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Stability of bright solitary-wave solutions to perturbed nonlinear Schrödinger equations

Todd Kapitula [a,1], Björn Sandstede [b,*]

[a] *Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131, USA*

[b] *Weierstraß-Institute for Applied Analysis and Stochastics, Mohrenstraße 39, 10117 Berlin, Germany*

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Abstract

The propagation of pulses in ideal nonlinear optical fibers without loss is governed by the nonlinear Schrödinger equation (NLS). When considering realistic fibers one must examine perturbed NLS equations, with the particular perturbation depending on the physical situation that is being modeled. A common example is the complex Ginzburg–Landau equation (CGL), which is a dissipative perturbation. It is known that some of the stable bright solitons of the NLS survive a dissipative perturbation such as the CGL. Given that a wave persists, it is then important to determine its stability with respect to the perturbed NLS. A major difficulty in analyzing the stability of solitary waves upon adding dissipative terms is that eigenvalues may bifurcate out of the essential spectrum. Since the essential spectrum of the NLS is located on the imaginary axis, such eigenvalues may lead to an unstable wave. In fact, eigenvalues can pop out of the essential spectrum even if the unperturbed problem has no eigenvalue embedded in the essential spectrum. Here we present a technique which can be used to track these bifurcating eigenvalues. As a consequence, we are able to locate the spectrum for bright solitary-wave solutions to various perturbed nonlinear Schrödinger equations, and determine precise conditions on parameters for which the waves are stable. In addition, we show that a particular perturbation, the parametrically forced NLS equation, supports stable multi-bump solitary waves. The technique for tracking eigenvalues which bifurcate from the essential spectrum is very general and should therefore be applicable to a larger class of problems than those presented here. © 1998 Elsevier Science B.V.

1. Introduction

The standard model for the propagation of pulses in an ideal nonlinear fiber without loss is the cubic nonlinear Schrödinger equation (NLS)

$$i\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi = 0 \tag{1.1}$$

* Corresponding author. Tel.: +49 30 20377 588; fax: +49 30 2044975; e-mail: sandstede@wias-berlin.de
[1] URL: http://www.math.unm.edu/~kapitula

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for $x \in \mathbb{R}$ with $\omega > 0$. It is known to support stable pulses given by

$$\Phi(x) = \sqrt{\frac{\omega}{2}}\,\mathrm{sech}(\sqrt{\omega}\,x). \tag{1.2}$$

If loss is present in the fiber, these pulses will cease to exist. Thus, amplifiers have to be used to compensate for the loss. The effects of linear loss in the fiber as well as other perturbations which account for amplifiers located along the fiber will then have to be incorporated into the model. The issue is whether pulses persist under the perturbation and what their stability might be. In this paper, we shall concentrate on the stability of pulses for two different perturbations of (1.1).

The first equation is the cubic–quintic Schrödinger equation (CQNLS)

$$\mathrm{i}\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi + 3\alpha|\phi|^4\phi = 0, \tag{1.3}$$

where $\alpha < 0$ is real and $x \in \mathbb{R}$. The CQNLS is the correct model to describe the propagation of pulses in dispersive materials with either a saturable or higher-order refraction index [8,10]. An optical fiber which satisfies this condition can be constructed, for example, by doping with two appropriate materials [5,29,30]. A physically realistic value for α is $3\alpha \approx -0.1$ [9,13,43], so the CQNLS cannot really be thought of as a small perturbation of the NLS. Eq. (1.3) describes an ideal fiber; therefore, it is natural to consider the perturbed CQNLS (PCQNLS)

$$\mathrm{i}\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi + 3\alpha|\phi|^4\phi = \mathrm{i}\epsilon(d_1\phi_{xx} + d_2\phi + d_3|\phi|^2\phi + d_4|\phi|^4\phi) \tag{1.4}$$

for $x \in \mathbb{R}$ where $\epsilon > 0$ is small and the other parameters are real and of O(1). The nonnegative parameter d_1 describes spectral filtering, d_2 describes the linear gain ($d_2 > 0$) or loss ($d_2 < 0$) due to the fiber, and d_3 and d_4 describe the nonlinear gain or loss due to the fiber.

The second equation is the parametrically forced Schrödinger equation (PFNLS)

$$\mathrm{i}\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi + \mathrm{i}\epsilon(\gamma\phi - \mu\phi^*) = 0 \tag{1.5}$$

for $x \in \mathbb{R}$ where $\epsilon > 0$ is not necessarily small, $\gamma > 0$ is the dissipation factor (linear loss), $\mu > 0$ is the parametric gain, and $2\omega > 0$ is the phase-mismatch coefficient of the parametric gain. Eq. (1.5) models the effect of linear loss and its compensation by phase-sensitive amplification with nonzero phase-mismatch [11,26,28,31,33]. The PFNLS equation is valid when discussing optical fiber rings in which the length of the fiber loop is much less than the dispersion and loss lengths [33].

Existence of solitary waves is known for these equations; in fact, there is an analytic expression for the wave for each of the above equations [32,33,40,44]. We shall be interested in their stability. Since these equations are posed on the unbounded real line, the spectrum of the linearization about a solitary-wave solution contains essential spectrum corresponding to radiation modes. In addition, the spectrum will contain several isolated eigenvalues of finite multiplicity. In particular, zero is such an eigenvalue by translation invariance. The essential spectrum of the nonlinear Schrödinger equations (1.1) and (1.3) is located on the imaginary axis since these equations are both infinite-dimensional Hamiltonian systems. Eqs. (1.4) and (1.5) are nonconservative perturbations of (1.3) and (1.1), respectively. In order to establish stability, it is necessary to compute the spectrum of the linearization about the solitary-wave solution upon adding dissipative terms. There are standard tools available in order to determine the fate of isolated eigenvalues. However, the essential spectrum is more difficult to handle. While the essential spectrum itself is readily computed upon perturbations [12, Appendix to Section 5], it is possible that eigenvalues may bifurcate from the essential spectrum. It is the problem of detecting such eigenvalues which is the primary issue of the present paper. We emphasize that eigenvalues may pop out of the essential spectrum even if the corresponding eigenfunctions in the unperturbed situation are not localized. Note also that the perturbations mentioned above are in general not bounded.

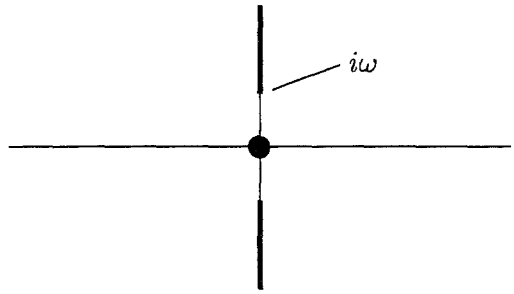

Fig. 1. The spectrum for the NLS. The point $\lambda = 0$ is an isolated eigenvalue with algebraic multiplicity 4. The rest of the spectrum is continuous spectrum, which is the curves $|\mathrm{Im}\,\lambda| \geq \omega$.

The spectrum for the NLS itself is completely understood. The point $\lambda = 0$ is an isolated eigenvalue with geometric multiplicity 2 and algebraic multiplicity 4, and the rest of the spectrum is continuous spectrum, which is the curve $|\mathrm{Im}\,\lambda| \geq \omega$ (see Fig. 1). Furthermore, there are no eigenvalues embedded in the continuous spectrum [23,24]. Now consider the generalized perturbed NLS equation

$$\mathrm{i}\partial_t\phi + (\partial_x^2 - \omega)\phi + 4|\phi|^2\phi + \epsilon f(|\phi|^2)\phi = \mathrm{i}\epsilon(d_1\partial_x^2\phi + R(\phi, \phi^*)), \tag{1.6}$$

where $f(\eta)$ is real-valued and smooth with $f(0) = 0$, $\epsilon \geq 0$, and $R(\mu, \eta)$ is real-valued and smooth. This equation comprises the aforementioned perturbations. We suppose that for $0 \leq \epsilon < \epsilon_0$ there exists a bright solitary-wave solution, $\Phi(x, \epsilon)$, which converges exponentially to zero as $|x| \to \infty$. The first result shows that if eigenvalues bifurcate from the essential spectrum for $\epsilon > 0$, they emanate at $\lambda = \pm\mathrm{i}\omega$. The specific form of the perturbation in (1.6) is actually *not* important for the conclusion of Theorem 1.1 to be true.

Theorem 1.1. Consider the linearization of (1.6) about the solitary wave $\Phi(x, \epsilon)$ for $\epsilon > 0$ sufficiently small. There are then four eigenvalues near zero. Any other isolated eigenvalue is close to $\lambda = \pm\mathrm{i}\omega$. In particular, if eigenvalues bifurcate from the essential spectrum, they do so only near $\lambda = \pm\mathrm{i}\omega$.

Remark 1.2. See Section 4 for a clarification of "close".

This result greatly simplifies our task since it suffices to investigate the region near $\lambda = \mathrm{i}\omega$ in order to detect eigenvalues popping out of the essential spectrum. In particular, it is not possible for eigenvalues to bifurcate from infinity.

For the PCQNLS, two of the eigenvalues near zero will leave the origin and be real and of $\mathrm{O}(\epsilon)$, while the other two will remain at the origin. Recently, Kapitula [19] was able to determine the location of the $\mathrm{O}(\epsilon)$ eigenvalues, and showed that in a certain region of the (d_1, d_2, d_3, d_4) parameter space they both move into the left-half of the complex plane. The continuous spectrum also moves into the left half-plane under perturbation provided $d_1 > 0$ and $d_2 < 0$. In this paper, we show that the eigenvalues do not bifurcate from the essential spectrum near $\lambda = \pm\mathrm{i}\omega$. Therefore, there exist stable pulses for the PCQNLS.

Theorem 1.3. Suppose that $d_2, d_4 < 0$, $d_3 > d_1 > 0$, and $(d_3 - d_1)^2 > \frac{24}{5}d_2d_4$. Let

$$\omega_\pm = \frac{5}{4|d_4|}\left(d_3 - d_1 \pm \sqrt{(d_3 - d_1)^2 - \frac{24}{5}d_2d_4}\right). \tag{1.7}$$

Then, for any $\epsilon > 0$ sufficiently small and any $\alpha \le 0$ with $|\alpha|$ sufficiently small, there exist solitary-wave solutions $\Phi_\pm$ of the PCQNLS for $\omega = \omega_\pm$ which are close to $\sqrt{\omega_\pm/2}\,\mathrm{sech}(\sqrt{\omega_\pm}\,x)$. The solitary wave Φ_- existing for $\omega = \omega_-$ is unstable, while the wave Φ_+ for $\omega = \omega_+$ is orbitally exponentially stable, i.e., if $\|\phi_0 - \Phi_+\|$ is sufficiently small, then there exists a $b > 0$ and constants $\tau, \theta \in \mathbb{R}$ such that $\|\phi(t,\cdot) - \Phi_+(\cdot + \tau)\mathrm{e}^{\mathrm{i}\theta}\| \le C\mathrm{e}^{-bt}$ for $t \ge 0$. Here $\|\cdot\|$ denotes the L^2-norm.

Remark 1.4. The theorem is also true when $\alpha > 0$ with $\alpha = \mathrm{O}(\epsilon^\gamma)$ for some $\gamma > \frac{1}{2}$, and we refer to Section 6 for details.

Remark 1.5. It is shown in Lemma 5.1 that an eigenvalue bifurcates out of the continuous spectrum if $\alpha > 0$ and $\epsilon = 0$. The wave will be stable for $1 \gg \alpha \ge \mathrm{O}(\sqrt{\epsilon}) > 0$ if it can be shown that this eigenvalue moves into the left half-plane for $\epsilon > 0$. Theorems 4.2 and 4.4 provide the framework for this calculation, which we leave to the interested reader.

The assumptions of the above result are reminiscent of an energy balance. The constants $d_1 > 0$ and $d_2 < 0$ correspond to dissipation of energy, while $d_3 > 0$ relates to a nonlinear gain of energy. Note that ω is related to the amplitude of the solitary wave. The wave with smaller amplitude ω_- is then always unstable, while the wave with larger amplitude ω_+ is stabilized due to nonlinear saturation (or loss) in the fiber represented by $d_4 < 0$. The amplitude of the stable pulse depends inversely on the strength of the nonlinear fifth-order saturation in the fiber. Finally, Eq. (1.7) reveals that if d_2 approaches $5(d_3 - d_1)^2/24d_4$, then the two waves coalesce and disappear in a saddle-node bifurcation.

If an initial value is chosen near the family (1.2) of solitary waves of the NLS such that its amplitude is larger than ω_-, then it is expected that the associated solution of the PCQNLS converges exponentially fast towards the stable pulse. It can be conjectured that the optical fiber therefore supports in a stable fashion only those pulses of the form $\mathrm{e}^{-\mathrm{i}(\omega_+ t + \beta)}\Phi(x)$, where ω_+ is given in (1.7) and $\beta \in \mathbb{R}$ is arbitrary.

We remark that the PCQNLS with $\alpha = 0$ has been investigated numerically. In [2], the region in the (d_1, d_2, d_3, d_4) parameter space inside which stable pulses exist has been explored numerically. The simulations presented in [2] are in good agreement with the theoretical results provided by Theorem 1.3. We refer to [2] for more details on the physical importance of the existence of stable pulses for the PCQNLS.

Now consider the PFNLS

$$\mathrm{i}\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi + \mathrm{i}\epsilon(\gamma\phi - \mu\phi^*) = 0.$$

Its solitary-wave solution is given by

$$\Phi(x,\omega,\epsilon) = \sqrt{\frac{\omega + \epsilon\mu\sin 2\theta}{2}}\,\mathrm{sech}(\sqrt{\omega + \epsilon\mu\sin 2\theta}\,x)\exp(i\Theta), \tag{1.8}$$

where

$$\cos 2\theta = \frac{\gamma}{\mu}.$$

When considering the PFNLS, three of the eigenvalues will leave the origin and be of $\mathrm{O}(\epsilon)$, and only one will remain. The reason that an extra eigenvalue leaves the origin is due to the fact that $\mu > 0$ breaks the rotational symmetry of the NLS. The location of the $\mathrm{O}(\epsilon)$ eigenvalues is known for all $\epsilon > 0$ [4,27]. If $\mu\sin 2\theta < 0$, then there will be a positive real eigenvalue, while if $\mu\sin 2\theta > 0$, there will be an eigenvalue located at $\lambda = -2\epsilon\gamma$ and a complex conjugate pair located on the line $\mathrm{Re}\,\lambda = -\epsilon\gamma$. When $\mu\sin 2\theta > 0$, we will locate any eigenvalues which move out of the continuous spectrum, at least for $\epsilon > 0$ sufficiently small. In particular, we will show that

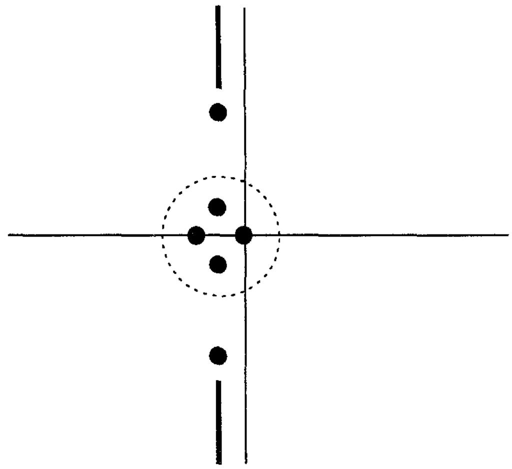

Fig. 2. The spectrum for the PFNLS for $\mu \sin 2\theta > 0$. There are four eigenvalues ϵ-close to the origin and two eigenvalues which are ϵ^2-close to the points $-\epsilon\gamma \pm \mathrm{i}\omega$ on the line $\mathrm{Re}\,\lambda = -\epsilon\gamma$. In fact, the spectrum is always symmetric with respect to the reflection across the line $\mathrm{Re}\,\lambda = -\epsilon\gamma$.

only one complex conjugate pair leaves the continuous spectrum for $\epsilon > 0$ sufficiently small. Due to the symmetries associated with PFNLS, we will then be able to conclude that these eigenvalues are located on the line $\mathrm{Re}\,\lambda = -\epsilon\gamma$. We refer to Fig. 2 for the spectrum of the linearization of the PFNLS about the solitary wave.

