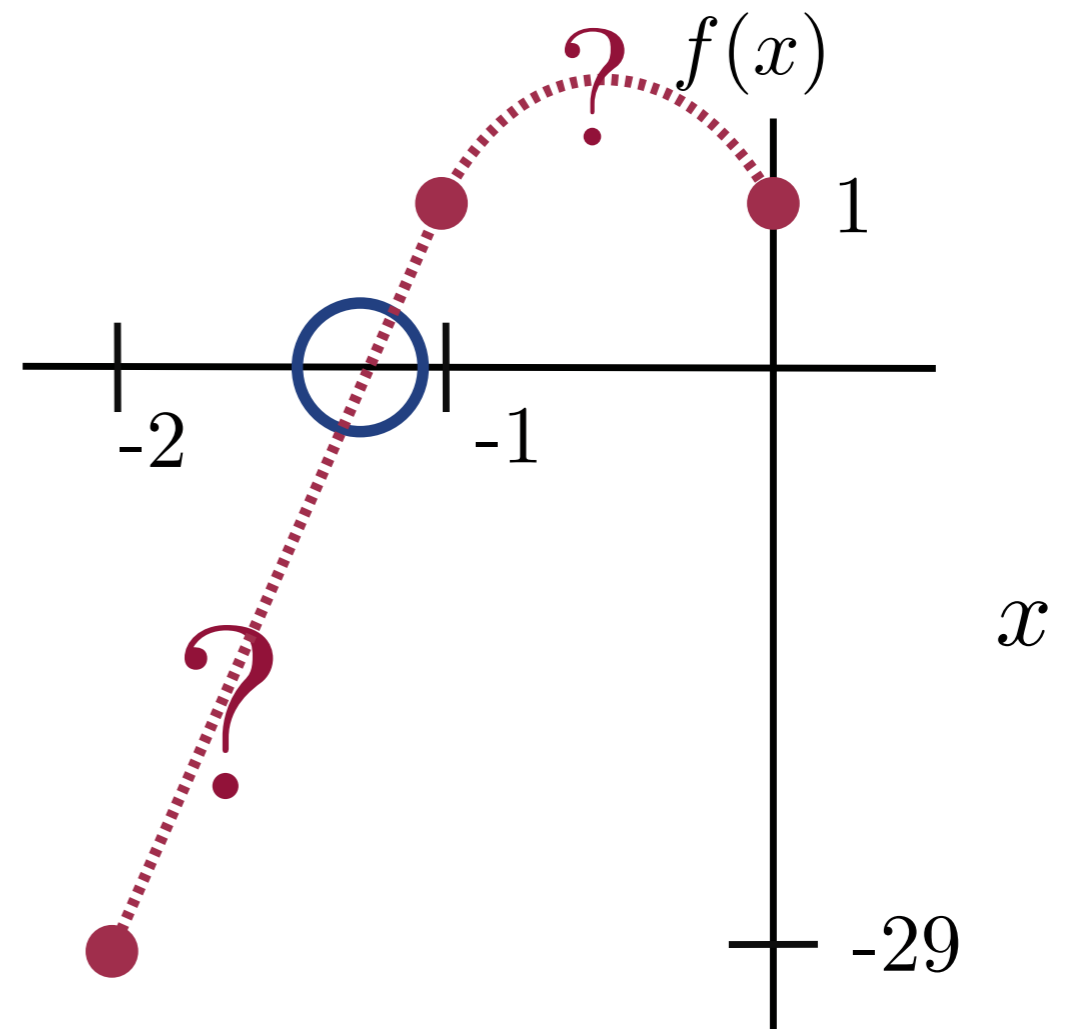
Try some more things:

- Also:

$$f(-1) = 1 > 0$$

$$f(-2) = -29 < 0$$

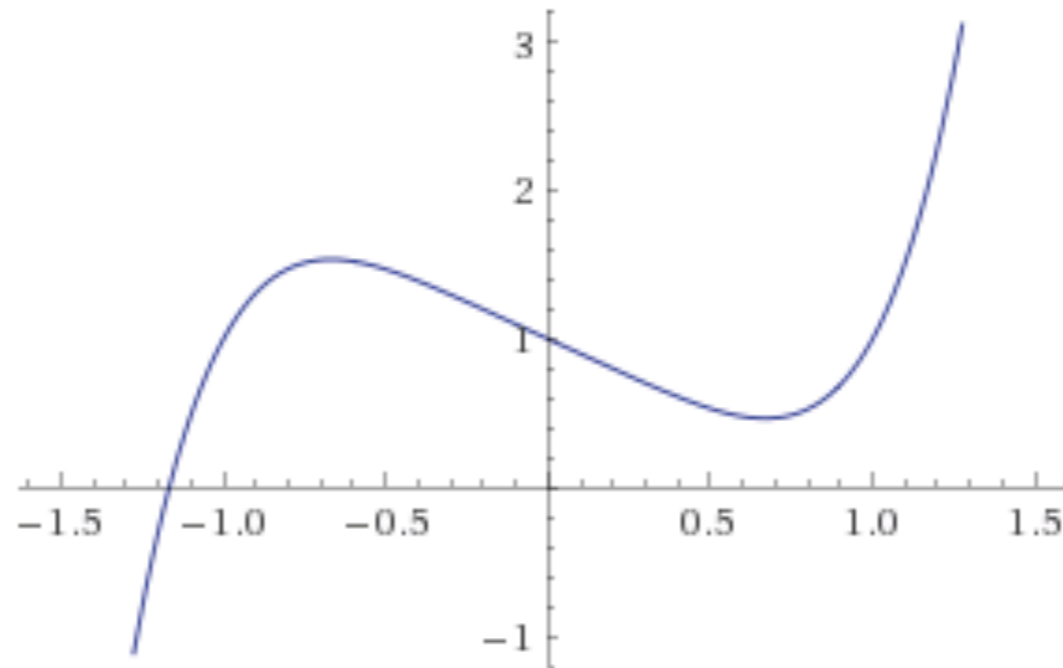
- So there is a solution between -1 and -2!



What do you mean by “Understanding solutions of equations that we can’t actually solve”?

Example: An equation we can’t solve.

$$f(x) = x^5 - x + 1 = 0$$



- Computers can help, but that’s not the point
- We’ve understood, mathematically, why there must be a solution between -2 and -1.
- Properties of the function force such a solution to exist.

What do you mean by “Understanding solutions of equations that we can’t actually solve”?

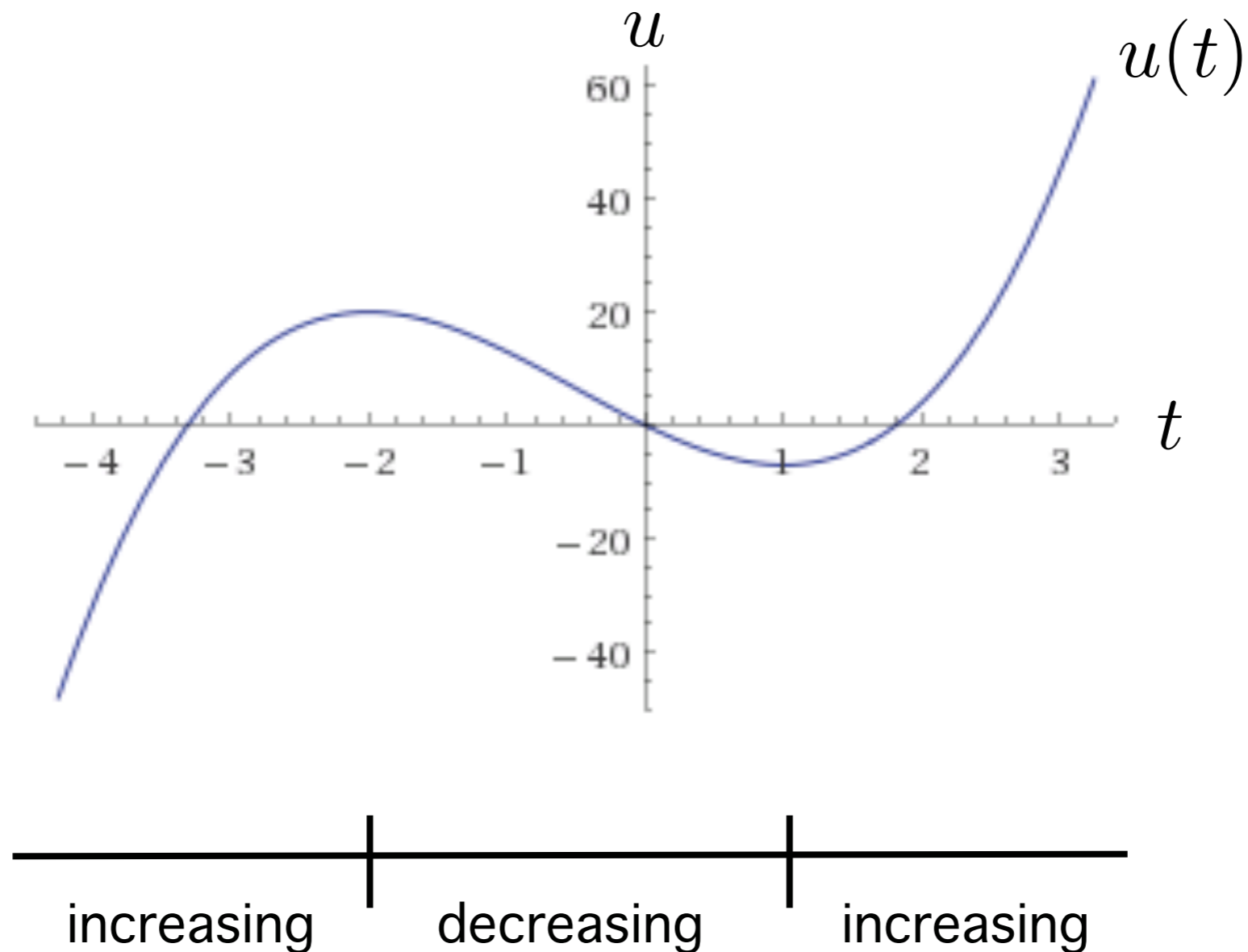
Example: Differential equation.

$$\frac{du}{dt} = f(u)$$

- Solution is a function $u(t)$
- Its **derivative** is $\frac{du}{dt}$
- We’re given information about the derivative only.
- So what is a derivative?

What do you mean by “Understanding solutions of equations that we can’t actually solve”?

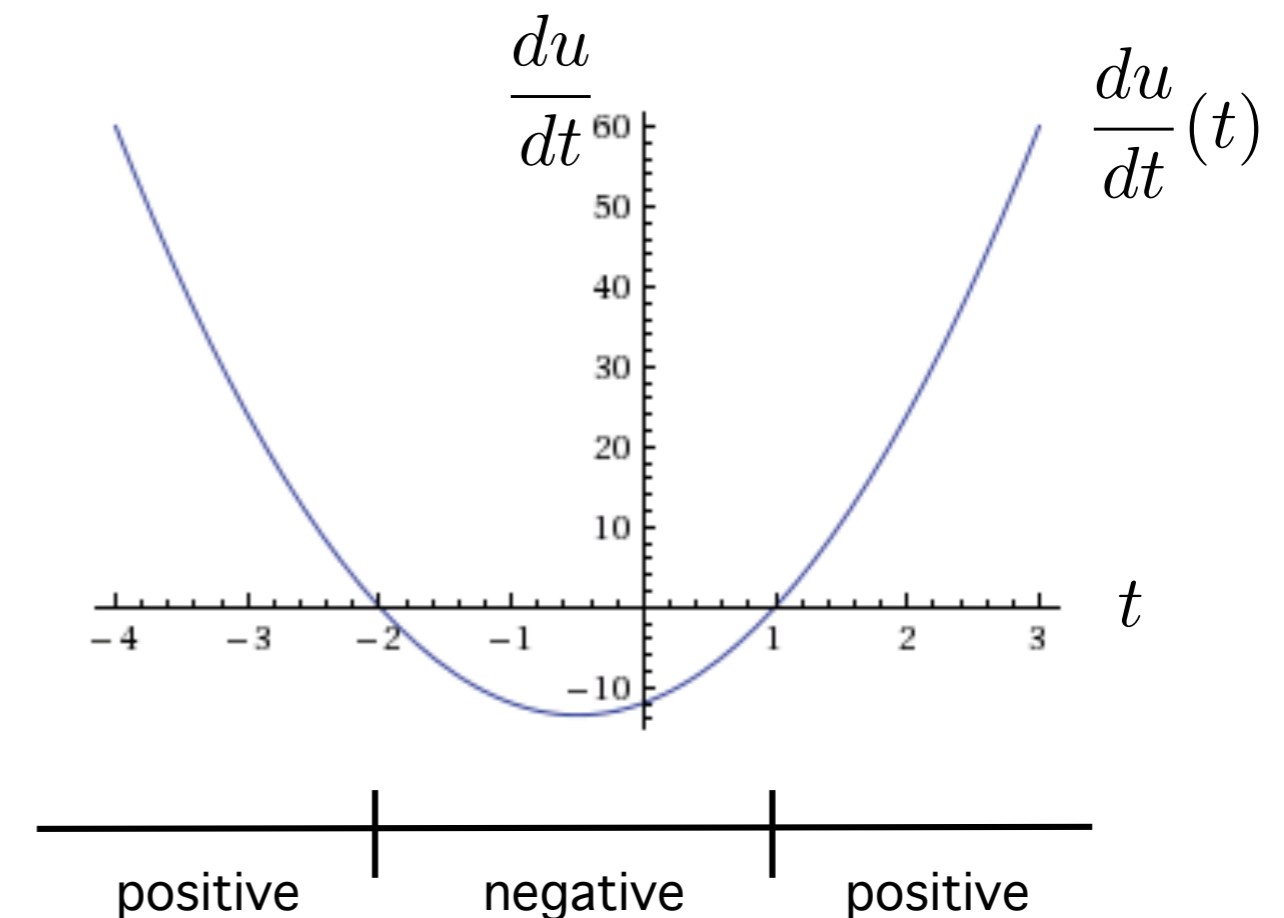
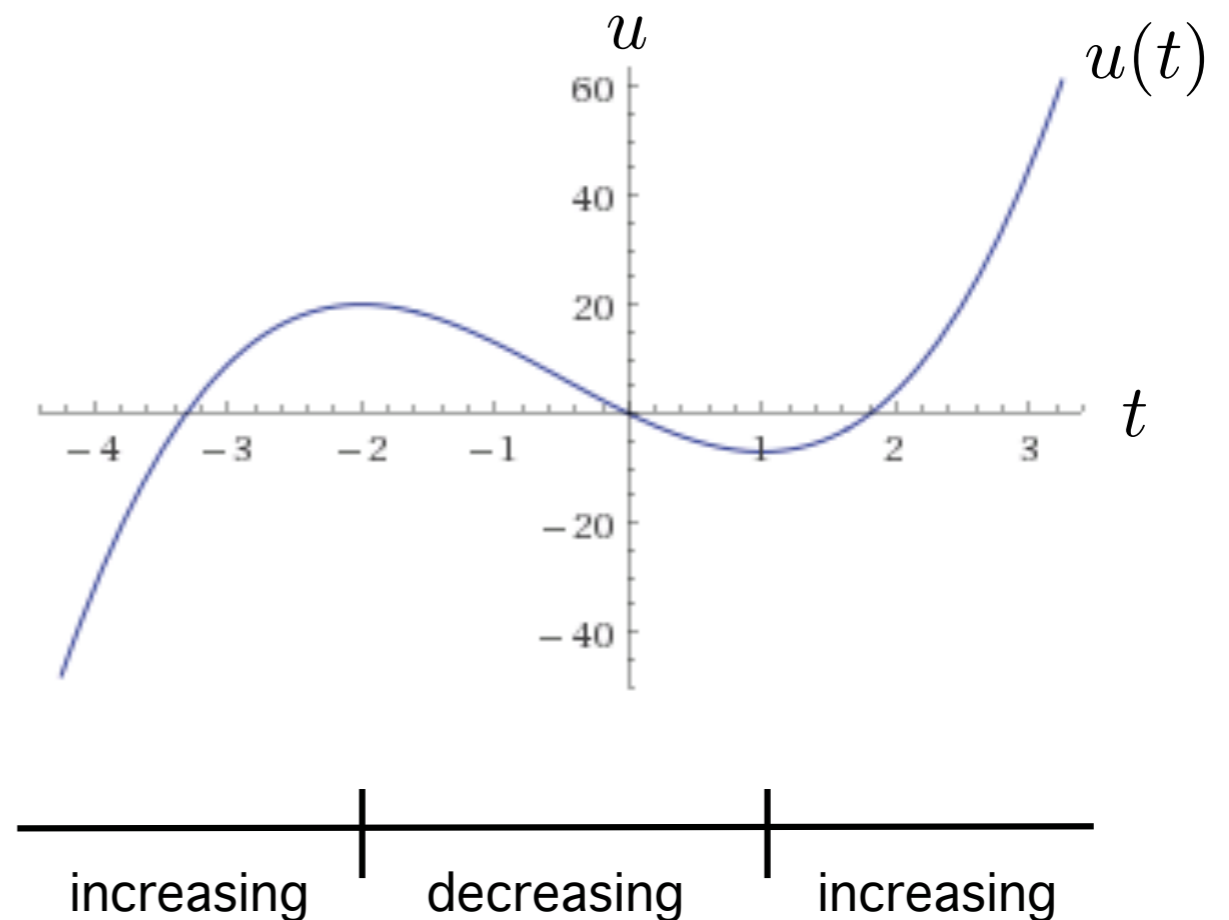
Example: Differential equation.



What is a derivative?

The derivative tells us when the original function is increasing or decreasing, and by how much.

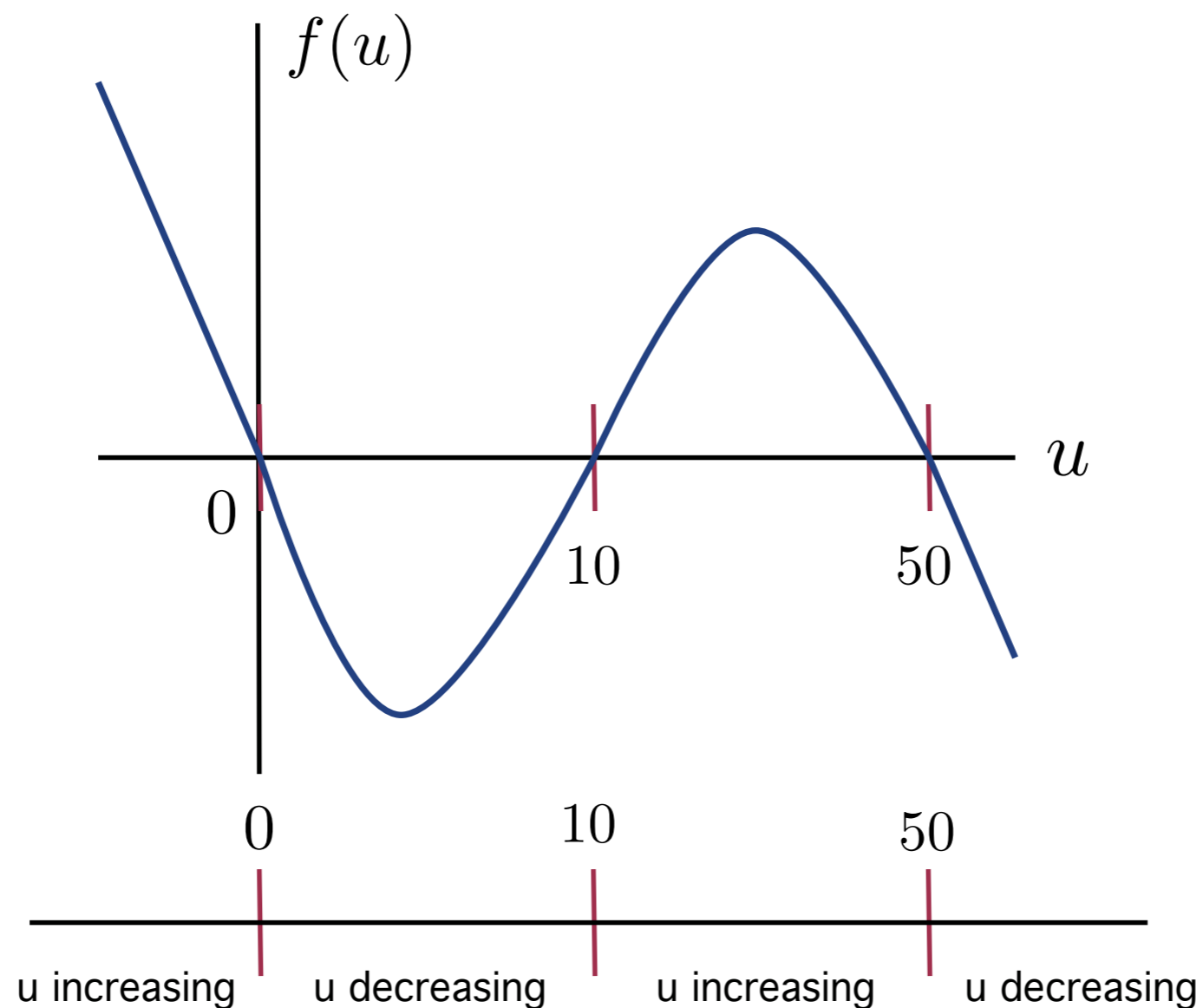
If the derivative is positive, the original function is increasing. The larger the derivative, the faster the function is increasing.



What do you mean by “Understanding solutions of equations that we can’t actually solve”?

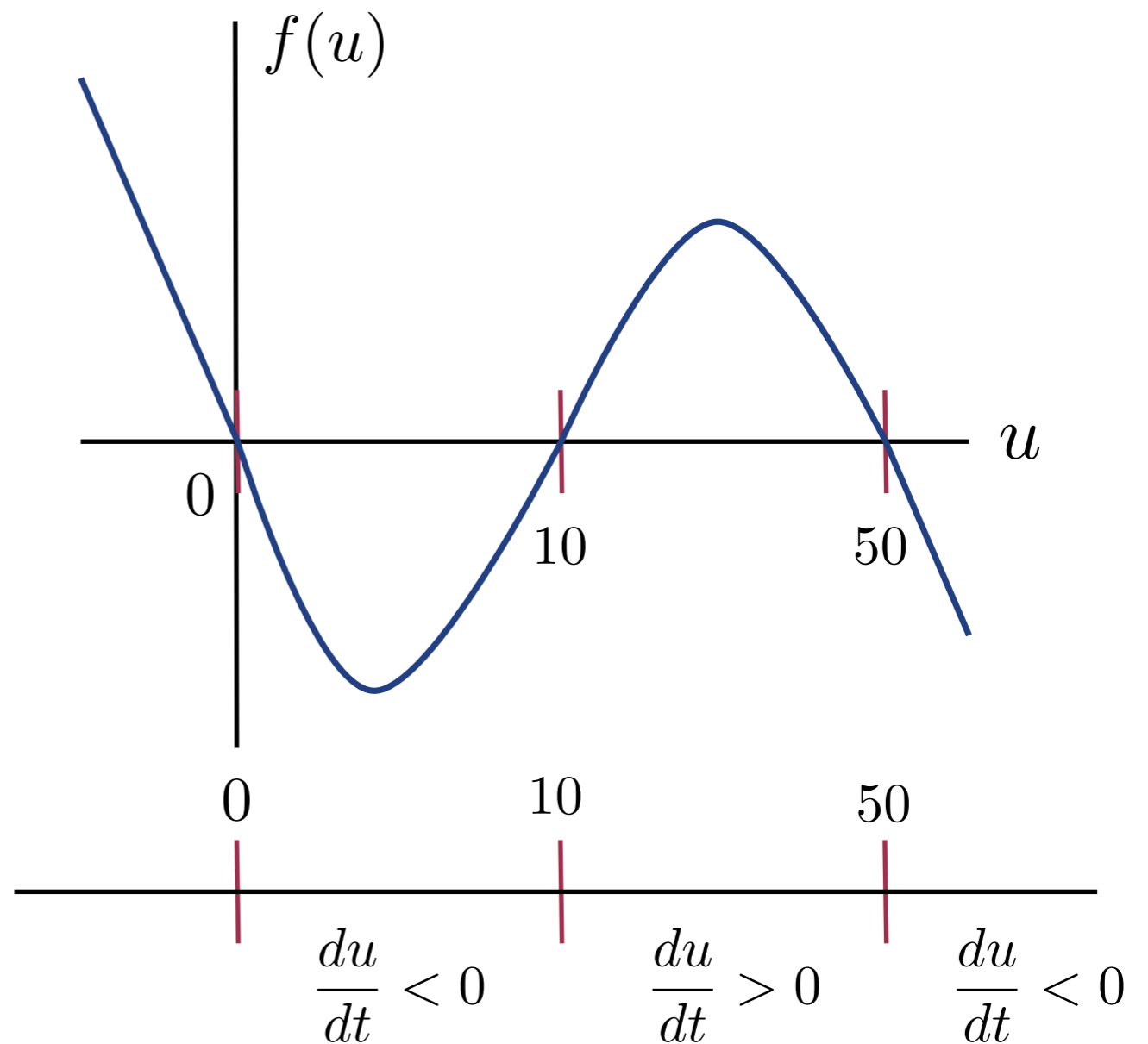
Example: Differential equation.

$$\frac{du}{dt} = f(u) = u(10 - u)(u - 50)$$



$$\frac{du}{dt} = f(u)$$

Suppose $u(t)$ is the population, as a function of time, of wolves in a given region.