When considering the PFNLS, having the spectrum located in the left half-plane is *not* sufficient to be able to conclude that the wave is stable. The PFNLS only generates a C^0-semigroup, so that the Spectral Theorem does not in general hold. Therefore, besides locating the spectrum, additional resolvent estimates, which are presented in Section 7, are necessary to prove the following stability theorem.

Theorem 1.6. Consider the PFNLS. If $\epsilon > 0$ is sufficiently small and $\mu \sin 2\theta > 0$, then the wave is orbitally exponentially stable, i.e., if $\|\phi_0 - \Phi\|$ is sufficiently small, then there exists a $b > 0$ and a constant $\tau \in \mathbb{R}$ such that $\|\phi(t, \cdot) - \Phi(\cdot + \tau)\| \leq C\mathrm{e}^{-bt}$ for $t \geq 0$.

Physically, this result implies that an optical storage loop under phase-sensitive amplification supports stable pulses. In fact, the pulse is exponentially stable. We therefore expect that additional noise in the fiber will not greatly affect the pulse. In [35], the effects of noise due to quantum fluctuations and guided acoustic-wave Brillouin scattering have been explored numerically for amplifiers with zero phase-mismatch, and the pulses were still found to be stable.

Numerical simulations in [6] show that the stable pulse destabilizes for larger values of ϵ due to a Hopf bifurcation. This bifurcation is created in the following way. As mentioned before, there are four eigenvalues located on the line $\mathrm{Re}\,\lambda = -\epsilon\gamma$ for $\epsilon > 0$ small (see Fig. 2). As ϵ increases these eigenvalues coalesce, leave the line $\mathrm{Re}\,\lambda = -\epsilon\gamma$, and finally two of them cross the imaginary axis. Note that two of the eigenvalues located on the line have popped out of the essential spectrum.

It turns out that adding a quintic term (representing higher-order saturation in the optical fiber) actually inhibits these eigenvalues from bifurcating from the essential spectrum. Thus, consider the PFNLS with an added quintic term, henceforth known as the parametrically forced cubic–quintic nonlinear Schrödinger equation (PFCQNLS):

$$\mathrm{i}\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi + 3\alpha|\phi|^4\phi + \mathrm{i}\epsilon(\gamma\phi - \mu\phi^*) = 0.$$

This equation describes the periodic parametric (phase-sensitive) amplification of solitary waves for fibers with a saturable or higher-order refraction index. It can be thought of as encompassing the effects of both the CQNLS and the PFNLS. It turns out that a balancing of the quintic term α with the forcing amplitude ϵ will control the number of eigenvalues which move out of the continuous spectrum. Specifically, as a consequence of Lemma 5.6, if $0 < |\alpha\omega|, \epsilon \ll 1$ and

$$\alpha < -\frac{12\mu \sin 2\theta}{\omega^2}\epsilon,$$

then no eigenvalues bifurcate out of the continuous spectrum. Otherwise, the picture is exactly as that given in Fig. 2. As far as we know, this balancing effect between the parametric forcing and possibly stabilizing effect of a negative α has not yet been documented in the literature.

We have shown above that the PFNLS admits a stable solitary wave having a single hump. Of interest is then whether there are stable multiple solitary waves. These are pulses having several humps, i.e., are concatenated copies of the single-hump solitary wave. In order for stable multiple (or multi-hump) solitary waves to exist, the interaction between single-hump waves must be suppressed. We show that stable multiple pulses exist provided spectral filtering is added to the physical situation governed by the PFNLS. Under this scenario, the perturbed equation is given by

$$\mathrm{i}\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi + \mathrm{i}\epsilon(\gamma\phi - \mu\phi^*) = \mathrm{i}\delta\phi_{xx}, \tag{1.9}$$

where $\delta > 0$.

Multiple solitary-wave solutions to (1.9) are formally constructed by concatenating N widely spaced copies of Φ or $-\Phi$, where $\Phi = \Phi_\delta$ is an $\mathrm{O}(\delta)$ correction to the expression given in (1.8). Since Φ and $-\Phi$ are concatenated, N-pulses can be obtained in a variety of ways. Denoting Φ and $-\Phi$ by "up" and "down", respectively, we may then consider arbitrary sequences of ups and downs corresponding to whether Φ or $-\Phi$ is used. Based upon an application of the work of Sandstede et al. [42, Theorems 1, 2, and 4], we have the following theorem concerning existence and stability of multiple solitary waves of (1.9).

Theorem 1.7. Fix $\epsilon > 0$ small and $N > 1$. Suppose that $\mu \sin 2\theta > 0$. For any $0 < \delta < \delta(\epsilon, N) \ll 1$ small there exists a unique multiple solitary wave of up–down–up–down–$\cdots$ type. These pulses are orbitally exponentially stable with respect to Eq. (1.9). Any other N-pulse consisting of copies of Φ or $-\Phi$ is unstable.

Remark 1.8. In fact, multiple solitary waves of up–down–up–$\cdots$ type exist for all $0 < \epsilon < \omega/\mu \sin 2\theta$ and for $\delta > 0$ small, and they are stable as long as Φ is stable when $\delta = 0$ (see Remark 7.7).

By Theorem 1.6, the condition $\mu \sin 2\theta > 0$ means that the primary pulse is stable. Therefore, if spectral filtering is used in the storage loop in addition to phase-sensitive amplification, the loop will also support stable multiple pulses. These pulses can be used for a more efficient data-encoding scheme replacing binary encoding (see [35]).

Having discussed the main motivation and results of this paper, we now turn to an outline of our approach. As mentioned above, the primary issue is the detection of eigenvalues which are either embedded in the essential spectrum or which bifurcate from the essential spectrum upon adding perturbations. The major tool which we use to accomplish this task is the Evans function (sometimes referred to as a transmission coefficient). The Evans function $E(\lambda)$ is a complex-valued function depending on $\lambda \in \mathbb{C}$ with the property that $E(\lambda) = 0$ whenever λ is an isolated eigenvalue. The Evans function is a priori defined only away from the essential spectrum; thus, it is not immediately clear that it can be used to locate embedded eigenvalues.

The issue of locating embedded eigenvalues is not new, and has been investigated by several authors. In doing a study of the stability of singular traveling waves to the FitzHugh–Nagumo equation, Jones [14] extended the Evans function through the essential spectrum in an analytic fashion. Pego and Weinstein [36] applied Jones' idea to the KdV equation and other related systems. The interested reader can also consult Jones et al. [15], Kapitula [16–18], and Rubin [39] for other problems in which an extended Evans function has been used in stability calculations.

In all these papers, an essential idea was to show that the extended Evans function had no zeros. Using continuity properties, it could then be concluded that the Evans functions stayed nonzero after adding perturbations, and therefore no eigenvalues bifurcated from the essential spectrum. The last is due to the fact that any isolated eigenvalue contributes a zero to the Evans function.

It is instructive to take a moment to understand the scenario under which the Evans function has been extended across the continuous spectrum. Writing the eigenvalue equation under consideration as a first-order system, one obtains

$$Y' = M(\lambda, x)Y, \quad Y \in \mathbb{R}^n,$$

where the matrix $M(\lambda, x)$ is analytic in λ. Since the solitary wave converges to a constant state as $|x| \to \infty$, the matrix $M_0(\lambda) = \lim_{|x|\to\infty} M(\lambda, x)$ exists and is also analytic in λ. By Henry's result [12, Appendix to, Section 5], λ is in the essential spectrum if, and only if, $M_0(\lambda)$ has eigenvalues on the imaginary axis. The Evans function is a priori defined only if the eigenvalues of $M_0(\lambda)$ have nonzero real part. Jones [14] was able to extend the Evans function across the essential spectrum under the condition that $M_0(\lambda)$ has precisely one eigenvalue with positive real part when λ is to the right of the essential spectrum (see [15–18,39] for an incremental generalization of the technique presented in [14]). If there are several eigenvalues of $M_0(\lambda)$ on the imaginary axis for λ in the essential spectrum, and if these eigenvalues do not all move into the same half plane when λ moves off the essential spectrum, this method fails. In particular, the method is not applicable to equations of Schrödinger-type.

The most theoretical part of this paper, namely Section 2, is concerned with constructing an extension of the Evans function for equations of Schrödinger-type. The extension is carried out under very general assumptions and is therefore applicable to quite general equations. It is quite technical, and on a first reading the reader may only wish to study the introduction to the section and Section 2.5. In Section 3, we explicitly calculate the extended Evans function for the nonlinear Schrödinger equation. An expansion of the Evans function near $\lambda = \pm i\omega$ and expressions for possibly bifurcating eigenvalues for perturbations of the cubic NLS are given in Section 4. In Section 5, these results are used to calculate eigenvalues bifurcating from the essential spectrum near its end points $\lambda = \pm i\omega$ for the cubic–quintic and the parametrically forced Schrödinger equation. Finally, we return to Eqs. (1.4) and (1.5) in Sections 6 and 7, respectively. The latter section also contains the proof of the existence of stable N-pulses for (1.9).

Remark 1.9. After this paper was submitted, we learned that Gardner and Zumbrun [7] independently and simultaneously obtained similar results on the extension of the Evans function.

2. The extension of the Evans function

In this section, the Evans function is extended across the essential spectrum. The extension is first developed for λ in compact sets. We then consider the case of large λ and show that the extended Evans function does not have any zeros for sufficiently large $|\lambda|$.

We shall first outline the basic ideas. Consider a linear equation

$$\mathbf{Y}' = A(\lambda, x)\mathbf{Y} = (A_0(\lambda) + R(\lambda, x))\mathbf{Y} \tag{2.1}$$

which is assumed to be the linear eigenvalue problem written as a first-order system. The parameter λ represents a prospective eigenvalue, and the $n \times n$ matrix $R(\lambda, x)$ is assumed to tend to zero exponentially fast as $|x| \to \infty$.

If λ is not in the essential spectrum, the matrix $A_0(\lambda)$ has no imaginary eigenvalues. Assume that $A_0(\lambda)$ has k eigenvalues with positive real part and $n-k$ eigenvalues with negative real part. It is then possible to define analytic solutions $\mathbf{Y}_i(\lambda, x)$ such that

$$\lim_{x\to-\infty} |\mathbf{Y}_j(\lambda, x)| \to 0 \text{ and } \mathbf{Y}_j(\lambda, 0) \text{ are linearly independent for } j = 1, \ldots, k,$$
$$\lim_{x\to+\infty} |\mathbf{Y}_l(\lambda, x)| \to 0 \text{ and } \mathbf{Y}_l(\lambda, 0) \text{ are linearly independent for } l = k+1, \ldots, n.$$

Therefore, it is possible to define the subspaces $E^{\mathrm{u}}(\lambda) = \mathrm{Span}\{\mathbf{Y}_1(\lambda, 0), \ldots, \mathbf{Y}_k(\lambda, 0)\}$ and $E^{\mathrm{s}}(\lambda) = \mathrm{Span}\{\mathbf{Y}_{k+1}(\lambda, 0), \ldots, \mathbf{Y}_n(\lambda, 0)\}$ of initial values leading to solutions of (2.1) which are exponentially decaying in backward or forward time, respectively. Hence, λ is an eigenvalue if, and only if, the spaces $E^{\mathrm{u}}(\lambda)$ and $E^{\mathrm{s}}(\lambda)$ have a nontrivial intersection, as this leads to a solution (the eigenfunction) of (2.1) which decays in both time-directions.

Following Alexander et al. [3], the Evans function is given by

$$E(\lambda) = \mathbf{Y}_1(\lambda, 0) \wedge \cdots \wedge \mathbf{Y}_n(\lambda, 0),$$

where the wedge product of n vectors measures the n-dimensional volume of the cube spanned by these vectors. In other words, the Evans function is a measurement of how far away the spaces $E^{\mathrm{u}}(\lambda)$ and $E^{\mathrm{s}}(\lambda)$ are from intersecting. Indeed, if there is any intersection, the vectors appearing in the above formula are no longer independent. The cube is then no longer n-dimensional, and hence its volume is zero. Therefore, the Evans function $E(\lambda)$ is zero for λ outside the essential spectrum if, and only if, λ is an eigenvalue.

If λ is in the essential spectrum, the matrix $A_0(\lambda)$ has imaginary eigenvalues and the above construction breaks down. The idea is to then analytically extend the spaces $E^{\mathrm{u}}(\lambda)$ and $E^{\mathrm{s}}(\lambda)$ so that they include solutions of (2.1) which either exponentially decay or which converge exponentially fast to eigenvectors corresponding to the imaginary eigenvalues of $A_0(\lambda)$. It is important to divide the set of imaginary eigenvalues into two disjoint groups, one of which is used for $E^{\mathrm{u}}(\lambda)$ and the other which is used for $E^{\mathrm{s}}(\lambda)$. Also, in order to make this construction unique, the exponential convergence towards eigenvectors must be fast enough. In the next section, we construct these solutions.

2.1. Rapidly decaying solutions of linear equations

In this section, we consider a linear system

$$u' = (B(\lambda, \mu) + R(\mu, x))u, \tag{2.2}$$

where $u \in \mathbb{C}^n$, $(\lambda, \mu) \in \Omega \times \mathbb{R}^p$, and $x \in \mathbb{R}$. Here, $\Omega \subset \mathbb{C}$ is open. Assume that the following condition is satisfied.

Hypothesis 2.1. There exists a vector $\eta(\lambda, \mu)$ such that $B(\lambda, \mu)\eta(\lambda, \mu) = 0$ for all (λ, μ) and $|\eta(\lambda, \mu)| \le M$ for some M. Moreover, there are numbers $K_1 \ge 1$, $K_2 \ge 0$, $\beta \in \mathbb{R}$, and $\gamma > 0$ with $\gamma > \beta$ such that

$$\|\mathrm{e}^{B(\lambda,\mu)x}\| \le K_1 \mathrm{e}^{\beta x}, \quad x \in \mathbb{R}, \qquad \|R(\mu, x)\| \le K_2 \mathrm{e}^{\gamma x}, \quad x \le 0.$$

We then have the following result which characterizes solutions decaying with the exponential rate γ to $\eta(\lambda, \mu)$ as $x \to -\infty$. It is necessary for the proofs in the subsequent sections.

Lemma 2.2. Assume that Hypothesis 2.1 is true. There exists a unique solution $u(\lambda, \mu)(x)$ of (2.2) defined for $x \le 0$ such that there exists a constant C with

$$|u(\lambda, \mu)(x) - \eta(\lambda, \mu)| \le C\mathrm{e}^{\gamma x}$$

as $x \to -\infty$. In addition, we have

$$|u(\lambda, \mu)(x) - \eta(\lambda, \mu)| \le \frac{2K_1K_2M}{\gamma - \beta}$$

uniformly for $x \in (-\infty, x_0]$ with $x_0 \le 0$ such that

$$\frac{K_1K_2M}{\gamma - \beta}\mathrm{e}^{\gamma x_0} \le \frac{1}{2}.$$

Furthermore, $u(\lambda, \mu)$ is analytic in λ if B and η are. Similarly, if B, η, and R are C^m in μ for some $m \ge 0$ and

$$\left\| \frac{\mathrm{d}^j}{\mathrm{d}\mu^j} R(\mu, x) \right\| \le C_j \mathrm{e}^{\gamma x}, \quad x \le 0$$

for $j = 1, \dots, m$, then $u(\lambda, \mu)$ is C^k in μ.

Proof. We seek the desired solution $u(\lambda, \mu)$ in the form $u(\lambda, \mu)(x) = \eta(\lambda, \mu) + v(x)$. The function v will be sought as a solution of the integral equation

$$v(x) = \int_{-\infty}^{x} \mathrm{e}^{B(\mu,\lambda)(x-y)} R(\mu, y)(\eta(\lambda, \mu) + v(y))\, \mathrm{d}y \tag{2.3}$$

for $x \in (-\infty, x_0]$ with $x_0 \le 0$, see also [36, Proposition 1.2]. Note that any solution v of (2.3) satisfies (2.2) by Hypothesis 2.1. We have

$$\left| \int_{-\infty}^{x} \mathrm{e}^{B(\mu,\lambda)(x-y)} R(\mu, y)\eta(\lambda, \mu)\, \mathrm{d}y \right| \le K_1K_2 \int_{-\infty}^{x} \mathrm{e}^{\beta(x-y)} \mathrm{e}^{\gamma y} |\eta(\lambda, \mu)|\, \mathrm{d}y \le \frac{K_1K_2M}{\gamma - \beta}\mathrm{e}^{\gamma x}.$$

Similarly, we obtain

$$\left| \int_{-\infty}^{x} \mathrm{e}^{B(\lambda,\mu)(x-y)} R(\mu, y)v(y)\, \mathrm{d}y \right| \le K_1K_2 \int_{-\infty}^{x} \mathrm{e}^{\beta(x-y)} \mathrm{e}^{\gamma y} |v(y)|\, \mathrm{d}y \le \frac{K_1K_2}{\gamma - \beta}\mathrm{e}^{\gamma x}\|v\|, \tag{2.4}$$

where

$$\|v\| := \sup_{y \le x_0} |v(y)|.$$

Set

$$V := C^0(-\infty, x_0).$$

The integral equation (2.3) can be written in the function space V as

$$v = F(\lambda, \mu)(\eta(\lambda, \mu) + v), \tag{2.5}$$

with

$$\|F(\lambda, \mu)v\| \le \frac{K_1K_2}{\gamma - \beta}\mathrm{e}^{\gamma x_0}\|v\|, \qquad \|F(\lambda, \mu)\eta(\lambda, \mu)\| \le \frac{K_1K_2M}{\gamma - \beta}.$$

Choose $x_0 \le 0$ such that

$$\frac{K_1 K_2}{\gamma - \beta} e^{\gamma x_0} \le \frac{1}{2}$$

so that $\|F(\lambda, \mu)\| \le \frac{1}{2}$ in the operator norm on V. Since $F(\lambda, \mu)$ is then a uniform contraction, we can solve (2.5) and obtain the fixed point v:

$$v = (\mathrm{id} - F(\lambda, \mu))^{-1} F(\lambda, \mu)\eta(\lambda, \mu).$$

In particular, we have

$$\|v\| \le 2\|F(\lambda, \mu)\eta(\lambda, \mu)\| \le \frac{2K_1 K_2 M}{\gamma - \beta}.$$

The estimates appearing in the lemma follow now immediately using (2.4). Finally, the statements about the dependence of the fixed point v on the parameters (λ, μ) are true since the operator $F(\lambda, \mu)$ is analytic in λ and C^m in μ. □

2.2. *Extension for λ in bounded sets*

We shall construct the stable and unstable subspaces $E^{\mathrm{u}}(\lambda)$ and $E^{\mathrm{s}}(\lambda)$ when λ is in the essential spectrum. Consider the linear system

$$\mathbf{Y}' = A(\lambda, x)\mathbf{Y}, \tag{2.6}$$

where $\mathbf{Y} \in \mathbb{C}^n$, and the matrix A is analytic in λ for each fixed x. Here, $\lambda \in \Omega$ where Ω will be specified later in (2.10).

Hypothesis 2.3. Assume that there exists a constant $\kappa > 0$ and matrices $A_\pm(\lambda)$ such that $A(\lambda, x) - A_\pm(\lambda)$ is independent of λ and

$$\lim_{x \to \pm\infty} |A(\lambda, x) - A_\pm(\lambda)| e^{\pm 5\kappa x} \le C, \tag{2.7}$$

where $C > 0$ is a fixed constant.

We begin with some hypotheses on the asymptotic matrices $A_\pm(\lambda)$.

Hypothesis 2.4. If $\mathrm{Re}\,\lambda > 0$, then for some $1 \le k < n$ both $A_\pm(\lambda)$ have k eigenvalues of positive real part and $n - k$ eigenvalues with negative real part.

For $\mathrm{Re}\,\lambda > 0$, define

$$\begin{aligned} \sigma_\pm^{\mathrm{u}}(\lambda) &= \sigma(A_\pm(\lambda)) \cap \{\mu \in \mathbb{C};\ \mathrm{Re}\,\mu > 0\}, \\ \sigma_\pm^{\mathrm{s}}(\lambda) &= \sigma(A_\pm(\lambda)) \cap \{\mu \in \mathbb{C};\ \mathrm{Re}\,\mu < 0\} \end{aligned} \tag{2.8}$$

to be the sets corresponding to the k $(n - k)$ eigenvalues of $A_\pm(\lambda)$ with positive (negative) real part.

Hypothesis 2.5. Let

$$\Gamma = \bigcup_{j=1}^{N} (\mathrm{i}a_j, \mathrm{i}b_j) \subset \mathrm{i}\mathbb{R},$$

where $a_j \le b_j \le a_{j+1}$ for $j = 1, \dots, N$ are real numbers, be such that if $\lambda \in \Gamma$, then the spectrum of $A_\pm(\lambda)$ is the disjoint union of two sets which are again denoted by $\sigma_\pm^{\mathrm{u}}(\lambda)$ and $\sigma_\pm^{\mathrm{s}}(\lambda)$. Moreover, $\sigma_\pm^{\mathrm{u}}(\lambda)$ and $\sigma_\pm^{\mathrm{s}}(\lambda)$ are the limits of $\sigma_\pm^{\mathrm{u}}(\tilde\lambda)$ and $\sigma_\pm^{\mathrm{s}}(\tilde\lambda)$, respectively, as $\tilde\lambda \to \lambda$ with $\mathrm{Re}\,\tilde\lambda > 0$.

If $\lambda = \mathrm{i}\tau \in \Gamma$, it therefore is required that the spectrum of $A_\pm(\lambda)$ is the disjoint union of $\sigma_\pm^{\mathrm{u}}(\mathrm{i}\tau)$ and $\sigma_\pm^{\mathrm{s}}(\mathrm{i}\tau)$. As a consequence, for fixed τ, there are neighborhoods $U_\pm^{\mathrm{u}}$ and $U_\pm^{\mathrm{s}}$ of $\sigma_\pm^{\mathrm{u}}(\mathrm{i}\tau)$ and $\sigma_\pm^{\mathrm{s}}(\mathrm{i}\tau)$, respectively, in $\mathbb{C}$ such that any eigenvalue of $A_\pm(\tilde\lambda)$ is contained in either U^{u} or U^{s} for any $\tilde\lambda$ close to λ. Indeed, eigenvalues depend continuously on parameters [22]. Hypothesis 2.5 then states that for all $\tilde\lambda$ close to λ with $\mathrm{Re}\,\tilde\lambda > 0$ any eigenvalue of $A_\pm(\tilde\lambda)$ which lies in U^{u} (U^{s}) has positive (negative) real part. In other words, the sets $\sigma_\pm^{\mathrm{u}}(\lambda)$ and $\sigma_\pm^{\mathrm{s}}(\lambda)$, which were originally defined for $\mathrm{Re}\,\lambda > 0$, can be continued as disjoint sets for λ in an open neighborhood of Γ, see Fig. 4.

In particular, there are numbers $\delta_j(\lambda) \ge 0$, $j = 1, \dots, n$, such that for any $\lambda \in \tilde\Sigma_j$ defined by

$$\tilde\Sigma_j := \{\lambda: a_j < \mathrm{Im}\,\lambda < b_j,\ -\delta_j(\lambda) < \mathrm{Re}\,\lambda \le 0\}$$

the spectrum of $A_\pm(\lambda)$ is the disjoint union of two sets $\sigma_\pm^{\mathrm{u}}(\lambda)$ and $\sigma_\pm^{\mathrm{s}}(\lambda)$ which are the continuation of $\sigma_\pm^{\mathrm{u}}(\mathrm{i}\tau)$ and $\sigma_\pm^{\mathrm{s}}(\mathrm{i}\tau)$ for $\mathrm{i}\tau \in (\mathrm{i}a_j, \mathrm{i}b_j)$.

Set $\Sigma_j \subset \tilde\Sigma_j$ to be such that if $\lambda \in \Sigma_j$, then

$$\min\{\mathrm{Re}\,\mu: \mu \in \sigma_\pm^{\mathrm{u}}(\lambda)\} > -\frac{\kappa}{n}, \qquad \max\{\mathrm{Re}\,\mu: \mu \in \sigma_\pm^{\mathrm{s}}(\lambda)\} < \frac{\kappa}{n}. \tag{2.9}$$

Finally, set Ω to be

$$\Omega = \left(\bigcup_{j=1}^{N} \Sigma_j\right) \cup \{\lambda: \mathrm{Re}\,\lambda > 0\}. \tag{2.10}$$

Note that Ω is open, simply connected, and $\Gamma \subset \Omega$. Some of the eigenvalues in the sets $\sigma_\pm^{\mathrm{s}}(\mathrm{i}\tau)$ and $\sigma_\pm^{\mathrm{u}}(\mathrm{i}\tau)$ might be contained in the imaginary axis and we will refer to these eigenvalues as those with small real part. Note that their number may depend on the interval (a_j, b_j) in which $\mathrm{i}\tau$ is contained.

The goal of this section is to construct an Evans function for $\lambda \in \Omega$ which is an analytic extension of that constructed by Alexander et al. [3]. Under the current setup, the Evans function is defined only for those λ with positive real part. The following discussion mirrors much of the presentation of Alexander et al. [3].

By setting

$$x = \frac{1}{2\kappa} \ln\left(\frac{1+\tau}{1-\tau}\right),$$

Eq. (2.6) becomes the autonomous system

$$\mathbf{Y}' = A(\lambda, \tau)\mathbf{Y}, \qquad \tau' = \kappa(1-\tau^2), \tag{2.11}$$

where $' = \mathrm{d}/\mathrm{d}\tau$. By Alexander et al. [3] we have the following.

Lemma 2.6. Assuming that Eq. (2.7) holds true, Eq. (2.11) is C^1 on $\mathbb{C}^n \times [-1, +1]$.

If $\mathbf{Y}_1, \dots, \mathbf{Y}_k$ are solutions of (2.6), then $\mathbf{Y}_1 \wedge \cdots \wedge \mathbf{Y}_k$ is a solution of

$$\mathbf{Y}' = A^{(k)}(\lambda, x)\mathbf{Y},$$

where $A^{(k)}(\lambda, x)$ is the linear derivation on $\Lambda^k\mathbb{C}^n$ induced by $A(\lambda, x)$. As above, this equation can be compactified to

$$\mathbf{Y}' = A^{(k)}(\lambda, \tau)\mathbf{Y}, \qquad \tau' = \kappa(1-\tau^2),$$

which is C^1 on $\Lambda^k\mathbb{C}^n \times [-1, +1]$.

Consider the asymptotic systems

$$\mathbf{Y}' = A_{\pm}^{(k)}(\lambda)\mathbf{Y}.$$

The eigenvalues of $A_{\pm}^{(k)}(\lambda)$ are the sums of any k-tuples of eigenvalues of $A_{\pm}(\lambda)$. For $\lambda \in \Omega$, the spectral sets $\sigma_-^{\mathrm{u}}(\lambda)$ and $\sigma_+^{\mathrm{s}}(\lambda)$ are well-defined. The spectral projection of $A_-(\lambda)$ associated with $\sigma_-^{\mathrm{u}}(\lambda)$ is denoted by $P_-^{\mathrm{u}}(\lambda)$. If $\mathrm{Re}\,\lambda > 0$, it is the spectral projection onto the sum of all generalized eigenspaces of eigenvalues with positive real part. Similarly, $P_+^{\mathrm{s}}(\lambda)$ denotes the spectral projection of $A_+(\lambda)$ associated with $\sigma_+^{\mathrm{s}}(\lambda)$. Both projections depend analytically on $\lambda \in \Omega$. Set

$$\alpha_-(\lambda) = \mathrm{trace}(A_-(\lambda)P_-^{\mathrm{u}}(\lambda)), \quad \alpha_+(\lambda) = \mathrm{trace}(A_+(\lambda)P_+^{\mathrm{s}}(\lambda)).$$

In particular, $\alpha_-(\lambda)$ and $\alpha_+(\lambda)$ are analytic in λ. Then $\alpha_-(\lambda)$ equals the sum of the eigenvalues (counted with multiplicity) contained in $\sigma_-^{\mathrm{u}}(\lambda)$. Similarly, $\alpha_+(\lambda)$ is the sum of the eigenvalues which lie in $\sigma_+^{\mathrm{s}}(\lambda)$. If $\mathrm{Re}\,\lambda > 0$, then $\alpha_-(\lambda)$ is the eigenvalue of $A_-^{(k)}(\lambda)$ with largest real part, and $\alpha_+(\lambda)$ is the eigenvalue of $A_+^{(k)}(\lambda)$ with least real part. In addition, if $\mathrm{Re}\,\lambda > 0$, then $\alpha_{\pm}(\lambda)$ are simple eigenvalues.