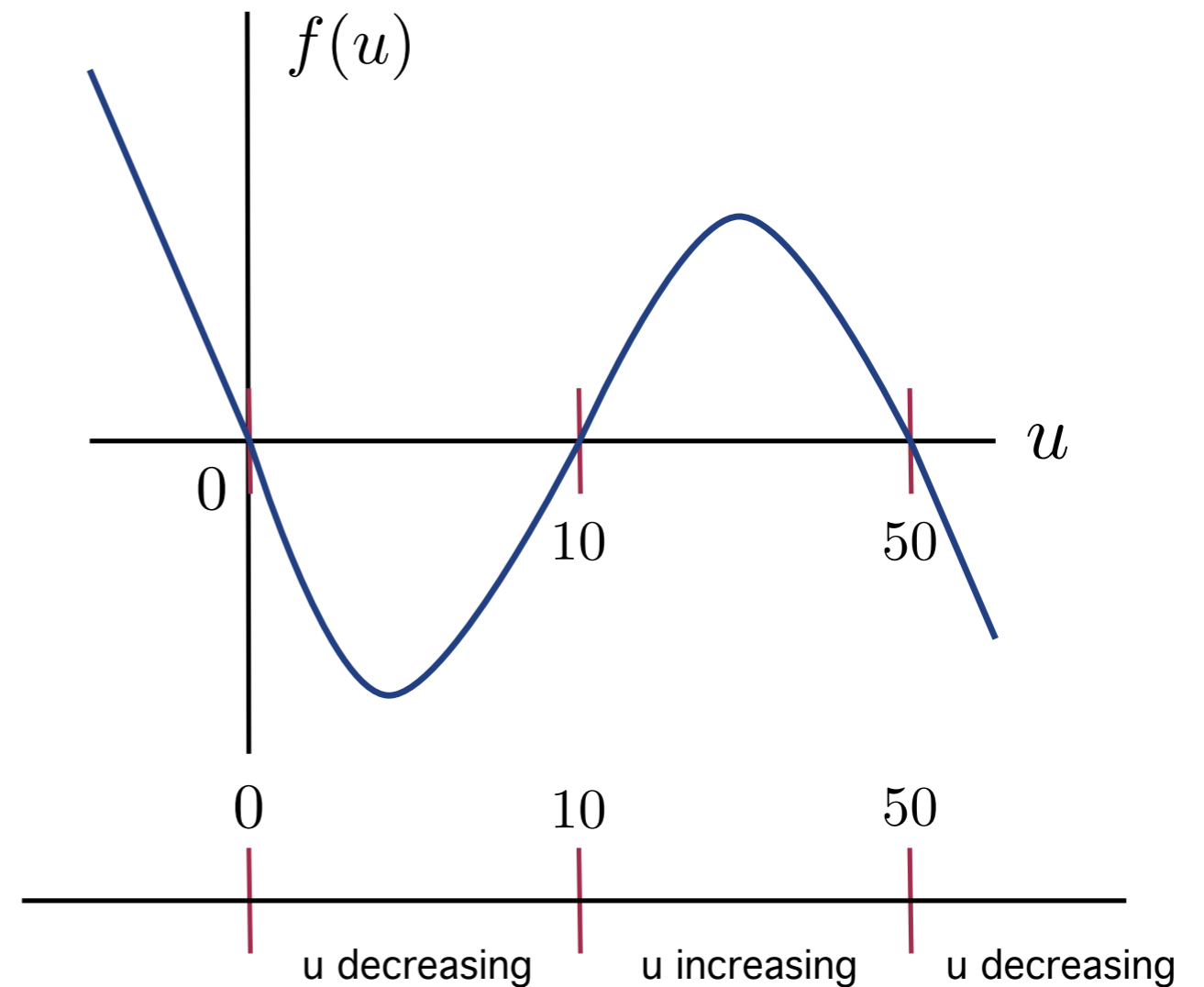


- If $\frac{du}{dt} < 0$ then the population decreases
- If $\frac{du}{dt} > 0$ then the population increases
- If $\frac{du}{dt} = 0$ then the population doesn't change

Example: Differential equation

$$\frac{du}{dt} = f(u)$$

- $\frac{du}{dt} < 0$ population decreases
- $\frac{du}{dt} > 0$ population increases
- $\frac{du}{dt} = 0$ population doesn't change



Equilibrium solution: solution that is independent of time

$$u(t) = 0, \quad u(t) = 10, \quad u(t) = 50$$

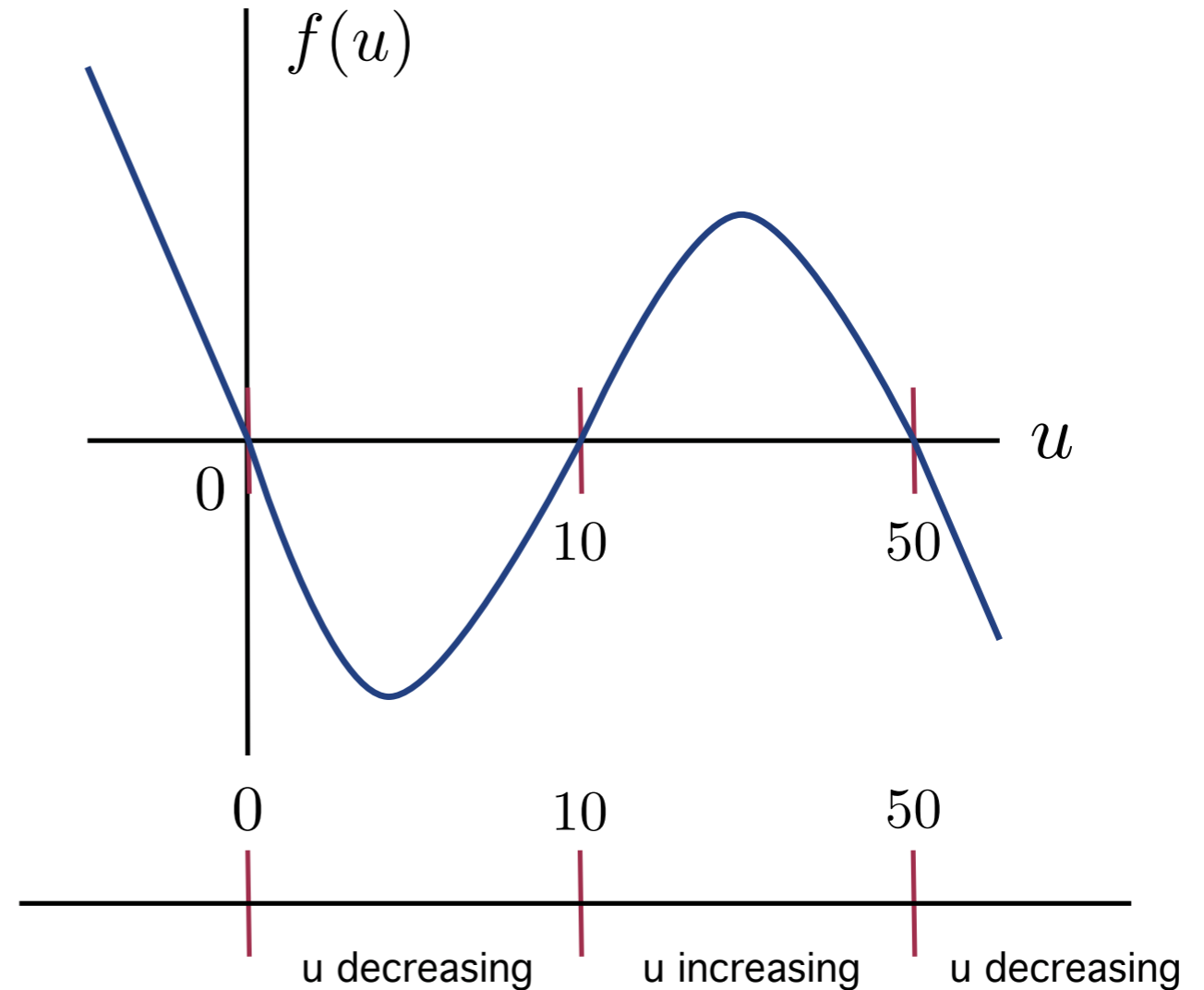
Example: Differential equation

Equilibrium solutions:

$$u(t) = 0$$

$$u(t) = 10$$

$$u(t) = 50$$



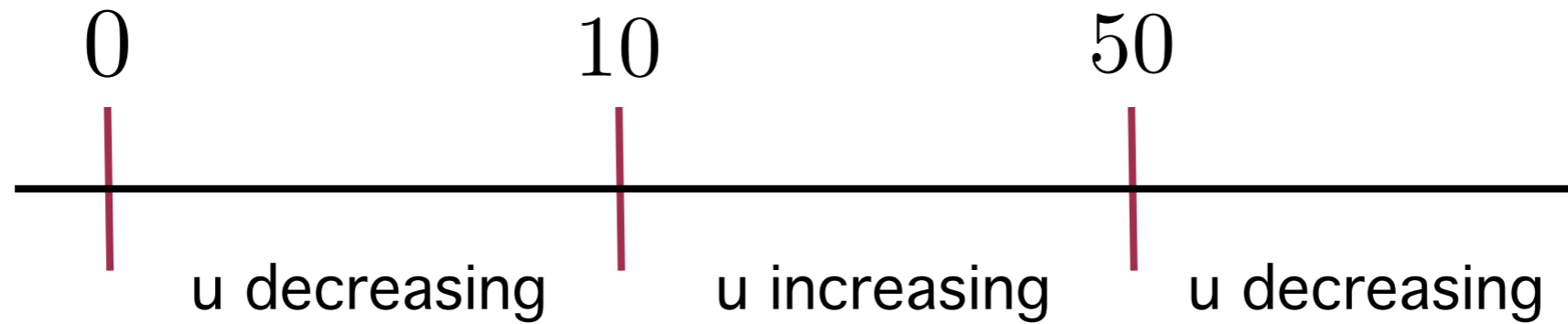
Stable equilibrium: start near it, converge to it

$$u(t) = 0 \quad u(t) = 50$$

Unstable equilibrium: start near it, move away from it

$$u(t) = 10$$

Example: Differential equation



Physical interpretation: 10 acts like a threshold. If the population is too small (less than 10), the wolves will die out (converge to 0). If the population is large enough (greater than 10), they will be able to sustain themselves at the level of 50 individuals.

Stability is important: stable states govern long-time behavior. Finding the stable states in a model tells you what behaviors you can expect to see in the system.