Set

$$\mathbf{Z}(\lambda, x) = \mathrm{e}^{-\alpha_-(\lambda)x}\mathbf{Y}(\lambda, x). \tag{2.12}$$

Then $\mathbf{Z}(\lambda, x)$ satisfies the ODE

$$\mathbf{Z}' = [A^{(k)}(\lambda, x) - \alpha_-(\lambda)\mathrm{id}]\mathbf{Z},$$

which, as above, can be compactified to

$$\mathbf{Z}' = [A^{(k)}(\lambda, \tau) - \alpha_-(\lambda)\mathrm{id}]\mathbf{Z}, \qquad \tau' = \kappa(1-\tau^2). \tag{2.13}$$

This again is a C^1 system on $\Lambda^k\mathbb{C}^n \times [-1, +1]$. In the invariant plane $\{\tau = -1\}$ this reduces to the autonomous system

$$\mathbf{Z}' = [A_-^{(k)}(\lambda) - \alpha_-(\lambda)\mathrm{id}]\mathbf{Z}.$$

The critical points are the eigenvectors, $\eta_-(\lambda)$, associated with $\alpha_-(\lambda)$, that is,

$$[A_-^{(k)}(\lambda) - \alpha_-(\lambda)\mathrm{id}]\eta_-(\lambda) = 0.$$

Since $\alpha_-(\lambda)$ is a simple eigenvalue of $A_-^{(k)}(\lambda)$ for $\mathrm{Re}\,\lambda > 0$, the associated eigenvector $\eta_-(\lambda)$ depends analytically on λ. However, $\alpha_-(\lambda)$ is not necessarily simple if $\mathrm{Re}\,\lambda \le 0$. Still, there is an analytic continuation of $\eta_-(\lambda)$ for $\lambda \in \Omega$. Indeed, we may choose $\eta_-(\lambda)$ as the $\Lambda^k\mathbb{C}^n$-representative of the generalized eigenspace $R(P_-^{\mathrm{u}}(\lambda))$ associated with the eigenvalues in $\sigma_-^{\mathrm{u}}(\lambda)$.

To be more precise, choose analytic functions $e_1(\lambda), \ldots, e_k(\lambda) \in R(P_-^{\mathrm{u}}(\lambda))$ for $\lambda \in \Omega$ such that these vectors are linearly independent for any $\lambda \in \Omega$. This is clearly possible, since $P_-^{\mathrm{u}}(\lambda)$ is analytic for $\lambda \in \Omega$ and Ω is simply connected. Then define

$$\eta_-(\lambda) := e_1(\lambda) \wedge \cdots \wedge e_k(\lambda) \in \Lambda^k\mathbb{C}^n,$$

and note that $\eta_-(\lambda)$ is analytic and an eigenvector of $A_-^{(k)}(\lambda)$ associated with the eigenvalue $\alpha_-(\lambda)$.

Now linearize (2.13) at the critical point $(\eta_-(\lambda), -1)$. If $\mathrm{Re}\,\lambda > 0$, then there is exactly one unstable eigenvalue, 2κ, and the associated eigenvector lies in the τ-direction. This is the key which has been used in [3] to define the Evans function. Suppose now that $\lambda \in \Sigma_j$ for some j. We claim that if $\lambda \in \Sigma_j$, any eigenvalue of $A_-^{(k)}(\lambda) - \alpha_-(\lambda)$ has real part strictly less than 2κ. Indeed, let β_- be the eigenvalue of $A_-^{(k)}(\lambda)$ with largest real part. Then β_- is the sum of the k eigenvalues of $A_-(\lambda)$ with largest real part. We number the eigenvalues of $A_-(\lambda)$ according to

$$\sigma_\pm^{\mathrm{u}}(\lambda) = \{\sigma_1^\pm(\lambda), \dots, \sigma_k^\pm(\lambda)\}, \qquad \sigma_\pm^{\mathrm{s}}(\lambda) = \{\sigma_{k+1}^\pm(\lambda), \dots, \sigma_n^\pm(\lambda)\}$$

and counted with multiplicity. Then β_- can be estimated by

$$\mathrm{Re}\,\beta_- - \sum_{i\in J^-(\lambda)} \mathrm{Re}\,\sigma_i^-(\lambda) < \frac{k}{n}\kappa, \tag{2.14}$$

where $J^-(\lambda)$ denotes the set of indices $1 \le i \le k$ which correspond to eigenvalues with positive real part. Indeed, for $\lambda \in \Sigma_j$, some of the $\sigma_i^-(\lambda)$ with $i \le k$ may have crossed the imaginary axis. They are then possibly replaced by eigenvalues $\sigma_i^-(\lambda)$ with $i > k$. However, the real part of each of these eigenvalues is less than κ/n by the choice of Σ_j, see (2.9). Therefore, their real parts adds up to at most $(k/n)\kappa$, and (2.14) is proved. Let

$$\beta_c^- = \beta_- - \alpha_-(\lambda).$$

For $\lambda \in \Sigma_j$, using the estimate (2.14) and (2.9), we obtain

$$\mathrm{Re}\,\beta_c^- = \mathrm{Re}\,\beta_- - \sum_{i=1}^{k} \mathrm{Re}\,\sigma_i^-(\lambda) < \frac{2k}{n}\kappa.$$

This proves our claim.

Therefore, if $\lambda \in \Sigma_j$, the unstable eigenvalue with largest real part is 2κ, with the eigenvector still pointing in the τ-direction. Thus, for $\lambda \in \Omega$ the point $(\eta_-(\lambda), -1)$ has a one-dimensional strong unstable manifold. Since the tangent vector to this manifold points in the τ-direction, the manifold can be written as a function of τ, say $\mathbf{Z}_-(\lambda, \tau)$, for $-1 \le \tau \ll 0$. It follows from Lemma 2.2 that $\mathbf{Z}_-(\lambda, \tau)$ is analytic in λ for $\lambda \in \Omega$. By applying the flow associated with (2.13), the solution $\mathbf{Z}_-(\lambda, \tau)$ is well-defined and analytic in λ for $\tau \in [-1, +1)$. By Eq. (2.12), this then defines a solution

$$\mathbf{Y}_-(\lambda, x) = \mathbf{Z}_-(\lambda, x)\mathrm{e}^{\alpha_-(\lambda)x},$$

which has the property that if $\mathrm{Re}\,\lambda > 0$, then $|\mathbf{Y}_-(\lambda, x)| \to 0$ exponentially fast as $x \to -\infty$. Note that $\mathbf{Y}_-(\lambda, x)$ is analytic in λ for $\lambda \in \Omega$.

Now set

$$\mathbf{Z} = \mathrm{e}^{-\alpha_+(\lambda)x}\mathbf{Y}(\lambda, x),$$

where $\mathbf{Y} \in \Lambda^{n-k}\mathbb{C}^n$. Then $\mathbf{Z}(\lambda, x)$ satisfies the ODE

$$\mathbf{Z}' = [A^{(n-k)}(\lambda, x) - \alpha_+(\lambda)\mathrm{id}]\mathbf{Z},$$

and in a manner similar to that described above a solution, $\mathbf{Z}_+(\lambda, \tau)$, can be constructed as the strong stable manifold of the point $(\eta_+(\lambda), +1)$, where $\eta_+(\lambda)$ is an analytic eigenvector of $A_+^{(n-k)}(\lambda) - \alpha_+(\lambda)\mathrm{id}$ constructed as before using $P_+^{\mathrm{s}}(\lambda)$ instead of $P_-^{\mathrm{u}}(\lambda)$. This in turn yields a solution

$$\mathbf{Y}_+(\lambda, x) = \mathbf{Z}_+(\lambda, x)\mathrm{e}^{\alpha_+(\lambda)x},$$

which has the property that if $\operatorname{Re}\lambda > 0$, then $|\mathbf{Y}_+(\lambda, x)| \to 0$ exponentially fast as $x \to +\infty$. Again, $\mathbf{Y}_+(\lambda, x)$ is analytic in λ for $\lambda \in \Omega$.

Define the Evans function to be

$$E(\lambda) = \exp\left(-\int^{x} \operatorname{trace} A(\lambda, s)\, \mathrm{d}s\right) \mathbf{Y}_-(\lambda, x) \wedge \mathbf{Y}_+(\lambda, x), \tag{2.15}$$

which for $\lambda \in \Omega$ has values in $\Lambda^n\mathbb{C}^n \cong \mathbb{C}$. It follows that $E(\lambda)$ is analytic for $\lambda \in \Omega$. We close with the following proposition.

Proposition 2.7. Suppose that Hypotheses 2.3 – 2.5 are true. Then the Evans function as described by Eq. (2.15) is analytic for $\lambda \in \Omega$, where Ω is described by Eq. (2.10). If λ is such that $\operatorname{Re}\lambda > 0$, then $E(\lambda)$ is the Evans function as constructed by Alexander et al. [3].

Corollary 2.8. Assume that the matrix $A(\lambda, \mu, x)$ depends in addition on a parameter $\mu \in \mathbb{R}^p$. Suppose that Hypothesis 2.3 is met for any μ and that Hypotheses 2.4 and 2.5 are satisfied for $\mu = 0$. In addition, suppose that $A(\lambda, \mu, x)$ is C^m in μ for some $m \geq 0$ and

$$\left\| \frac{\mathrm{d}^j}{\mathrm{d}\mu^j}(A(\lambda, \mu, x) - A_\pm(\lambda, \mu)) \right\| \mathrm{e}^{\pm 5\kappa x} \leq C_j, \quad x \to \pm\infty,$$

for $j = 1, \ldots, m$. Take any open subset $\tilde{\Omega}$ of Ω with $\operatorname{clos}\tilde{\Omega} \subset \Omega$. The Evans function $E(\lambda, \mu)$ exists then for μ close to zero and $\lambda \in \tilde{\Omega}$. Moreover, $E(\lambda, \mu)$ is analytic in λ and C^m in μ.

Proof. The statements follow easily from the above discussion and Lemma 2.2. □

2.3. Extension through branch points

Thus far, we considered regions in the complex plane such that the spectrum of the matrices $A_\pm(\lambda)$ was the disjoint union of the sets $\sigma^{\mathrm{u}}_\pm(\lambda)$ and $\sigma^{\mathrm{s}}_\pm(\lambda)$. In this section, we consider the case that the decomposition ceases to exist at an *isolated* point $\lambda \in \mathbb{C}$. In other words, we study neighborhoods of the points $\mathrm{i}a_j$ and $\mathrm{i}b_j$ appearing in the definition of the set Γ in Hypothesis 2.5.

We do not strive for the most general result possible, but instead restrict ourselves to cases which will arise in the analysis of perturbations of the cubic nonlinear Schrödinger equation. Therefore, let $n = 4$. Consider the linear system

$$\mathbf{Y}' = A(\lambda, \mu, x)\mathbf{Y},$$

where $\mathbf{Y} \in \mathbb{C}^4$, and the matrix A is analytic in λ and smooth in $\mu \in \mathbb{R}^p$ for each fixed x. We assume that Hypotheses 2.3 and 2.4 are met with $k = 2$ for any small μ. In addition, suppose that $A_\pm(\lambda, \mu) = A(\lambda, \mu)$.

We start with the following assumption on the asymptotic matrix $A(\lambda, \mu)$. Set

$$K := \{\lambda\colon |\lambda - \mathrm{i}\omega| \leq \delta,\ \operatorname{Re}\lambda \geq 0\}, \qquad \hat{K} := K \setminus \{\mathrm{i}\omega\}.$$

The point $\mathrm{i}\omega$ should be thought of as a point $a_j = b_j = \omega$ in Hypothesis 2.5.

Hypothesis 2.9. For $\lambda \in \hat{K}$ and any μ close to zero, the eigenvalues of $A(\lambda, \mu)$ can be written as continuous functions such that

$$\sigma_1(\lambda, \mu),\ \sigma_2(\lambda, \mu) \in \sigma^{\mathrm{u}}(\lambda, \mu), \quad \sigma_3(\lambda, \mu),\ \sigma_4(\lambda, \mu) \in \sigma^{\mathrm{s}}(\lambda, \mu)$$

are disjoint. Moreover,

$$\mathrm{Re}\,\sigma_2(\lambda,\mu) \geq \delta > 0, \quad -\mathrm{Re}\,\sigma_4(\lambda,\mu) \geq \delta > 0$$

uniformly in $\lambda \in K$ and μ. Suppose that $\sigma_1(\lambda,0), \sigma_3(\lambda,0) \to 0$ as $\lambda \to \mathrm{i}\omega$ such that the kernel of $A(\mathrm{i}\omega,0)$ is one-dimensional. Also, assume that

$$\mathrm{Re}\,\sigma_1(\lambda,\mu) > 0, \quad -\mathrm{Re}\,\sigma_3(\lambda,\mu) > 0$$

for $\mu \neq 0$ and $\lambda \in K$.

We can then extend the Evans function $E(\lambda,\mu)$ as a continuous function in $\lambda \in K$ and μ.

Lemma 2.10. Assume that Hypothesis 2.9 is met. There exists then an extension of the Evans function $E(\lambda,\mu)$ defined for $\lambda \in K$ and any μ close to zero such that $E(\lambda,\mu)$ is continuous in $\lambda \in K$ and μ.

Proof. The eigenvalues of the matrix $A(\lambda,\mu)$ are simple for $(\lambda,\mu) \neq (\mathrm{i}\omega,0)$ by Hypothesis 2.9. For $(\lambda,\mu) \neq (\mathrm{i}\omega,0)$, denote the normalized eigenvectors of the matrix $A(\lambda,\mu)$ associated with $\sigma_j(\lambda,\mu)$ by $v_j^{\mathrm{u}}(\lambda,\mu)$, where $j=1,2$. It is clear from Hypothesis 2.9 that the eigenvector $v_2^{\mathrm{u}}(\lambda,\mu)$ is continuous in $(\lambda,\mu) \in K \times \mathbb{R}^p$.

The kernel of $A(\mathrm{i}\omega,0)$ is one-dimensional by Hypothesis 2.9 and therefore spanned by the normalized vector $\hat{v}_1^{\mathrm{u}}$. We have

$$(A(\lambda,\mu) - A(\mathrm{i}\omega,0))v_1^{\mathrm{u}}(\lambda,\mu) + A(\mathrm{i}\omega,0)v_1^{\mathrm{u}}(\lambda,\mu) = A(\lambda,\mu)v_1^{\mathrm{u}}(\lambda,\mu) = \sigma_1(\lambda,\mu)v_1^{\mathrm{u}}(\lambda,\mu).$$

Since $\sigma_1(\lambda,\mu) \to 0$ as $(\lambda,\mu) \to (\mathrm{i}\omega,0)$, $|v_1^{\mathrm{u}}(\lambda,\mu)| = 1$, and $A(\lambda,\mu)$ is smooth in (λ,μ), we see that $A(\mathrm{i}\omega,0)v_1^{\mathrm{u}}(\lambda,\mu) \to 0$ as $(\lambda,\mu) \to (\mathrm{i}\omega,0)$. Therefore, possibly after multiplying $\hat{v}_1^{\mathrm{u}}$ with -1, the limit

$$\lim_{(\lambda,\mu)\to(\mathrm{i}\omega,0)} v_1^{\mathrm{u}}(\lambda,\mu) = \hat{v}_1^{\mathrm{u}}$$

exists. Indeed, without loss of generality, the restriction of $A(\mathrm{i}\omega,0)$ to its generalized kernel is given by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and so the sign of $\langle v_1^{\mathrm{u}}(\lambda,\mu), \hat{v}_1^{\mathrm{u}}\rangle$ is not zero for μ small.

Therefore, we can extend $v_1^{\mathrm{u}}(\lambda,\mu)$ continuously to $(\lambda,\mu) = (\mathrm{i}\omega,0)$ by setting $v_1^{\mathrm{u}}(\mathrm{i}\omega,0) = \hat{v}_1^{\mathrm{u}}$. We can then proceed as in Section 2.2 upon defining

$$\eta_-(\lambda,\mu) = v_1^{\mathrm{u}}(\lambda,\mu) \wedge v_2^{\mathrm{u}}(\lambda,\mu).$$

Continuity of the resulting Evans function follows from Lemma 2.2. □

Finally, we consider differentiable extensions of the Evans function. Set

$$U := \{\lambda;\ |\lambda - \mathrm{i}\omega| \leq \delta\} \setminus \{\lambda;\ \mathrm{Im}\,\lambda = \mathrm{i}\omega,\ \mathrm{Re}\,\lambda < 0\}, \quad \hat{U} := U \setminus \{\mathrm{i}\omega\}.$$

Hypothesis 2.11. For $\lambda \in U$ and any μ, the eigenvalues of $A(\lambda,\mu)$ can be written as continuous functions such that

$$\sigma_1(\lambda,\mu),\ \sigma_2(\lambda,\mu) \in \sigma^{\mathrm{u}}(\lambda), \quad \sigma_3(\lambda,\mu),\ \sigma_4(\lambda,\mu) \in \sigma^{\mathrm{s}}(\lambda)$$

are disjoint for $\lambda \in \hat{U}$. Moreover,

$$\operatorname{Re} \sigma_2(\lambda, \mu) \geq \delta > 0, \qquad -\operatorname{Re} \sigma_4(\lambda, \mu) \geq \delta > 0$$

uniformly in $\lambda \in U$, and $\sigma_1(\lambda), \sigma_3(\lambda)$ are independent of μ. Suppose that $\sigma_1(\lambda), \sigma_3(\lambda) \to 0$ as $\lambda \to \mathrm{i}\omega$ such that the kernel of $A(\mathrm{i}\omega, 0)$ is one-dimensional and spanned by the nonzero vector $\hat{v}_1(\mu)$.

Lemma 2.12. Assume that Hypothesis 2.11 is met. There exists then an extension of the Evans function $E(\lambda, \mu)$ defined for $\lambda \in U$ and μ close to zero such that $E(\lambda, \mu)$ is continuous in $\lambda \in U$ and μ. Moreover, $E(\lambda, \mu)$ is differentiable in μ, and its derivative is continuous in (λ, μ).

Proof. Again, we want to extend the 2-form $\eta_-(\lambda, \mu) = v_1^{\mathrm{u}}(\lambda, \mu) \wedge v_2^{\mathrm{u}}(\lambda, \mu)$ in a smooth fashion to the point $\lambda = \mathrm{i}\omega$. A priori, the above 2-form is defined for $\lambda \in \hat{U}$ and $\mu \in \mathbb{R}^p$, and it is C^1 in μ with its derivative being continuous in λ. We can extend $\eta_-(\lambda, \mu)$ to $\lambda = \mathrm{i}\omega$ by

$$\eta_-(\mathrm{i}\omega, \mu) := \hat{v}_1(\mu) \wedge v_2^{\mathrm{u}}(\mathrm{i}\omega, \mu).$$

Note that $\hat{v}_1(\mu)$ is smooth in μ. It suffices therefore to show that $\eta_-(\lambda, \mu)$ is C^1 in μ for any $\lambda \in U$ with its derivative being continuous in (λ, μ).

On account of Hypothesis 2.11, we may assume that

$$A(\mathrm{i}\omega, \mu) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

for any small μ with $\hat{v}_1(\mu) = (1, 0)^{\mathrm{T}}$. Writing $v_1^{\mathrm{u}}(\lambda, \mu) = (1, 0)^{\mathrm{T}} + w(\lambda, \mu)$, we shall show that $w(\lambda, \mu)$ can be chosen such that it is C^1 in μ and continuous in λ. Set

$$B(\lambda, \mu) := A(\lambda, \mu) - A(\mathrm{i}\omega, \mu),$$

and consider the following system:

$$\langle (1, 0)^{\mathrm{T}}, w \rangle = 0, \qquad \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + B(\lambda, \mu) - \sigma_1(\lambda)\mathrm{id} \right] w = (B(\lambda, \mu) + \sigma_1(\lambda)\mathrm{id})(1, 0)^{\mathrm{T}}.$$

Since $\sigma_1(\lambda)$ is a simple eigenvalue of $A(\lambda, \mu)$ for $\lambda \in \hat{U}$ and any μ, we know that the above system has a unique solution. This solution can be easily obtained using the implicit function theorem and the claim follows. We omit the details. □

2.4. No large eigenvalues

Consider the linear eigenvalue problem $LP = \lambda P$, where

$$L = D(\mu)\partial_x^2 + N(\mu, x).$$

The goal of this section is to show that if $|\lambda|$ is large, then the Evans function can be constructed as in Section 2.2. Furthermore, it will be shown that the extended Evans function will be nonzero for $|\lambda|$ large uniformly in μ. We assume that the $n \times n$ matrix $N(\mu, x)$ is smooth in x, and that there exist asymptotic matrices $N_\pm(\mu)$ and a $\kappa > 0$ such that

$$\lim_{x \to \pm\infty} |N(\mu, x) - N_\pm(\mu)| \mathrm{e}^{\pm 5\kappa x} \leq C. \tag{2.16}$$

Assume that the matrices $N(\mu, x)$, $N_\pm(\mu)$, and $D(\mu)$ are continuous in μ.

Hypothesis 2.13. The eigenvalues $\gamma_1(\mu), \dots, \gamma_n(\mu)$ of $D(\mu)$ are nonzero and satisfy

$$|\arg \gamma_i(\mu)| \leq \pi/2$$

for all μ. Furthermore, assume that $D(\mu)$ is diagonalizable for any μ.

If $\mathbf{Y} = [P, Q]^{\mathrm{T}}$, where $Q = P'$, the eigenvalue problem can be rewritten as the system

$$\mathbf{Y}' = A(\lambda, \mu, x)\mathbf{Y}, \tag{2.17}$$

where

$$A(\mu, \lambda, x) = \begin{bmatrix} 0 & \mathrm{id}_n \\ D^{-1}(\mu)(\lambda \mathrm{id}_n - N(\mu, x)) & 0 \end{bmatrix}.$$

As a consequence of (2.16), the matrix $A(\mu, \lambda, x)$ satisfies Eq. (2.7); therefore, (2.17) can be compactified as

$$\mathbf{Y}' = A(\mu, \lambda, \tau)\mathbf{Y}, \qquad \tau' = \kappa(1 - \tau^2). \tag{2.18}$$

Set

$$r = |\lambda|^{-1/2}, \qquad z = \frac{x}{r}, \qquad \tilde{Q} = rQ.$$

Upon setting $\tilde{\mathbf{Y}} = [P, \tilde{Q}]^{\mathrm{T}}$, Eq. (2.18) becomes

$$\tilde{\mathbf{Y}}' = A(\mu, \lambda, r, \tau)\tilde{\mathbf{Y}}, \qquad \tau' = r\kappa(1 - \tau^2),$$

where now $' = \mathrm{d}/\mathrm{d}z$ and

$$A(\mu, \lambda, r, \tau) = \begin{bmatrix} 0 & \mathrm{id}_n \\ D^{-1}(\mu)(\mathrm{e}^{\mathrm{i}\arg\lambda}\mathrm{id}_n - r^2 N(\mu, \tau)) & 0 \end{bmatrix}. \tag{2.19}$$

Note that $A(\mu, \lambda, r, \tau)$ is smooth in the last three parameters. Letting $\nu_i(\mu) = 1/\gamma_i(\mu)$, $i = 1, \dots, n$, denote the eigenvalues of $D^{-1}(\mu)$, we have the following lemma. Note that $\arg \nu_i = -\arg \gamma_i$, and that $|\nu_i| = 1/|\gamma_i|$.

Lemma 2.14. Set

$$A_{\pm}(\mu, \lambda, r) = \lim_{\tau \to \pm 1} A(\mu, \lambda, r, \tau).$$

The eigenvalues of $A_{\pm}(\mu, \lambda, 0)$ are given by

$$\begin{aligned} \sigma_j^-(\mu, \lambda, 0) &= +|\nu_j(\mu)|^{1/2} \exp(\mathrm{i}\,(\arg \nu_j(\mu) + \arg\lambda)/2), && j = 1, \dots, n, \\ \sigma_j^-(\mu, \lambda, 0) &= -|\nu_j(\mu)|^{1/2} \exp(\mathrm{i}\,(\arg \nu_j(\mu) + \arg\lambda)/2), && j = n+1, \dots, 2n, \\ \sigma_j^+(\mu, \lambda, 0) &= \sigma_j^-(\mu, \lambda, 0). && j = 1, \dots, 2n. \end{aligned}$$

Furthermore, for $j = 1, \dots, n$

$$\sigma_j^{\pm}(\mu, \lambda, r) = \sigma_j^{\pm}(\mu, \lambda, 0) + \mathrm{O}(r^2).$$

Proof. The eigenvalues σ of $A_{\pm}(\lambda, 0)$ satisfy the characteristic equation

$$de + (D^{-1}(\mu)\mathrm{e}^{i\arg\lambda} - \sigma^2 id_n) = 0,$$

from which one immediately gets the first part of the proposition. The second part follows from [22, Theorem II.5.11], since by Hypothesis 2.13 the matrix $D^{-1}(\mu)$ is diagonalizable. □

As a consequence of Lemma 2.14, if $\operatorname{Re}\lambda > 0$, then the eigenvalues $\sigma_j^{\pm}(0,\lambda,0)$ are ordered according to equation (2.8), that is, $\operatorname{Re}\sigma_j^{\pm}(0,\lambda,0) > 0$ for $j = 1,\dots,n$ and $\operatorname{Re}\sigma_j^{\pm}(0,\lambda,0) < 0$ for $j = n+1,\dots,2n$. Following the previous argument, in order to extend the Evans function across the imaginary axis, we must have the following: there exists a smooth positive function $\theta(r)$, with $\theta(r) \to 0$ as $r \to 0^+$, such that if $|\arg\lambda| < \pi/2 + \theta(r)$, then

$$\min_{j=1,\dots,n} \operatorname{Re}\sigma_j^{\pm}(\mu,\lambda,r) > -\frac{\kappa}{2n}r, \qquad \max_{j=n+1,\dots,2n} \operatorname{Re}\sigma_j^{\pm}(\mu\lambda,r) < \frac{\kappa}{2n}r \tag{2.20}$$

uniformly in μ.

Lemma 2.15. There exists an $r_0 > 0$ such that for any λ with

$$|\arg\lambda| < \frac{\pi}{2} + \frac{\kappa}{4n\nu^*}r$$

and $r < r_0$, Eq. (2.20) is satisfied. Here,

$$\nu^* = \max_{j=1,\dots n} |\nu_j|^{1/2} = \min_{j=1,\dots n} |\gamma_j|^{1/2}. \tag{2.21}$$

In other words, we may take

$$\theta(r) = \frac{\kappa}{4n\nu^*}r.$$

Proof. Without loss of generality, assume that $1 \le j \le n$. As a consequence of Lemma 2.14,

$$\operatorname{Re}\sigma_j^{\pm}(\mu,\lambda,r) = |\nu_j|^{1/2}\cos(\tfrac{1}{2}(\arg\nu_j + \arg\lambda)) + \mathrm{O}(r^2),$$

so that Eq. (2.20) will be satisfied if for $0 \le r \ll 1$,

$$\cos\left(\frac{1}{2}(\arg\nu_j + \arg\lambda)\right) > -\frac{\kappa}{4n|\nu_j|^{1/2}}r. \tag{2.22}$$

Eq. (2.22) will in turn be satisfied if

$$|\arg\lambda| < 2\cos^{-1}\left(-\frac{\kappa}{4|\nu_j|^{1/2}n}r\right) - |\arg\nu_j| = \pi - |\arg\nu_j| + \frac{\kappa}{2|\nu_j|^{1/2}n}r + \mathrm{O}(r^2). \tag{2.23}$$

Using the definition

$$\nu^* = \max_{j=1,\dots n} |\nu_j|^{1/2} = \min_{j=1,\dots n} |\gamma_j|^{1/2},$$

one can immediately see that if

$$|\arg\lambda| < \frac{\pi}{2} + \frac{\kappa}{4n\nu^*}r,$$

then (2.23) is satisfied. Thus, the function $\theta(r)$ discussed previously can be written as

$$\theta(r) = \frac{\kappa}{4n\nu^*}r,$$

and the lemma is proved. □

Remark 2.16. Note that the definition of $\arg\lambda$ yields that

$$\left|\frac{\mathrm{Re}\,\lambda}{\mathrm{Im}\,\lambda}\right| < \theta(r) \iff |\arg\lambda| < \frac{\pi}{2} + \theta(r). \tag{2.24}$$

With Lemma 2.15 in hand, the n-form $\mathbf{Y}_-(\mu, \lambda, r, x)$ can now be constructed as in Section 2.2. In a similar manner, the n-form $\mathbf{Y}_+(\mu, \lambda, r, x)$ can be constructed. Thus, for $0 \le r < r_0$ and $|\arg\lambda| < \pi/2 + \theta(r)$ the Evans function

$$E(\mu, \lambda, r) = \mathbf{Y}_-(\mu, \lambda, r, x) \wedge \mathbf{Y}_+(\mu, \lambda, r, x)$$

is well-defined. Since $\tau' = 0$ when $r = 0$, the n-forms $\mathbf{Y}_\pm(\lambda, 0, x)$ can be constructed for any λ. As another consequence of Lemma 2.14, it is not difficult to see that if $|\arg\lambda| \le \pi/2$, then $E(\mu, \lambda, 0) \ne 0$. We claim that the Evans function is nonzero for all r sufficiently small and $|\arg\lambda| \le \pi/2 + \theta(r)$.

To prove this claim, we proceed as in Section 2.2 and consider the equation

$$\mathbf{Z}' = [A^{(n)}(\mu, \lambda, r, \tau) - \alpha_-(\lambda)\mathrm{id}]\mathbf{Z}, \qquad \tau' = r\kappa(1 - \tau^2). \tag{2.25}$$

Here $A^{(n)}(\mu, \lambda, r, \tau)$ is induced by the matrix $A(\mu, \lambda, r, \tau)$ given in (2.19). When $r = 0$ the vector field (2.25) is autonomous and a solution is given by $(\eta_-(\lambda), \tau)$ for $\tau \in [-1, 1]$. As in Section 2.2, for $r \ne 0$ the eigenvector $\eta_-(\mu, \lambda, r)$ extends. We seek the strong unstable manifold of the point $(\eta_-(\mu, \lambda, r), -1)$ and claim that it is a small perturbation of $\{(\eta_-(\lambda), \tau);\ \tau \in [-1, 0]\}$.