Example: Differential equation

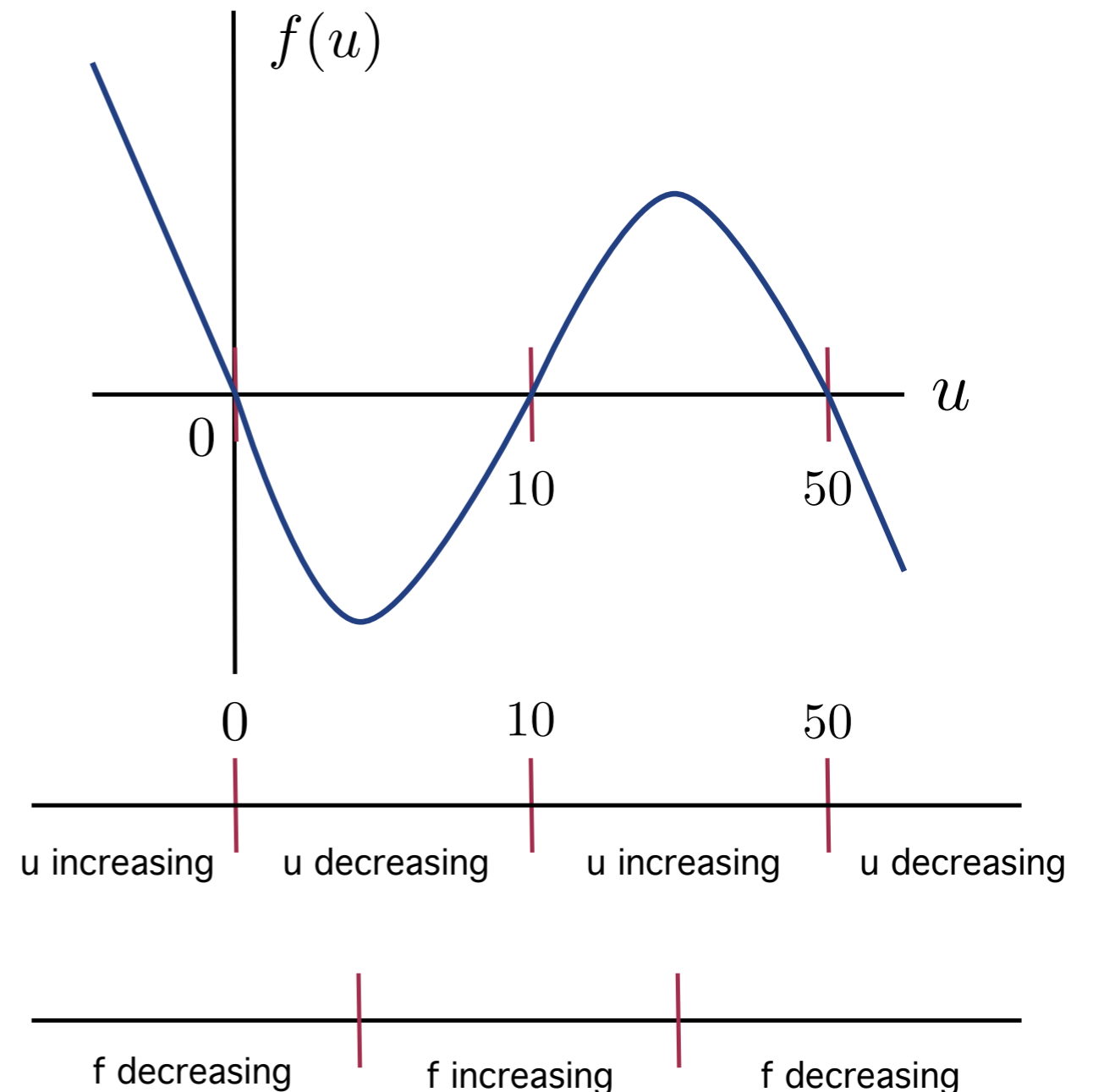
Question: Mathematically, why are 0 and 50 stable, while 10 is unstable? How can we predict stability?

- f is decreasing at 0, 50

$$\frac{df}{du}(0) < 0, \quad \frac{df}{du}(50) < 0$$

- f is increasing at 10

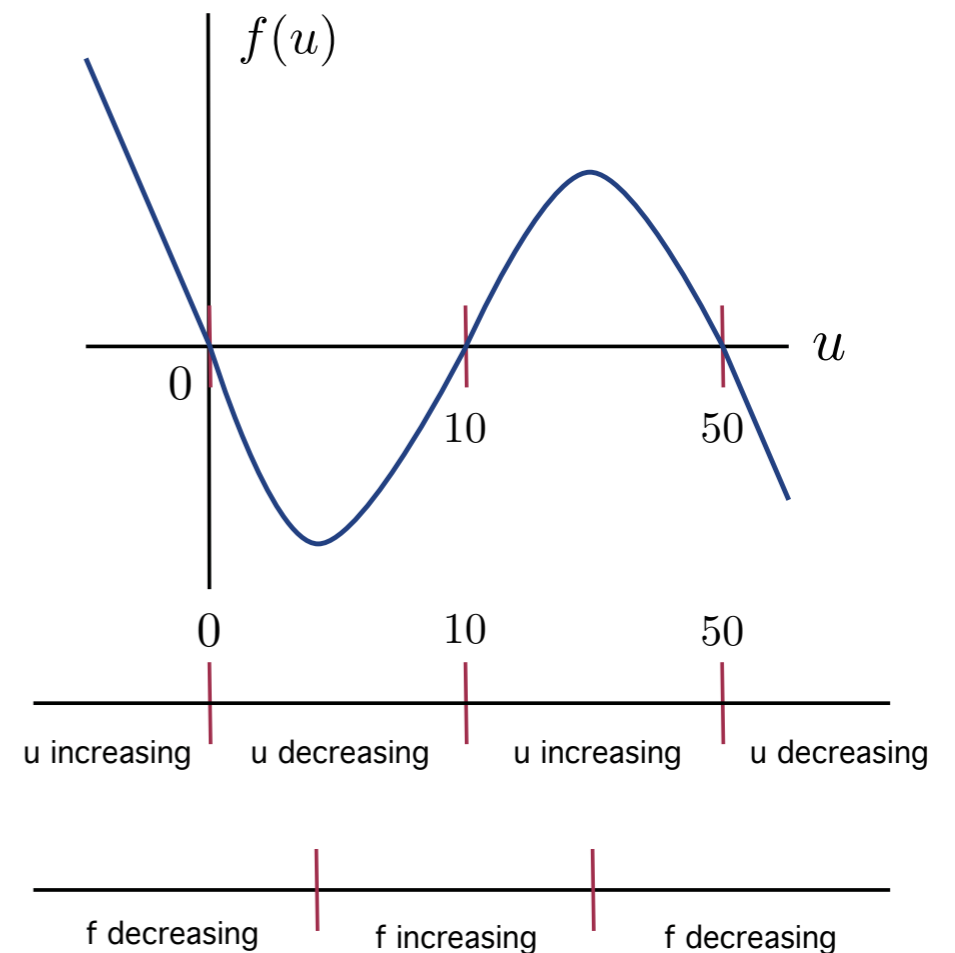
$$\frac{df}{du}(10) > 0$$



What do you mean by “Understanding solutions of equations that we can’t actually solve”?

Example: Differential equation.

$$\frac{du}{dt} = f(u)$$



Summary:

- Equilibrium: value of u such that $f(u) = 0$
- Stable equilibrium: need $\frac{df}{du} < 0$
- Unstable equilibrium: need $\frac{df}{du} > 0$
- Determined without a formula for the solution.

What do you mean by “Understanding solutions of equations that we can’t actually solve”?

Strategy for predicting the behavior of solutions to differential equations:

$$\frac{du}{dt} = f(u)$$

- Identify any equilibrium solutions
- Determine if they are stable or unstable
- Solutions will converge to stable equilibria

Moral: stable equilibria of mathematical models determine the behaviors we expect to see in the real world

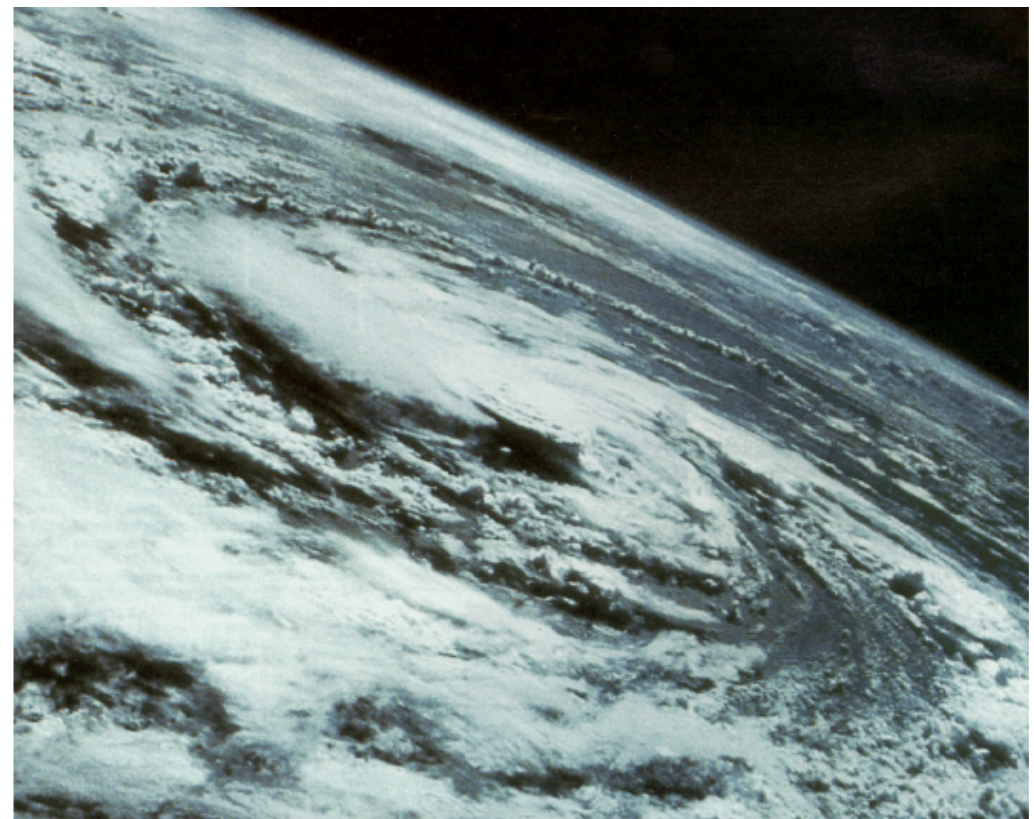
Can we apply this strategy to the Navier-Stokes equations? YES!

Main ideas:

- Vortices can be viewed as equilibria
- Vortices are stable equilibria
- Therefore, we see vortices everywhere!



Jupiter's Great Red Spot, seen from Voyager 1



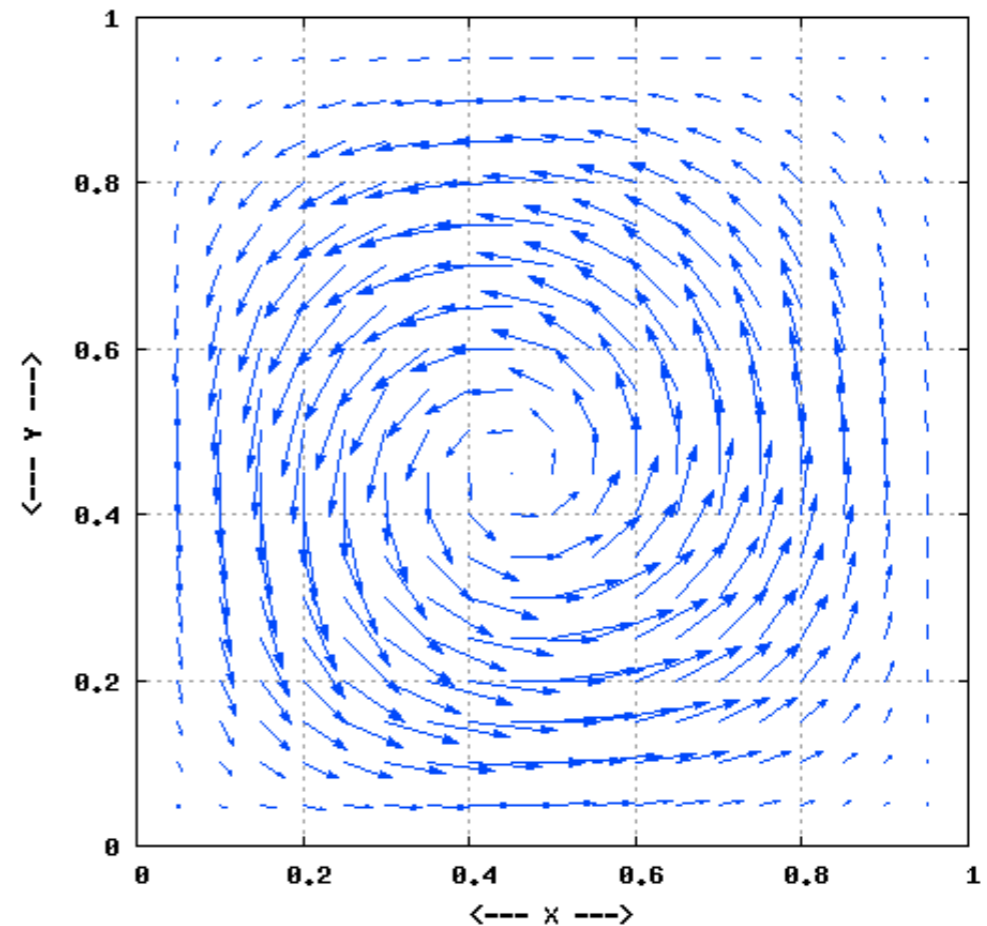
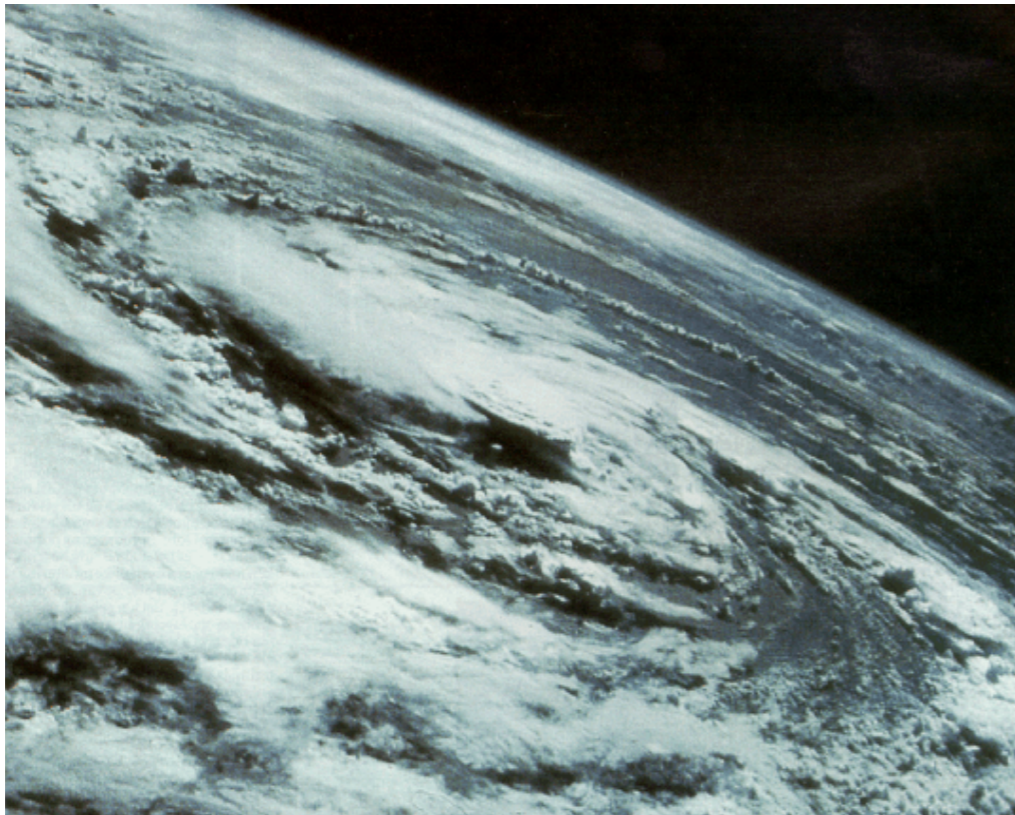
Hurricane Gladys, 1968, seen from Apollo 7