Going back to the time variable z, we obtain the system

$$\mathbf{Z}' = [\tilde{A}^{(n)}(\mu, \lambda, r, rz) - \alpha_-(\mu, \lambda, r)\mathrm{id}]\mathbf{Z} \tag{2.26}$$

on $\Lambda^{2n}\mathbb{C}^n$, where

$$\tilde{A}(\mu, \lambda, r, rz) = \begin{bmatrix} 0 & \mathrm{id}_n \\ D^{-1}(\mu)(\mathrm{e}^{\mathrm{i}\arg\lambda}\mathrm{id}_n - r^2 N(\mu, rz)) & 0 \end{bmatrix}.$$

Let $\tilde{A}_-^{(n)}(\mu, \lambda, r)$ be the limit of $\tilde{A}^{(n)}(\mu, \lambda, r, rz)$ as $z \to -\infty$. It is a consequence of the definition of the derivation $A^{(n)}$ and Eq. (2.16) that

$$\|\tilde{A}^{(n)}(\mu, \lambda, r, rz) - \tilde{A}^{(n)}(\mu, \lambda, r)\| \le Cr^2\mathrm{e}^{5r\kappa z}$$

as $z \to -\infty$, where the constant C can be chosen independently of (μ, λ, r). In other words, we may write (2.26) according to

$$\mathbf{Z}' = [B(\mu, \lambda, r) + R(\mu, \lambda, r, z)]\mathbf{Z},$$

with

$$B(\mu, \lambda, r) = \tilde{A}^{(n)}(\mu, \lambda, r) - \alpha_-(\mu, \lambda, r)\mathrm{id}, \qquad \|R(\mu, \lambda, r, z)\| \le Cr^2\mathrm{e}^{5r\kappa z}.$$

For $|\arg\lambda| < \pi/2 + \theta(r)$, any eigenvalue of the matrix $B(\mu, \lambda, r)$ has real part less than $r\kappa$; therefore,

$$|\mathrm{e}^{B(\mu,\lambda,r)z}| \le C\mathrm{e}^{r\kappa z}.$$

Also, zero is an eigenvalue of $B(\mu, \lambda, r)$ with eigenvector $\eta_-(\mu, \lambda, r)$.

We may therefore apply Lemma 2.2 with $K_1 = C$, $K_2 = Cr^2$, $\beta = r\kappa$, and $\gamma = 5r\kappa$. As the result, the strong unstable manifold of $\eta_-(\mu, \lambda, r)$ is given by

$$\eta_-(\mu, \lambda, r) + \mathrm{O}(r)$$

on $(-\infty, 0]$, since with the above choices we have

$$\frac{K_1 K_2}{\gamma - \beta} = C^2 r \frac{1}{4\kappa}$$

and $\eta_-(\mu, \lambda, r)$ is bounded uniformly in (μ, λ, r).

Thus, since $\mathbf{Y}_- = \mathrm{e}^{\alpha_- z}\mathbf{Z}_-$, we have that

$$\mathbf{Y}_-(\mu, \lambda, r, 0) = \eta_-(\mu, \lambda, r) + \mathrm{O}(r).$$

In a similar manner, one can show that

$$\mathbf{Y}_+(\mu, \lambda, r, 0) = \eta_+(\mu, \lambda, r) + \mathrm{O}(r).$$

Therefore, from the definition of the Evans function we have that

$$E(\mu, \lambda, r) = (\mathbf{Y}_- \wedge \mathbf{Y}_+)(\mu, \lambda, r, 0) = (\eta_- \wedge \eta_+)(\mu, \lambda, 0) + \mathrm{O}(r) \neq 0$$

for r sufficiently small (a consequence of Lemma 2.14).

Note that the above approach is still valid if the initial estimate on R is weakened to

$$\|R(\mu, \lambda, r, z)\| \leq Cr\mathrm{e}^{5r\kappa z},$$

for in this case a unique solution is initially guaranteed for $z < z_0 = \mathrm{O}((\ln r)/r) \ll 0$, and can be continued for $z > z_0$ by applying the flow. However, the error term in the above identity of $E(\mu, \lambda, r)$ is then $\mathrm{O}(1)$ instead of $\mathrm{O}(r)$; hence, it is not clear that $E(\mu, \lambda, r) \neq 0$ for small r.

Upon going back to the original variables, we can close the discussion in this section with the following proposition which is a consequence of Lemma 2.15, (2.24) and the above discussion.

Proposition 2.17. Suppose that Hypothesis 2.13 and Eq. (2.16) are met. There then exists an $L > 0$ such that if

$$|\lambda| > L, \qquad \left|\frac{\mathrm{Re}\,\lambda}{\mathrm{Im}\,\lambda}\right| < \frac{\kappa}{4n\nu^*}|\lambda|^{-1/2},$$

where ν^* is defined in (2.21), then the extended Evans function is well-defined and nonzero.

Remark 2.18. In particular, the Evans function can then be extended in a nonzero fashion into the strip

$$0 \geq \mathrm{Re}\,\lambda \geq -\frac{q\kappa}{4n\nu^*}, \qquad |\mathrm{Im}\,\lambda| \geq L$$

for some $q = q(L) < 1$.

2.5. Example: Perturbed nonlinear Schrödinger equations

Finally, we apply the results of the previous sections to the generalized perturbed nonlinear Schrödinger equation

$$\mathrm{i}\partial_t\phi + (\partial_x^2 - \omega)\phi + f(|\phi|^2, \alpha)\phi = \mathrm{i}\epsilon d_1\partial_x^2\phi + \mathrm{i}\epsilon R(\phi, \phi^*). \tag{2.27}$$

Here $f(\eta, \alpha)$ is real-valued and smooth function with $f(0, \alpha) = 0$, ϵ is nonnegative, and $R(\mu, \eta)$ is real-valued and smooth. Let $\mu = (\alpha, \epsilon)$. Note that this equation encompasses both the perturbed cubic–quintic NLS and the parametrically forced NLS.

Hypothesis 2.19. There exists a smooth function $\Phi(x, \mu)$ which is a steady-state solution to (2.27) and satisfies the condition that $|\Phi(x, \mu)| \to 0$ at rate $\mathrm{O}(\mathrm{e}^{-5\kappa|x|})$ as $|x| \to \infty$. The same estimate is true for the derivative of $\Phi(x, \mu)$ with respect to μ. Furthermore, $\Phi_0(x) = \Phi(x, 0)$ is real-valued.

Remark 2.20. In order for the wave to decay exponentially fast, it must be true that when ϵ is small, then $\omega > 0$.

The goal of this section is to prove the following two theorems.

Theorem 2.21. Assume that $\epsilon = 0$. Let

$$\Sigma_1 = \{\lambda: \mathrm{Re}\,\lambda > 0\}, \qquad \Sigma_2 = \{\lambda: |\mathrm{Im}\,\lambda| < \omega\},$$
$$\Sigma_3 = \{\lambda: |\mathrm{Im}\,\lambda| > \omega,\ -L \le \mathrm{Re}\,\lambda \le 0\},$$

where $L > 0$ is a constant which will be determined later, and set

$$\Omega = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \tag{2.28}$$

(see Fig. 3). The Evans function $E(\lambda, \alpha)$ is defined and analytic for $\lambda \in \Omega$, and is an analytic extension of that constructed by Alexander et al. [3]. It is nonzero for sufficiently large $|\lambda|$, and has a continuous limit at $\lambda = \pm\mathrm{i}\omega$. Finally, it is C^1 in α for $\lambda \in \Omega \cup \{\pm\mathrm{i}\omega\}$, and the derivative with respect to α is continuous in λ.

Now suppose that $\epsilon > 0$ is small.

Theorem 2.22. Choose $\tilde{\Omega} \subset \Omega$ such that $\mathrm{clos}\,\tilde{\Omega} \subset \Omega$, where Ω is given in (2.28). There then exists an $\epsilon_0 > 0$ such that the Evans function $E(\lambda, \alpha, \epsilon)$ is defined for $0 < \epsilon < \epsilon_0$ and for $\lambda \in \tilde{\Omega}$. It is analytic for $\lambda \in \tilde{\Omega}$, smooth in ϵ, and is an extension of that constructed by Alexander et al. [3]. Furthermore, it is nonzero for sufficiently large $|\lambda|$.

Now suppose that the Evans function can be shown to be nonzero if $\epsilon = 0$ and $|\mathrm{Im}\,\lambda| > \omega$. Then it will necessarily be true that for $0 < \epsilon < \epsilon_0$ there exists a $\delta = \delta(\epsilon) > 0$ such that the extended Evans function will be nonzero for $|\mathrm{Im}\,\lambda| > \omega + \delta$. Under this scenario it will only be possible for eigenvalues to bifurcate out of the continuous spectrum near $\lambda = \pm\mathrm{i}\omega$. It turns out that the Evans function can be extended up to $\lambda = \pm\mathrm{i}\omega$ such that it is differentiable in ϵ. A local bifurcation analysis near $\lambda = \pm\mathrm{i}\omega$ will then reveal whether and how many eigenvalues bifurcate out of the essential spectrum. This idea will be exploited in the upcoming sections.

In the remaining part of this section, we prove the theorems. By setting $\phi = u + \mathrm{i}v$, where u and v are real, Eq. (2.27) can be rewritten as the system

$$\begin{aligned} \partial_t u + (\partial_x^2 - \omega)v + f(u^2 + v^2, \alpha)v &= \epsilon d_1 \partial_x^2 u + \epsilon R_1(u, v), \\ \partial_t v - (\partial_x^2 - \omega)u - f(u^2 + v^2, \alpha)u &= \epsilon d_1 \partial_x^2 v + \epsilon R_2(u, v), \end{aligned} \tag{2.29}$$

where

$$R_1(u, v) = \mathrm{Re}\, R(u + \mathrm{i}v, u - \mathrm{i}v), \qquad R_2(u, v) = \mathrm{Im}\, R(u + \mathrm{i}v, u - \mathrm{i}v).$$

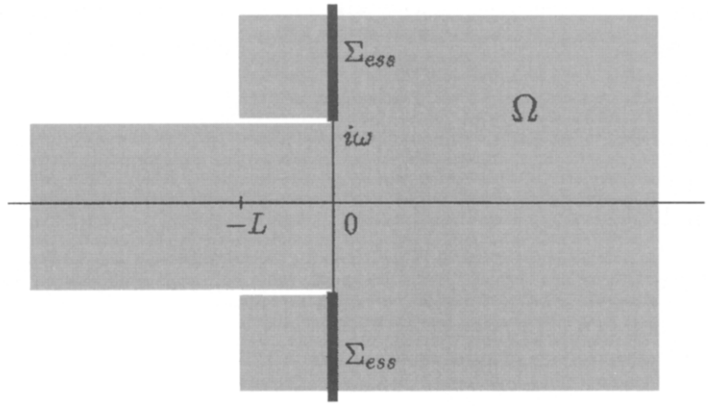

Fig. 3. The Evans function for the perturbed cubic NLS is defined for $\lambda \in \Omega \subset \mathbb{C}$.

It will be assumed that $d_1 \geq 0$, so that (2.29) will have a well-posed initial-value problem. Upon setting $P = [u, v]^{\mathrm{T}}$ and linearizing, we get the eigenvalue problem

$$\lambda P = D(\epsilon)\partial_x^2 P + (N_0(x, \alpha) + \epsilon N_1(x))P,$$

where

$$D(\epsilon) = \begin{pmatrix} \epsilon d_1 & -1 \\ 1 & \epsilon d_1 \end{pmatrix}, \qquad N_0(x, \alpha) = \begin{pmatrix} 0 & \omega - f(\Phi_0^2, \alpha) \\ -\omega + f(\Phi_0^2, \alpha) + 2\Phi_0^2 + f'(\Phi_0^2, \alpha) & 0 \end{pmatrix},$$

and $N_1(x)$ is uniformly bounded and approaches an asymptotic matrix N_1^0 exponentially fast as $|x| \to \infty$. In addition,

$$N_0(x, \alpha) \to \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \qquad |x| \to \infty,$$

and the limiting matrix is independent of α. When $\epsilon = 0$, the continuous spectrum is given by

$$\Sigma_{\text{ess}} = \{\lambda;\ \mathrm{Re}\,\lambda = 0,\ |\mathrm{Im}\,\lambda| \geq \omega\}.$$

We are now ready to prove the following lemma.

Lemma 2.23. Assume that $d_1 \geq 0$. There exist $\alpha_0 > 0$ and $\epsilon_0 > 0$ (not necessarily small), and positive constants L_1 and L_2 which are independent of α and ϵ, such that in the region

$$|\lambda| \geq L_1, \quad \mathrm{Re}\,\lambda \geq -L_2,\ 0 < \epsilon < \epsilon_0,\ |\alpha| < \alpha_0,$$

the Evans function $E(\lambda, \alpha, \epsilon)$ for Eq. (2.27) is defined and nonzero.

Proof. It is a simple matter to check that the eigenvalues of $D(\epsilon)$ satisfy Hypothesis 2.13. The extension of the Evans function and the fact that it will be nonzero for large $|\lambda|$ then follows immediately from Proposition 2.17. □

Remark 2.24. Since the zeros of the Evans function locate those eigenvalues with localized eigenfunctions, we know that there will be no large eigenvalues, even if there is no diffusion present.

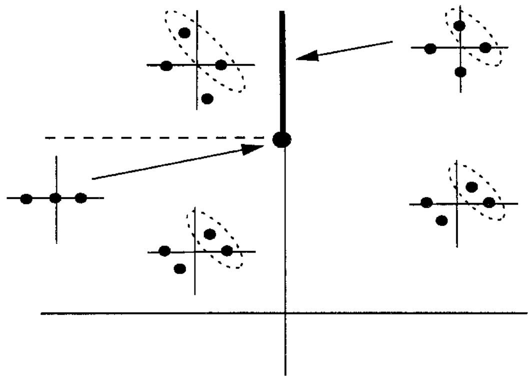

Fig. 4. Here, the location of the eigenvalues $\sigma_j^{\pm}(0, \lambda)$ of $A_0(0, \lambda)$ with $j = 1, 2$ is indicated for λ in various regions of the complex plane. Eigenvalues inside the dotted ellipsoids belong to the unstable spectral set $\sigma^{\mathrm{u}}(\lambda)$. The point $\lambda = \mathrm{i}\omega$ corresponds to a branch point where the spectral decomposition ceases to exist. The dashed line emanating from the branch point indicates the cut defined in (2.30).

Following the procedure of the previous section, the matrix $A(\mu, \lambda, x)$ is given by

$$A(\mu, \lambda, x) = \begin{pmatrix} 0 & \mathrm{id}_2 \\ D^{-1}(\epsilon)(\lambda\,\mathrm{id}_2 - N_0(x, \alpha) - \epsilon N_1(x)) & 0 \end{pmatrix},$$

where $\mu = (\alpha, \epsilon)$. As before, set

$$A_0(\epsilon, \lambda) = \lim_{|x| \to \infty} A(\mu, \lambda, x),$$

and note that $A_0(\epsilon, \lambda)$ does not depend on α.