Vortices and the Navier-Stokes equations

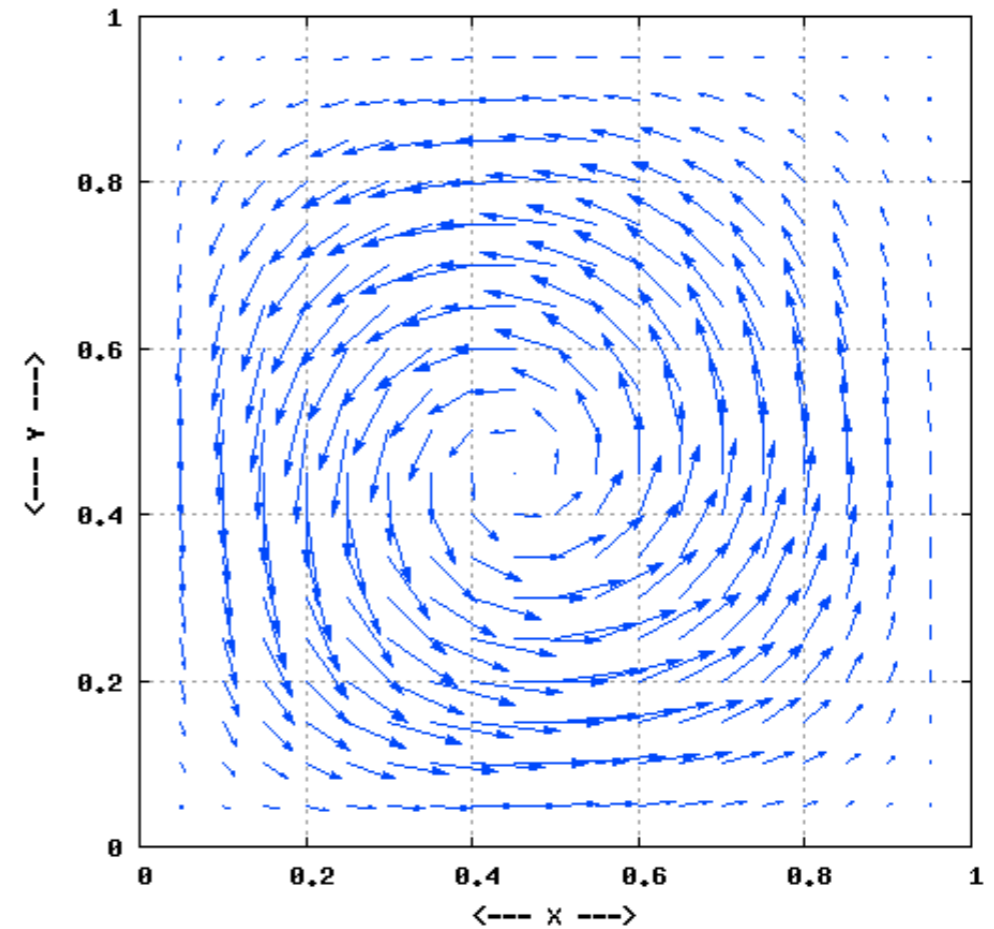
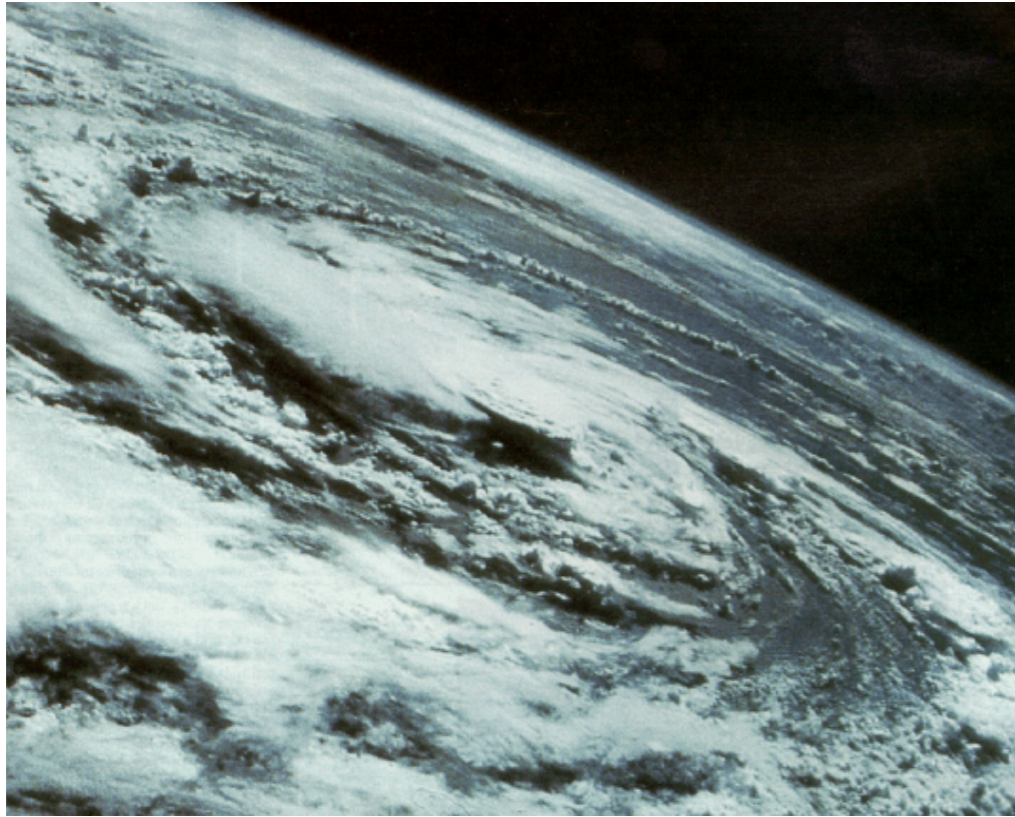
The Navier-Stokes equation is a type of differential equation:

$$\frac{\partial u}{\partial t} = \mathcal{F}(u)$$

The unknown function $u(x,y,t)$ is the velocity of the fluid at a given point in space, (x,y) , and time, t .



Vortices and the Navier-Stokes equations

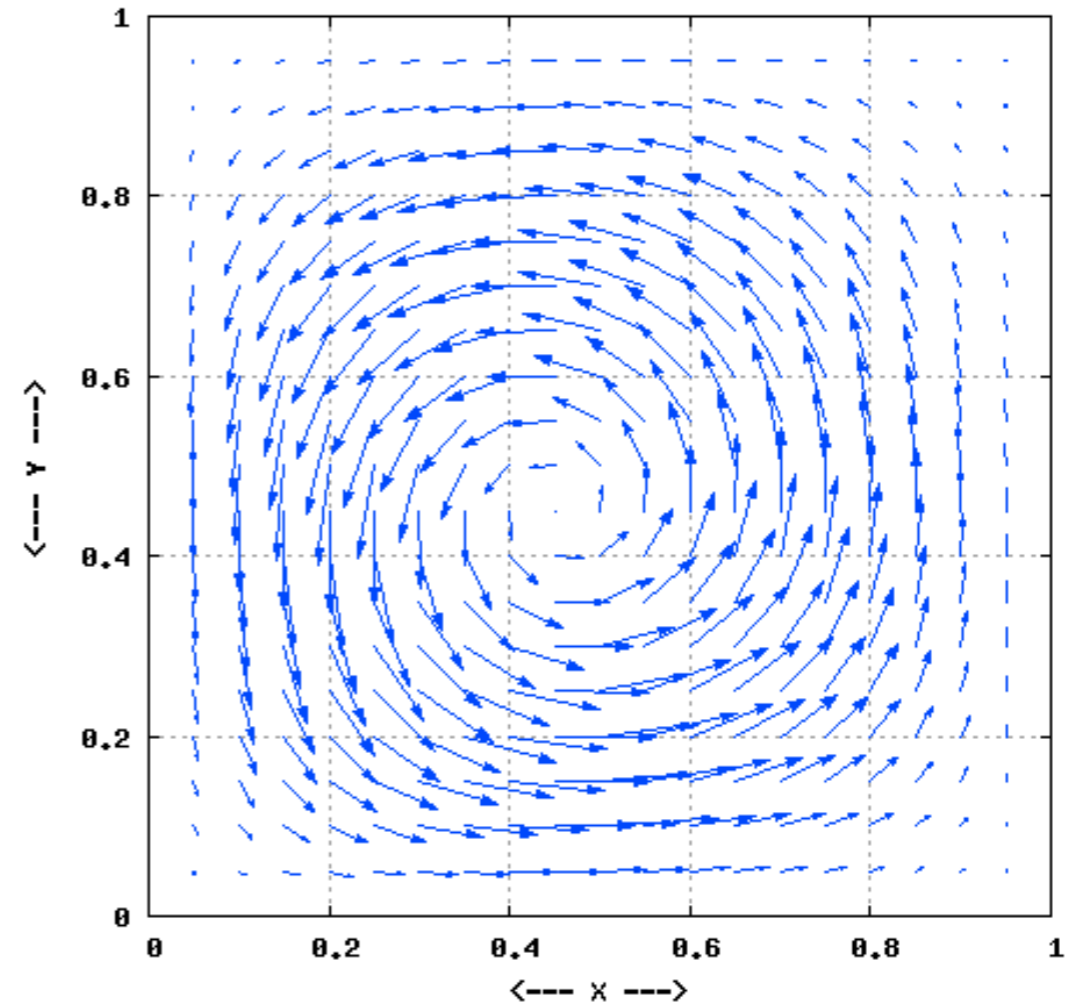


At each point in space, u is like an arrow, that points in the direction that the fluid is moving. This arrow depends on where we look at the vortex in space (x,y) and at what time t we look at it.

Vortices and the Navier-Stokes equations

$$\frac{\partial u}{\partial t} = \mathcal{F}(u)$$

$$u(x, y, t) = (u_1(x, y, t), u_2(x, y, t))$$



- Velocity u depends on where you measure it (x, y) and when you measure it (t)
- Velocity has two components, the first tells you how fast the fluid moves left/right, and the second tells you how fast it moves up/down.
- Partial differential equation: derivatives in x, y , and t

Vortices and the Navier-Stokes equations

$$\frac{\partial u}{\partial t} = \mathcal{F}(u)$$

$$u(x, y, t) = (u_1(x, y, t), u_2(x, y, t))$$

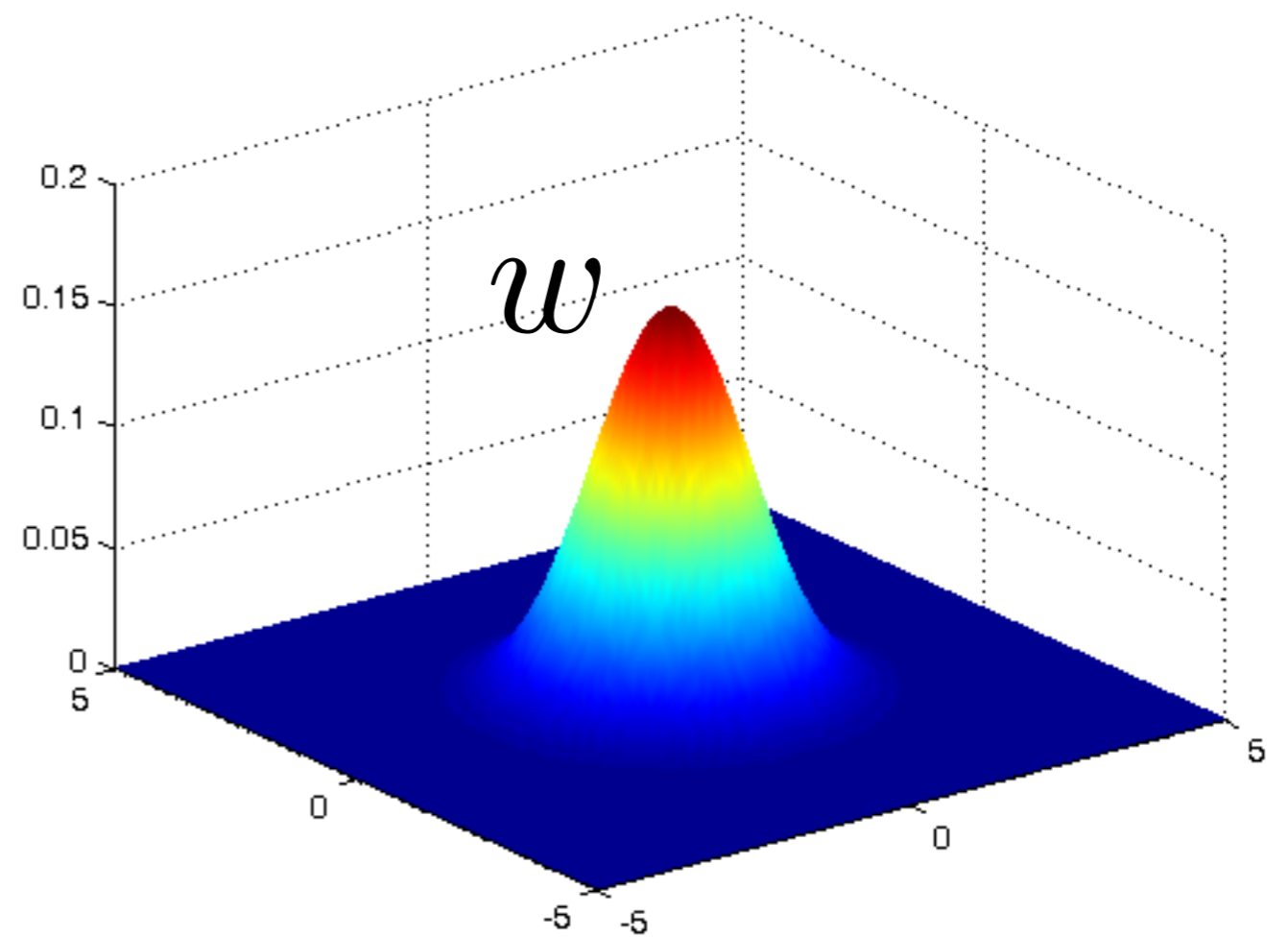
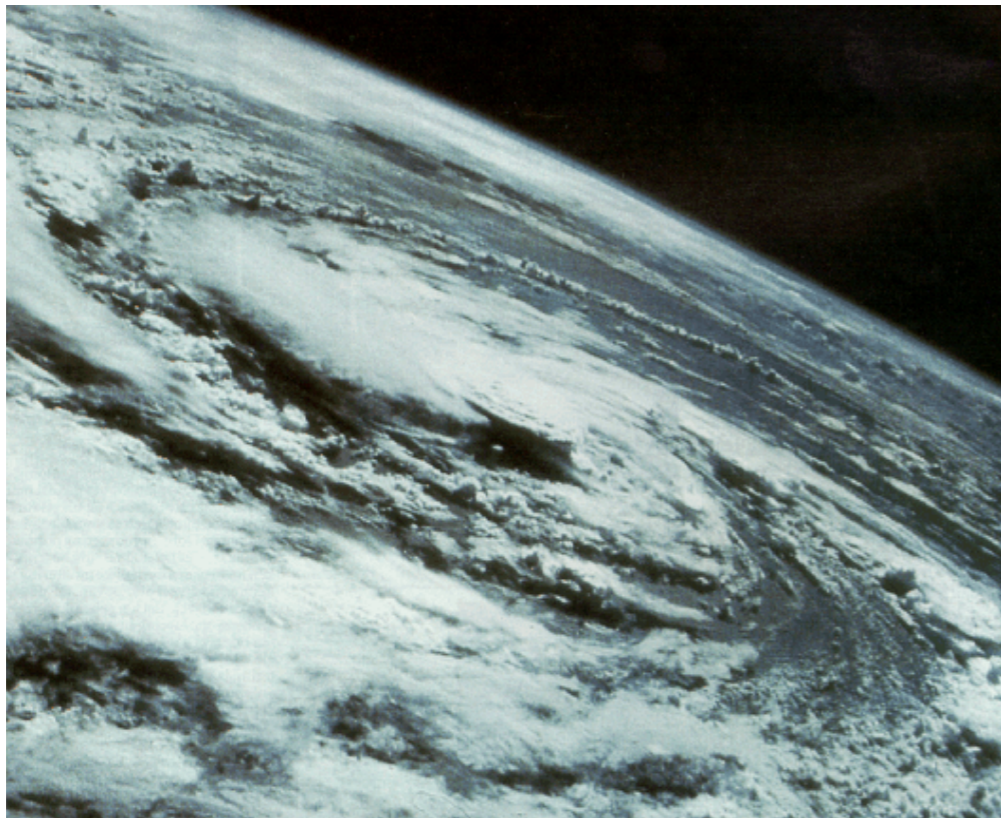
Newton's Law: Force = mass \times acceleration

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} - u_1 \frac{\partial u_1}{\partial x} - u_2 \frac{\partial u_1}{\partial y} - \frac{\partial p}{\partial x} \\ \frac{\partial u_2}{\partial t} &= \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} - u_1 \frac{\partial u_2}{\partial x} - u_2 \frac{\partial u_2}{\partial y} - \frac{\partial p}{\partial y} \end{aligned}$$

$$\text{Incompressibility: } \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0$$

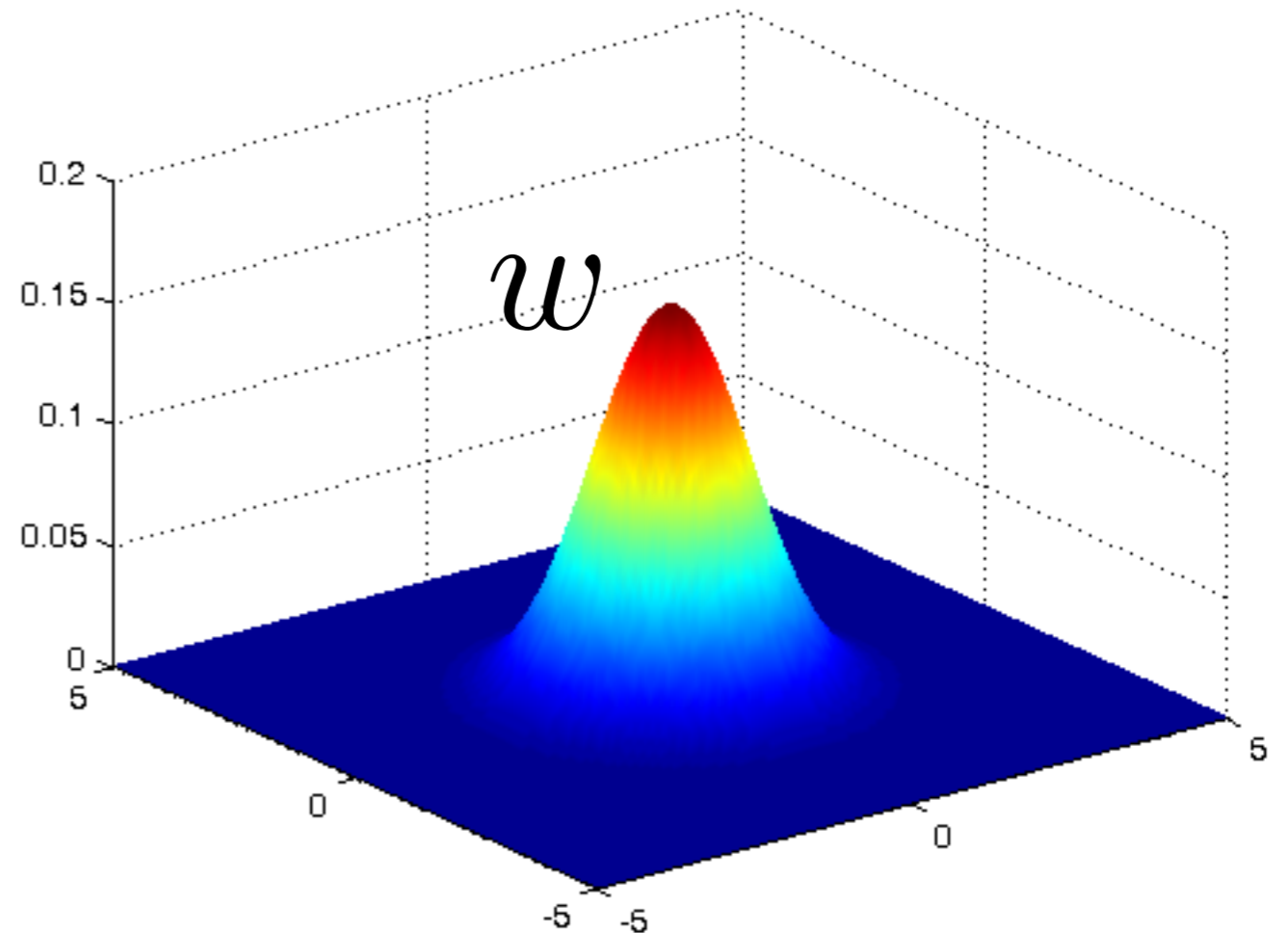
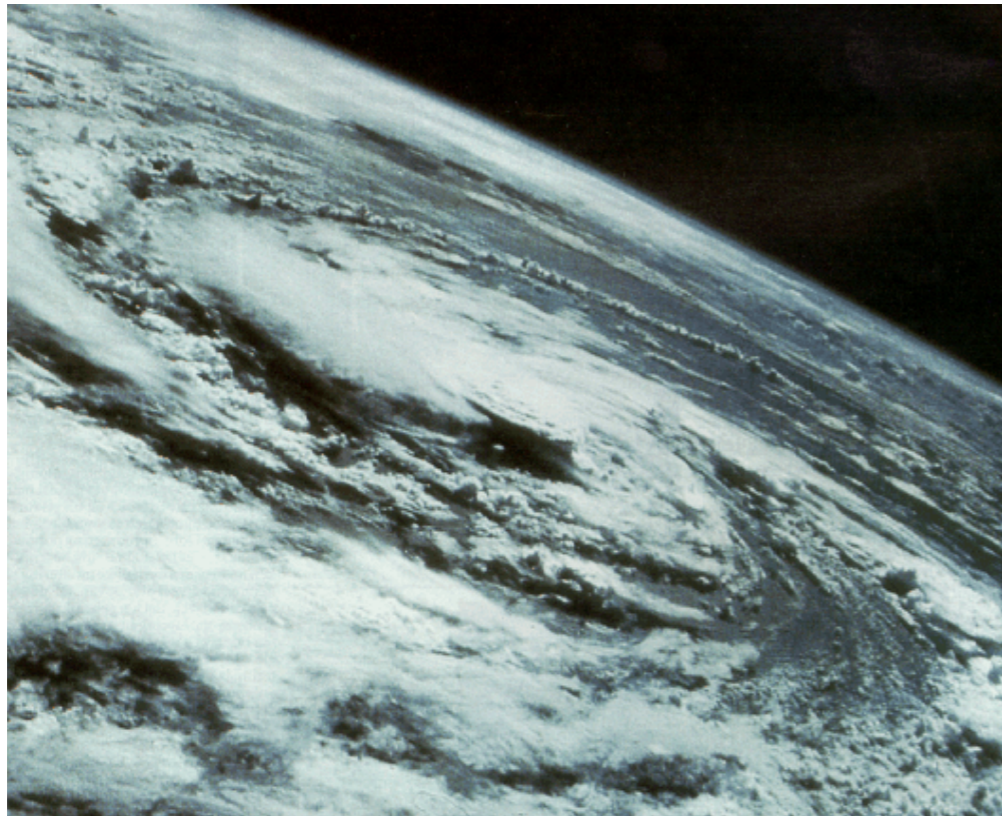
Vortices and the Navier-Stokes equations

Vortices aren't stationary. So how can we view them as equilibria?