For the moment, assume that $\epsilon = 0$. A routine calculation shows that the eigenvalues of $A_0(0, \lambda)$ are given by

$$\begin{aligned} \sigma_1^{\pm}(0, \lambda) &= \pm\sqrt{|\omega - \mathrm{i}\lambda|}\mathrm{e}^{(\mathrm{i}/2)\arg(\omega - \mathrm{i}\lambda)}, \quad \arg(\omega - \mathrm{i}\lambda) \in \left[-\frac{3\pi}{2}, \frac{\pi}{2}\right), \\ \sigma_2^{\pm}(0, \lambda) &= \pm\sqrt{|\omega + \mathrm{i}\lambda|}\mathrm{e}^{(\mathrm{i}/2)\arg(\omega + \mathrm{i}\lambda)}, \quad \arg(\omega + \mathrm{i}\lambda) \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \end{aligned} \tag{2.30}$$

(see Fig. 4). A simple observation reveals that if $\mathrm{Re}\,\lambda > 0$, then for $i = 1, 2$

$$\mathrm{Re}\,\sigma_i^{+}(0, \lambda) > 0, \quad \mathrm{Re}\,\sigma_i^{-}(0, \lambda) < 0,$$

and $\sigma_i^{\pm}(0, \lambda)$ are analytic across Σ_{ess}. As a consequence of Proposition 2.7, we now have the following lemma.

Lemma 2.25. Assume that $\epsilon = 0$. Then the Evans function $E(\lambda, \alpha)$ can be extended across Σ_{ess} onto the strip

$$\omega < |\mathrm{Im}\,\lambda| \le L_1, \quad -L_3 < \mathrm{Re}\,\lambda \le 0,$$

for some $L_3 > 0$.

Corollary 2.26. Assume that $\epsilon = 0$, and set

$$L = \min\{L_2, L_3\},$$

where L_2 is given in Lemma 2.23. Then the Evans function can be extended across Σ_{ess} onto the strip

$$\omega < |\mathrm{Im}\,\lambda|, \quad -L < \mathrm{Re}\,\lambda \le 0.$$

Furthermore, the extended Evans function will be nonzero for $|\lambda| > L_1$.

Remark 2.27. As it will be seen in Section 3, if $f(\eta, \alpha) = 4\eta$, i.e., if one looks at the cubic NLS, then $L = \infty$.

When $\epsilon = 0$, it is straightforward to prove that Hypotheses 2.9 and 2.11 are met with respect to the parameter α. Indeed, the limiting matrix does not depend on α at all. Applying Lemmas 2.10 and 2.12 then shows that the Evans function $E(\lambda, \alpha)$ is differentiable in α and can be extended to $\lambda = \mathrm{i}\omega$. Combining the results obtained so far, we have proved Theorem 2.21. Theorem 2.22 is true as a consequence of Corollary 2.8 and Lemma 2.23.

3. The Evans function for the cubic NLS

The goal in this section is to explicitly construct the extended Evans function for the cubic NLS. Once this is accomplished, we will then be able to locate its zeros, and hence be able to determine the location of the eigenvalues which may bifurcate out of the continuous spectrum. The calculation of the Evans function is possible since a complete set of eigenfunctions to the NLS has been given by Kaup [23] and Kaup et al. [24] using Inverse Scattering Theory.

Instead of using the formulation in Eq. (2.29), we will write the cubic NLS as the system

$$\mathrm{i}\phi_t + (\partial_x^2 - \omega)\phi + 4\phi^2\psi = 0, \qquad -\mathrm{i}\psi_t + (\partial_x^2 - \omega)\psi + 4\phi\psi^2 = 0,$$

where ψ is defined by $\psi = \phi^*$. The system is written in this way so that the results in [23,24] can be more easily exploited.

The bright solitary-wave solution is given by

$$\Phi(x, \omega) = \sqrt{\frac{\omega}{2}}\,\mathrm{sech}(\sqrt{\omega}\,x).$$

Linearization yields the system

$$\mathrm{i}P_t + LP = 0,$$

where

$$L = (\partial_x^2 - \omega)\sigma_3 + 4\Phi^2(2\sigma_3 + \mathrm{i}\sigma_2).$$

Here σ_2 and σ_3 are the Pauli spin matrices

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Setting $P(x, t) \to P(x)\mathrm{e}^{\rho t}$, one then gets the linear eigenvalue problem

$$(L + \mathrm{i}\rho)P = 0.$$

Upon setting

$$\rho = \mathrm{i}\lambda,$$

we then get the more conventional eigenvalue problem

$$(L-\lambda)P = 0. \tag{3.1}$$

Remark 3.1. The wave will be unstable if there exists an eigenvalue with $\mathrm{Im}\,\lambda < 0$, that is, $\mathrm{Re}\,\rho > 0$.

Since the wave is unstable for $\mathrm{Im}\,\lambda < 0$, we define the Evans function for $\mathrm{Im}\,\lambda < 0$, and extend it across $\mathrm{Im}\,\lambda = 0$.

Let $\mathbf{Y} = [P, Q]^{\mathrm{T}}$, where $Q = P'$. Then $\mathbf{Y}$ satisfies the equation

$$\mathbf{Y}' = M(\lambda, x)\mathbf{Y}, \tag{3.2}$$

where

$$M(\lambda, x) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega+\lambda-g(x) & -h(x) & 0 & 0 \\ -h(x) & \omega-\lambda-g(x) & 0 & 0 \end{bmatrix}, \tag{3.3}$$

and

$$g(x) = 8\Phi^2(x,\omega), \qquad h(x) = 4\Phi^2(x,\omega).$$

Set

$$M_0(\lambda) = \lim_{|x|\to\infty} M(\lambda, x).$$

The eigenvalues of $M_0(\lambda)$ are given by $\pm\gamma_{\mathrm{f}}(\lambda)$ and $\pm\gamma_{\mathrm{s}}(\lambda)$, where

$$\gamma_{\mathrm{f}}(\lambda) = \sqrt{|\omega+\lambda|}\,\mathrm{e}^{(\mathrm{i}/2)\arg(\omega+\lambda)}, \qquad \arg(\omega+\lambda) \in \left[-\frac{3\pi}{2}, \frac{\pi}{2}\right),$$
$$\gamma_{\mathrm{s}}(\lambda) = \sqrt{|\omega-\lambda|}\,\mathrm{e}^{(\mathrm{i}/2)\arg(\omega-\lambda)}, \qquad \arg(\omega-\lambda) \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

and the associated eigenvectors are $[1, 0, \pm\gamma_{\mathrm{f}}(\lambda), 0]^{\mathrm{T}}$ and $[0, 1, 0, \pm\gamma_{\mathrm{s}}(\lambda)]^{\mathrm{T}}$. The branch cuts of the above functions are being taken so that $\gamma_{\mathrm{s}}(\lambda) > 0$ for $\lambda \in (-\infty, \omega)$, while $\gamma_{\mathrm{f}}(\lambda) > 0$ for $\lambda \in (-\omega, \infty)$. Note that

$$\mathrm{Re}\,\lambda > 0 \;\Rightarrow\; \mathrm{Re}\,\gamma_{\mathrm{f}}(\lambda) > \mathrm{Re}\,\gamma_{\mathrm{s}}(\lambda), \qquad \mathrm{Re}\,\lambda < 0 \;\Rightarrow\; \mathrm{Re}\,\gamma_{\mathrm{f}}(\lambda) < \mathrm{Re}\,\gamma_{\mathrm{s}}(\lambda),$$

and that the functions are analytic if $\mathrm{Im}\,\lambda < 0$.

As a consequence of Theorem 2.21, we have the following lemma.

Lemma 3.2. Let

$$\Sigma_1 = \{\lambda\colon \mathrm{Im}\,\lambda < 0\}, \qquad \Sigma_2 = \{\lambda\colon |\mathrm{Re}\,\lambda| < \omega\},$$
$$\Sigma_3 = \{\lambda\colon |\mathrm{Re}\,\lambda| > \omega,\ 0 \le \mathrm{Im}\,\lambda < L\},$$

and set

$$\Omega = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3.$$

There is an $L > 0$ such that the Evans function is defined and analytic for $\lambda \in \Omega$, and is an analytic extension of that constructed by Alexander et al. [3]. Furthermore, it is nonzero for sufficiently large $|\lambda|$. Finally, it has a continuous limit at $\lambda = \pm\omega$.

Before continuing, we need a couple of preliminary results.

Lemma 3.3. Let $\mathbf{Y}(\lambda, x) = [P(\lambda, x), Q(\lambda, x)]^{\mathrm{T}}$ be a solution to (3.2). Another solution to (3.2) is then $\mathbf{Y}(\lambda, x) = [P(\lambda, -x), -Q(\lambda, -x)]^{\mathrm{T}}$. A solution to

$$\mathbf{Y}' = M(\lambda^*, x)\mathbf{Y}$$

is given by $\mathbf{Y}^*(\lambda, x)$. Finally, if $\lambda \in \mathbb{R}$, then a solution to the adjoint problem

$$\mathbf{Z}' = -M^{\mathrm{T}}(\lambda, x)\mathbf{Z}$$

is given by $\mathbf{Z}(\lambda, x) = [-Q(\lambda, x), P(\lambda, x)]^{\mathrm{T}}$.

Proof. The first part follows immediately from the fact that both $g(x)$ and $h(x)$ are even functions. The second part follows as soon as one notices that

$$M(\lambda^*, x)^* = M(\lambda, x).$$

The third part is a simple calculation, and is left to the interested reader. □

Lemma 3.4 (Kaup [23], Kaup et al. [24]). When $\operatorname{Re}\lambda > 0$, a solution to (3.1) is given by

$$P^+(\lambda, x) = -\frac{\mathrm{e}^{-\gamma_{\mathrm{s}}(\lambda)x}}{(\gamma_{\mathrm{s}}(\lambda) - \sqrt{\omega})^2}\left\{(\lambda - 2\omega + 2\sqrt{\omega}\,\gamma_{\mathrm{s}}(\lambda)\tanh(\sqrt{\omega}\,x))\begin{bmatrix}0\\1\end{bmatrix} + 2\Phi^2(x, \omega, 0)\begin{bmatrix}1\\1\end{bmatrix}\right\}.$$

When $\operatorname{Re}\lambda < 0$, a solution to (3.1) is given by

$$P^-(\lambda, x) = \frac{\mathrm{e}^{-\gamma_{\mathrm{f}}(\lambda)x}}{(\gamma_{\mathrm{f}}(\lambda) + \sqrt{\omega})^2}\left\{(\lambda + 2\omega + 2\sqrt{\omega}\,\gamma_{\mathrm{f}}(\lambda)\tanh(\sqrt{\omega}\,x))\begin{bmatrix}0\\1\end{bmatrix} - 2\Phi^2(x, \omega, 0)\begin{bmatrix}1\\1\end{bmatrix}\right\}.$$

Furthermore, besides the functions $P^+(\omega + k^2, x)$ and $P^-(-(\omega + k^2), x)$, where $k \in \mathbb{R}^+$, along with the eigenfunctions of L at $\lambda = 0$, there are no other bounded eigenfunctions of L.

Since λ is an eigenvalue if, and only if, $-\lambda$ is, it suffices to calculate the Evans function only for $\operatorname{Re}\lambda > 0$. For the rest of this discussion assume therefore that $\operatorname{Re}\lambda > 0$. The following arguments can easily be modified for the case $\operatorname{Re}\lambda < 0$.

First set

$$\mathbf{Y}_{\mathrm{s}}^-(\lambda, x) = \begin{bmatrix}P^+(\lambda, -x)\\-Q^+(\lambda, -x)\end{bmatrix}, \qquad \mathbf{Y}_{\mathrm{s}}^+(\lambda, x) = \begin{bmatrix}P^+(\lambda, x)\\Q^+(\lambda, x)\end{bmatrix}, \tag{3.4}$$

where $P^+(\lambda, x)$ is defined in the above lemma. Note that

$$\begin{aligned}\lim_{x\to\mp\infty} \mathbf{Y}_{\mathrm{s}}^{\mp}(\lambda, x)\mathrm{e}^{\mp\gamma_{\mathrm{s}}(\lambda)x} &= [0, 1, 0, \pm\gamma_s(\lambda)],\\ \lim_{x\to\pm\infty} \mathbf{Y}_{\mathrm{s}}^{\mp}(\lambda, x)\mathrm{e}^{\mp\gamma_{\mathrm{s}}(\lambda)x} &= \frac{\lambda - 2\omega - 2\sqrt{\omega}\,\gamma_{\mathrm{s}}(\lambda)}{\lambda - 2\omega + 2\sqrt{\omega}\,\gamma_{\mathrm{s}}(\lambda)}[0, 1, 0, \pm\gamma_s(\lambda)].\end{aligned} \tag{3.5}$$

There exists a unique solution $\mathbf{Y}_\mathrm{f}^-$ to (3.2) such that

$$\lim_{x\to-\infty} \mathbf{Y}_\mathrm{f}^-(\lambda, x)\mathrm{e}^{-\gamma_\mathrm{f}(\lambda)x} = \begin{bmatrix} 1 \\ 0 \\ \gamma_\mathrm{f}(\lambda) \\ 0 \end{bmatrix}. \tag{3.6}$$

This is due to the fact that $\gamma_\mathrm{f}(\lambda)$ is the positive eigenvalue of $M_0(\lambda)$ with largest real part. Similarly, there exists a unique solution $\mathbf{Z}_\mathrm{f}^+(\lambda, z)$ to the adjoint problem with the asymptotics

$$\lim_{x\to\infty} \mathbf{Z}_\mathrm{f}^+(\lambda, x)\mathrm{e}^{\gamma_\mathrm{f}(\lambda)x} = \begin{bmatrix} \gamma_\mathrm{f}(\lambda) \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Define the reduced Evans function

$$E_\mathrm{f}(\lambda) = \mathbf{Y}_\mathrm{f}^-(\lambda, x)\cdot \mathbf{Z}_\mathrm{f}^+(\lambda, x).$$

Before continuing, we need the following information regarding the reduced Evans function.

Lemma 3.5. $E_\mathrm{f}(\lambda)$ is analytic and nonzero for $\mathrm{Re}\,\lambda > 0$.

Proof. The analyticity follows from the fact that the eigenvalue $\gamma_\mathrm{f}(\lambda)$ is simple and thus analytic for $\mathrm{Re}\,\lambda > 0$ (see Lemma 2.2). In the following, it is important to note that if $E_\mathrm{f}(\lambda) = 0$, then

$$\lim_{x\to\infty} |\mathbf{Y}_\mathrm{f}^-(\lambda, x)\mathrm{e}^{-\gamma_\mathrm{f}(\lambda)x}| = 0.$$

First suppose that $\lambda \in (\omega, \infty)$. If $E_\mathrm{f}(\lambda) = 0$, then $\mathbf{Y}_\mathrm{f}^-$ is a uniformly bounded function which decays exponentially fast as $x \to -\infty$. However, Lemma 3.4 precludes the existence of such a solution.