Instead of studying the velocity, u , we'll study the vorticity, w , which is essentially the “shell” of the vortex.

Vortices and the Navier-Stokes equations



- $w < 0$: fluid spins counterclockwise
- $w > 0$: fluid spins clockwise
- The larger w is, the faster the fluid is spinning

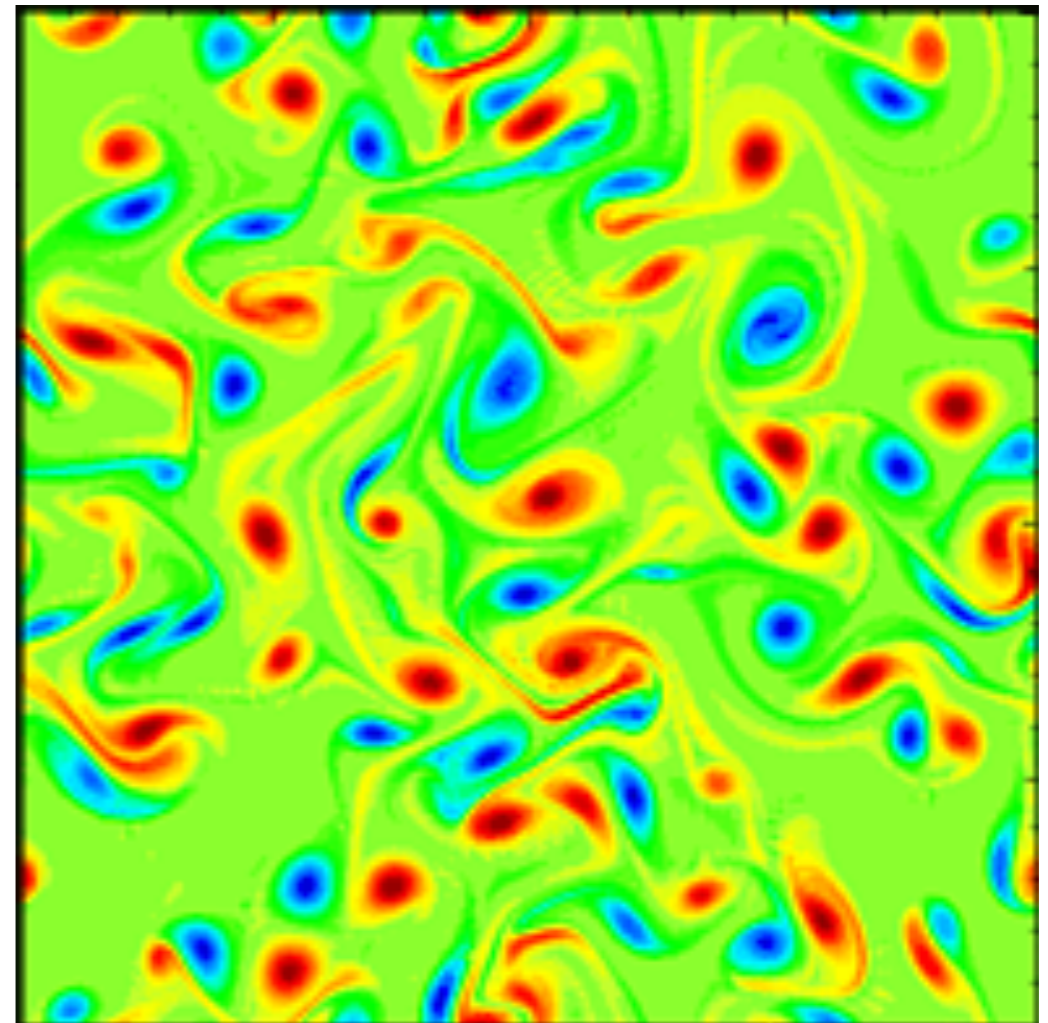
Vortices and the Navier-Stokes equations

Vorticity: $w = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}$

- $w < 0$: fluid spins counterclockwise
- $w > 0$: fluid spins clockwise

Vortices appear to organize the dynamics of fluids: why?

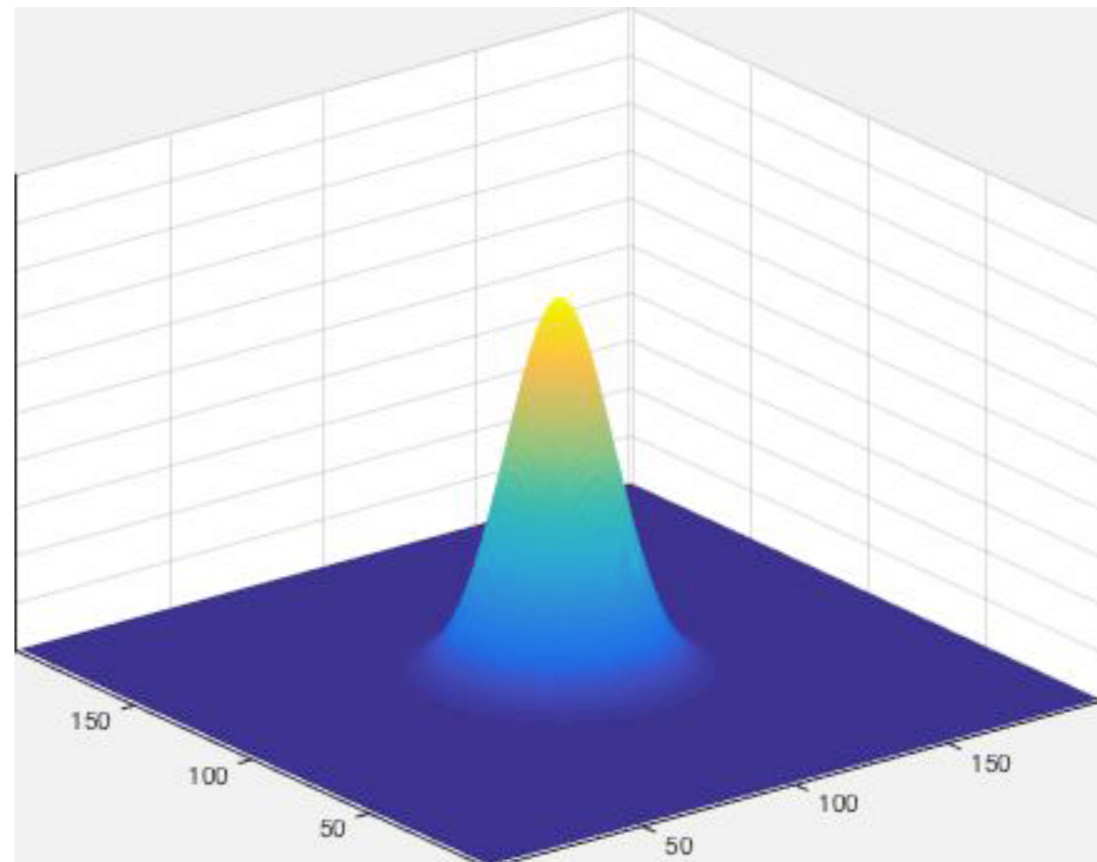
Movie: from fluid dynamics lab at the University of Technology in Eindhoven



Vortices and the Navier-Stokes equations

Behavior of a vortex:

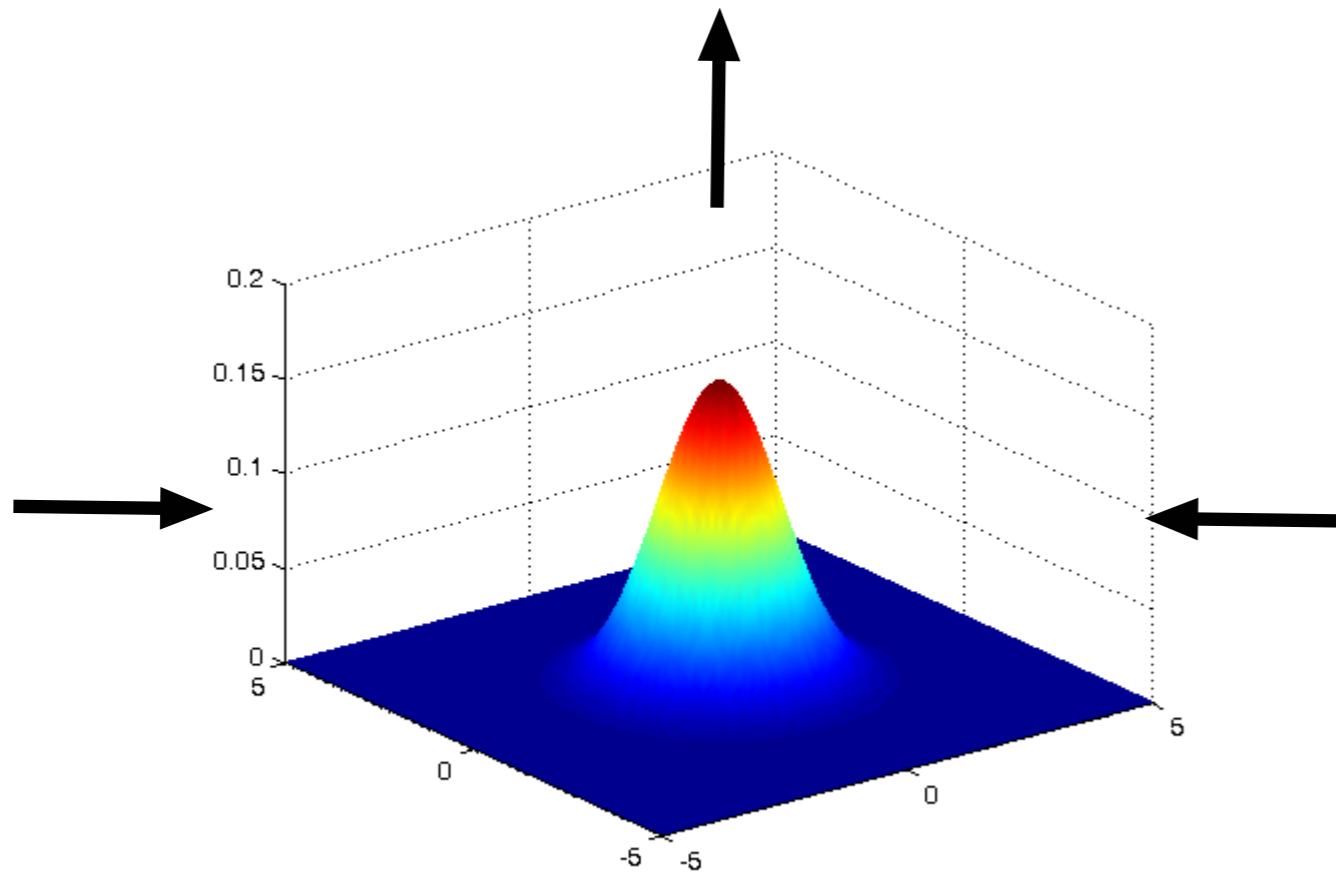
- Stir a fluid at rest, then let it evolve
- Observe a single vortex
- Vortex shell is “Gaussian”: like a symmetric hill
- Vortex will spread out and become weaker as time evolves



Vortices and the Navier-Stokes equations

For the vortex to be an equilibrium

- rescale space to prevent it from spreading out
- rescale the height to prevent it from decaying



Vortex no longer
changes as
time evolves:

**Equilibrium
Vortex**

Vortices and the Navier-Stokes equations

Rewrite the Navier-Stokes equation in these new variables:

$$\frac{\partial \omega}{\partial \tau} = \mathcal{G}(\omega)$$

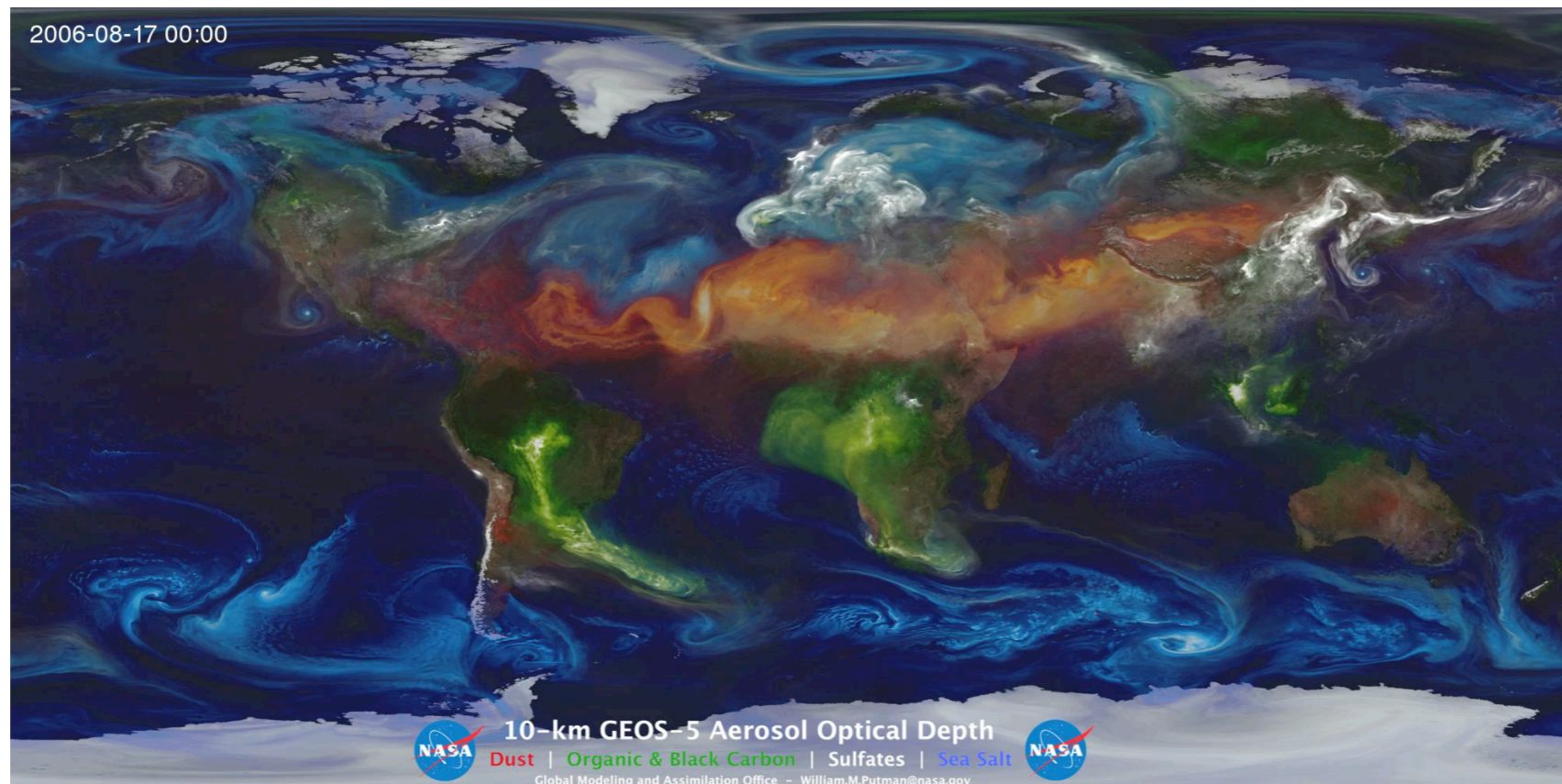
- Equilibrium vortex solution: $\mathcal{G}(\omega_{\text{vortex}}) = 0$
- Equilibrium **vortex is stable**: $\frac{\partial \mathcal{G}}{\partial \omega}(\omega_{\text{vortex}}) < 0$

Intuitively, the Navier-Stokes equation is similar to the previous example of a basic differential equation. We can't solve it, but we've found a stable equilibrium solution: a vortex.

Vortices and the Navier-Stokes equations

Vortices are stable, and attract anything “nearby”. Wherever there is any rotation in the fluid, we can zoom in to that location and see a little vortex. Hence, we are locally “near” a vortex, and so the fluid will become more vortex-like in that location, until outside influences break the local vortex structure.

Result: we see vortices everywhere!



Other physical phenomena that can be viewed as stable equilibria of mathematical models:

John Scott Russell, on solitary waves, or solitons:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; ... assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and ... after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".

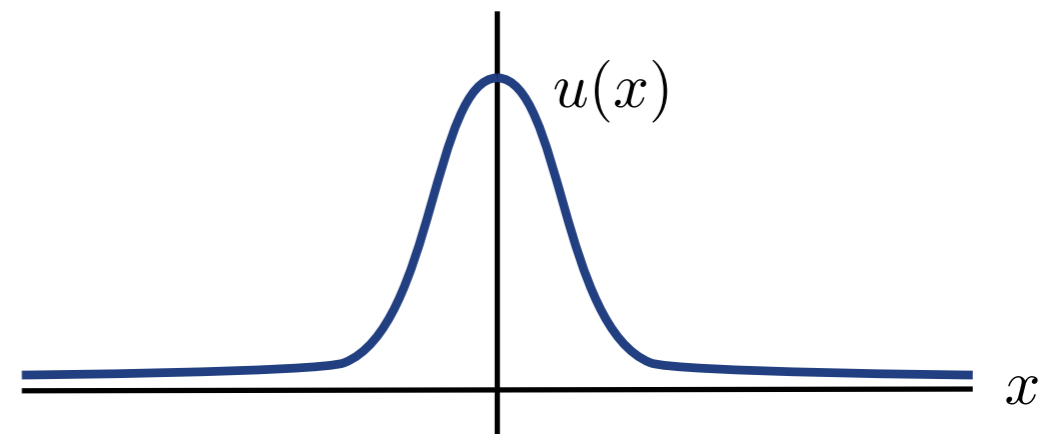
Recreation near Heriot-Watt University in 1995



KdV Equation:

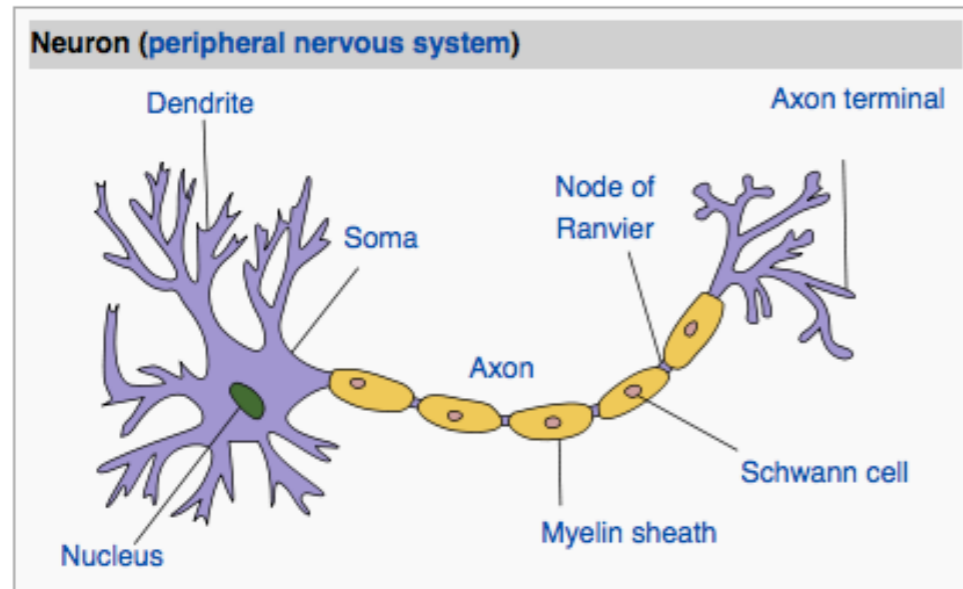
$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3} - u \frac{\partial u}{\partial x}$$

u is the height of the wave in a one-dimensional channel



Other physical phenomena that can be viewed as stable equilibria of mathematical models:

Electrical impulse moving along the axon in a nerve cell:



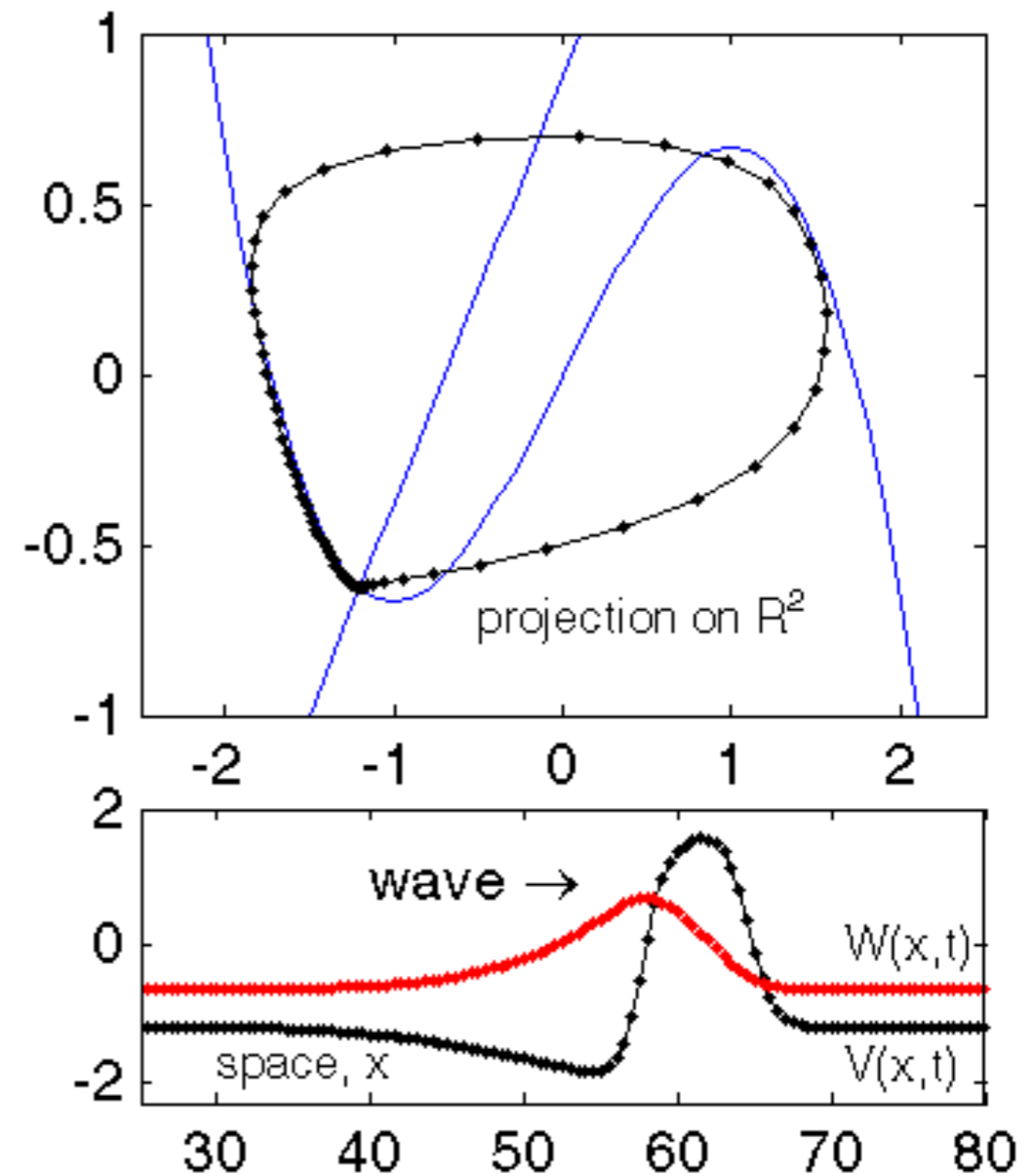
"Neuron Hand-tuned" by Quasar Jarosz

FitzHugh-Nagumo equation:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + f(v) - w$$

$$\frac{\partial w}{\partial t} = \epsilon(u - \gamma w)$$

v is the voltage, and w is a recovery variable modeling negative feedback



Thanks for your
attention!