Now suppose that $\lambda = \omega$. If $E_\mathrm{f}(\omega) = 0$, then

$$\lim_{x\to\infty} |\mathbf{Y}_\mathrm{f}^-(\omega, x)\mathrm{e}^{-\gamma_\mathrm{f}(\omega)x}| = 0.$$

Consider the 3-form $\mathbf{Y}_\mathrm{f}^- \wedge \mathbf{Y}_\mathrm{s}^- \wedge \mathbf{Y}_\mathrm{f}^+$. This 3-form induces a solution to the adjoint equation, $\mathbf{Z}$. Since $\mathbf{Y}_\mathrm{f}^+(\lambda, x) = \mu[P_\mathrm{f}^-(\lambda, -x), -Q_\mathrm{f}^-(\lambda, -x)]^\mathrm{T}$ for some nonzero constant μ, where $\mathbf{Y}_\mathrm{f}^-(\lambda, x) = [P_\mathrm{f}^-(\lambda, x), Q_\mathrm{f}^-(\lambda, x)]^\mathrm{T}$, the adjoint solution then satisfies

$$\lim_{|x|\to\infty} |\mathbf{Z}(\omega, x)| = 0.$$

By Lemma 3.3, this then implies that there exists a solution to (3.2) which decays as $|x| \to \infty$. However, this contradicts Lemma 3.4.

Now suppose that $\lambda \in \{\lambda \in \mathbb{C}\colon \mathrm{Im}\,\lambda \geq 0,\ \lambda \notin [\omega, \infty)\}$. It is known that there are no eigenvalues to L, which implies by the result of Alexander et al. [3] that

$$\lim_{x\to\infty} \mathbf{Y}_\mathrm{f}^-(\lambda, x) \wedge \mathbf{Y}_\mathrm{s}^-(\lambda, x)\mathrm{e}^{-(\gamma_\mathrm{f}(\lambda)+\gamma_\mathrm{s}(\lambda))x} = \mu[1, 0, \gamma_\mathrm{f}(\lambda), 0]^\mathrm{T} \wedge [0, 1, 0, \gamma_\mathrm{s}(\lambda)]^\mathrm{T} \tag{3.7}$$

for some nonzero constant μ. By Eq. (3.5) we have

$$\lim_{x\to\infty} \mathbf{Y}_\mathrm{s}^-(\lambda, x)\mathrm{e}^{-\gamma_\mathrm{s}(\lambda)x} = \frac{\lambda - 2\omega - 2\sqrt{\omega}\,\gamma_\mathrm{s}(\lambda)}{\lambda - 2\omega + 2\sqrt{\omega}\,\gamma_\mathrm{s}(\lambda)}[0, 1, 0, \gamma_\mathrm{s}(\lambda)]^\mathrm{T}.$$

If $E_{\mathrm{f}}(\lambda) = 0$, then

$$\lim_{x\to\infty} |\mathbf{Y}_{\mathrm{f}}^{-}(\omega, x)\mathrm{e}^{-\gamma_{\mathrm{f}}(\omega)x}| = 0.$$

Thus, in this case

$$\lim_{x\to\infty} |\mathbf{Y}_{\mathrm{f}}^{-}(\lambda, x) \wedge \mathbf{Y}_{\mathrm{s}}^{-}(\lambda, x)\mathrm{e}^{-(\gamma_{\mathrm{f}}(\lambda)+\gamma_{\mathrm{s}}(\lambda))x}| = 0,$$

which violates (3.7).

It is now known that $E_{\mathrm{f}}(\lambda) \neq 0$ for Im $\lambda \geq 0$. By Lemma 3.3

$$E_{\mathrm{f}}(\lambda^*) = \mathbf{Y}_{\mathrm{f}}^{-}(\lambda^*, x) \cdot \mathbf{Z}_{\mathrm{f}}^{+}(\lambda^*, x) = (\mathbf{Y}_{\mathrm{f}}^{-}(\lambda, x))^* \cdot (\mathbf{Z}_{\mathrm{f}}^{+}(\lambda, x))^* = E_{\mathrm{f}}(\lambda)^*.$$

Thus, $E_{\mathrm{f}}(\lambda) \neq 0$ for Im $\lambda \geq 0$ necessarily implies that the same holds true for Im $\lambda \leq 0$. □

Remark 3.6. The function $E_{\mathrm{f}}(\lambda)$ can be extended to include the imaginary axis.

Remark 3.7. $E_{\mathrm{f}}(\lambda) \to 0$ as $\lambda \to 0^+$.

Using the definition of $E_{\mathrm{f}}(\lambda)$ it is easy to check that

$$\lim_{x\to\infty} \mathbf{Y}_{\mathrm{f}}^{-}(\lambda, x)\mathrm{e}^{-\gamma_{\mathrm{f}}(\lambda)x} = \frac{E_{\mathrm{f}}(\lambda)}{2\gamma_{\mathrm{f}}(\lambda)}[1, 0, \gamma_{\mathrm{f}}(\lambda), 0]^{\mathrm{T}}.$$

Since $E_{\mathrm{f}}(\lambda) \neq 0$, the solution

$$\mathbf{Y}_{\mathrm{f}}^{+}(\lambda, x) = 2\gamma_{\mathrm{f}}(\lambda)[P_{\mathrm{f}}^{-}(\lambda, -x), -Q_{\mathrm{f}}^{-}(\lambda, -x)]^{\mathrm{T}},$$

where $\mathbf{Y}_{\mathrm{f}}^{-}(\lambda, x) = [P_{\mathrm{f}}^{-}(\lambda, x), Q_{\mathrm{f}}^{-}(\lambda, x)]^{\mathrm{T}}$, is well-defined for Re $\lambda > 0$. Note that

$$\lim_{x\to-\infty} \mathbf{Y}_{\mathrm{f}}^{+}(\lambda, x)\mathrm{e}^{\gamma_{\mathrm{f}}(\lambda)x} = E_{\mathrm{f}}(\lambda)[1, 0, -\gamma_{\mathrm{f}}(\lambda), 0]^{\mathrm{T}}. \tag{3.8}$$

For Re $\lambda > 0$ the Evans function is given by

$$E(\lambda) = (\mathbf{Y}_{\mathrm{f}}^{-} \wedge \mathbf{Y}_{\mathrm{s}}^{-} \wedge \mathbf{Y}_{\mathrm{f}}^{+} \wedge \mathbf{Y}_{\mathrm{s}}^{+})(\lambda, x).$$

Based upon the above discussion, the Evans function can be explicitly calculated.

Proposition 3.8. For Re $\lambda > 0$ the Evans function is given by

$$E(\lambda) = 4E_{\mathrm{f}}(\lambda)\gamma_{\mathrm{f}}(\lambda)\gamma_{\mathrm{s}}(\lambda)\frac{\lambda - 2\omega - 2\sqrt{\omega}\,\gamma_{\mathrm{s}}(\lambda)}{\lambda - 2\omega + 2\sqrt{\omega}\,\gamma_{\mathrm{s}}(\lambda)}.$$

The analytic function $E_{\mathrm{f}}(\lambda)$ is nonzero for Re $\lambda > 0$, and can be scaled such that $E_{\mathrm{f}}(\omega) = 1$.

Proof. By Eqs. (3.5), (3.6), and (3.8), the behavior as $x \to -\infty$ is well-understood for all the functions comprising $E(\lambda)$. The result then follows immediately after evaluating

$$\lim_{x\to-\infty} (\mathbf{Y}_{\mathrm{f}}^{-} \wedge \mathbf{Y}_{\mathrm{s}}^{-} \wedge \mathbf{Y}_{\mathrm{f}}^{+} \wedge \mathbf{Y}_{\mathrm{s}}^{+})(\lambda, x),$$

and rescaling $\mathbf{Z}_{\mathrm{f}}^{+}(\omega, x)$ such that $E_{\mathrm{f}}(\omega) = 1$. □

Theorem 1.1 is a consequence of Proposition 3.8 and the discussion following Theorem 2.22. Indeed, the expression for the Evans function of the cubic NLS vanishes only at $\lambda = 0$ and $\lambda = \pm\omega$ which correspond to eigenvalues at zero and $\rho = \pm\mathrm{i}\omega$, respectively, by Remark 3.1.

4. The Evans function for the cubic NLS near $\lambda = \omega$

As a consequence of Proposition 3.8, we now know that an eigenvalue may bifurcate out of the continuous spectrum only at $\lambda = \pm\omega$ for perturbations of the cubic NLS. In this section, an expression for the derivative $\partial_{\tilde{\epsilon}} E(\omega, 0)$ of the Evans function at the edge $\lambda = \omega$ of the essential spectrum is derived. Using a Taylor expansion of the Evans function, we are then able to state precise conditions under which an eigenvalue bifurcates out of the essential spectrum. The results are applied in Section 5 to concrete perturbations.

Remark 4.1. We restrict ourselves to perturbations of the cubic NLS only for the sake of clarity. The results can be easily generalized to other equations.

We write the perturbed eigenvalue equation as

$$\mathbf{Y}' = M(\lambda, x, \tilde{\epsilon})\mathbf{Y}, \tag{4.1}$$

where $M(\lambda, x, 0)$ is the matrix given in Eq. (3.3). Assume that

$$M_0(\lambda, \tilde{\epsilon}) = \lim_{|x|\to\infty} M(\lambda, x, \tilde{\epsilon})$$

exists with exponential convergence. Furthermore, we assume that Hypothesis 2.11 is met. In particular, the matrix $M_0(\omega, \tilde{\epsilon})$ has eigenvalues $\sigma_1(\tilde{\epsilon}), \sigma_4(\tilde{\epsilon})$ with modulus bigger than zero and $\sigma_2 = \sigma_3 = 0$, while the kernel of $M_0(\omega, \tilde{\epsilon})$ is one-dimensional. Lemma 2.12 shows that the Evans function $E(\lambda, \tilde{\epsilon})$ is then differentiable in $\tilde{\epsilon}$ for $\lambda = \omega$. The first result gives a formula for the derivative $\partial_{\tilde{\epsilon}} E(\omega, 0)$.

Theorem 4.2. Consider (4.1) and assume that Hypothesis 2.11 is met. The derivative of the Evans function with respect to $\tilde{\epsilon}$ at $(\lambda, \tilde{\epsilon}) = (\omega, 0)$ is then given by

$$\partial_{\tilde{\epsilon}} E(\omega, 0) = \int_{-\infty}^{\infty} \mathbf{Z}_{\mathrm{s}}(\omega, x, 0) \cdot \partial_{\tilde{\epsilon}} M(\omega, x, 0)\mathbf{Y}_{\mathrm{s}}^{+}(\omega, x, 0)\,\mathrm{d}x,$$

where

$$\mathbf{Z}_{\mathrm{s}}(\omega, x, 0) = 2\sqrt{2\omega}\begin{bmatrix} -\partial_x P^{+}(\omega, x) \\ P^{+}(\omega, x) \end{bmatrix}, \quad \mathbf{Y}_{\mathrm{s}}^{+}(\omega, x, 0) = \begin{bmatrix} P^{+}(\omega, x) \\ \partial_x P^{+}(\omega, x) \end{bmatrix},$$

and

$$P^{+}(\omega, x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{2}{\omega}\Phi^2(x, \omega, 0)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

has been defined in Lemma 3.4.

Proof. Postponed until Section 4.1. □

Remark 4.3. The interested reader should consult Kapitula [20] for a related result in the circumstance that the zero of the Evans function is not located in the essential spectrum.

Next, we exploit differentiability of the Evans function and expand it into a Taylor series. Let

$$U := \{\lambda\colon |\lambda - \omega| \le \delta\} \setminus \{\lambda\colon \mathrm{Re}\,\lambda = \omega,\ \mathrm{Im}\,\lambda > 0\}.$$

For $\lambda \in U$, we can write

$$E(\lambda,\tilde{\epsilon}) = E(\lambda,0) + \partial_{\tilde{\epsilon}}E(\lambda,0)\tilde{\epsilon} + \mathrm{o}(\tilde{\epsilon}) = E(\lambda,0) + (\partial_{\tilde{\epsilon}}E(\omega,0) + g_1(\lambda,\tilde{\epsilon}))\tilde{\epsilon},$$

where g_1 is continuous and $g_1(\omega,0) = 0$. Using the expression for the Evans function for $\tilde{\epsilon} = 0$ given in Proposition 3.8, we then see that for $\lambda \in U$ the Evans function is given by

$$E(\lambda,\tilde{\epsilon}) = 4\sqrt{2\omega}\,\gamma_{\mathrm{s}}(\lambda)(1+g_2(\lambda)) + (\partial_{\tilde{\epsilon}}E(\omega,0) + g_1(\lambda,\tilde{\epsilon}))\tilde{\epsilon},$$

where $g_2(\lambda)$ is continuous and $g_2(\omega) = 0$.

Due to the branch cut taken for $\gamma_{\mathrm{s}}(\lambda)$, we then see that

$$-\tfrac{1}{4}\pi < \arg(\partial_{\tilde{\epsilon}}E(\omega,0)\tilde{\epsilon}) < \tfrac{3}{4}\pi \quad\Longrightarrow\quad E(\lambda,\tilde{\epsilon}) \neq 0$$

for $\lambda \in U$, and hence no eigenvalue bifurcates out of the continuous spectrum. Otherwise, a single eigenvalue bifurcates out of the continuous spectrum, and $E(\lambda^*(\tilde{\epsilon}),\tilde{\epsilon}) = 0$, where

$$\lambda^* = \omega\left(1 - \frac{(\partial_{\tilde{\epsilon}}E(\omega,0))^2}{32\omega^2}\tilde{\epsilon}^2\right) + \mathrm{o}(\tilde{\epsilon}^2).$$

We summarize the above discussion in the following theorem.

Theorem 4.4. Assume that Hypothesis 2.11 is satisfied.

(i) If $-\pi/4 < \arg(\partial_{\tilde{\epsilon}}E(\omega,0)\tilde{\epsilon}) < 3\pi/4$, then no eigenvalue bifurcates out of the essential spectrum.

(ii) If $3\pi/4 < \arg(\partial_{\tilde{\epsilon}}E(\omega,0)\tilde{\epsilon}) < 7\pi/4$, then a single eigenvalue bifurcates out of the essential spectrum, and its location given by

$$\lambda^* = \omega\left(1 - \frac{(\partial_{\tilde{\epsilon}}E(\omega,0))^2}{32\omega^2}\tilde{\epsilon}^2\right) + \mathrm{o}(\tilde{\epsilon}^2).$$

Remark 4.5. Any bifurcating eigenvalue is $\mathrm{O}(\tilde{\epsilon}^2)$ close to $\lambda = \omega$.

Remark 4.6. In particular, if $\partial_{\tilde{\epsilon}}E(\omega,0)\tilde{\epsilon} > 0$, no eigenvalue pops out of the essential spectrum, while for $\partial_{\tilde{\epsilon}}E(\omega,0)\tilde{\epsilon} < 0$ a single eigenvalue bifurcates.

4.1. *Proof of Theorem 4.2*

Eq. (4.1) above induces the perturbed solutions $\mathbf{Y}_{\mathrm{f}}^{\pm}(\lambda,x,\tilde{\epsilon})$ and $\mathbf{Y}_{\mathrm{s}}^{\pm}(\lambda,x,\tilde{\epsilon})$, where $\mathbf{Y}_{\mathrm{f}}^{\pm}(\lambda,x,0)$ and $\mathbf{Y}_{\mathrm{s}}^{\pm}(\lambda,x,0)$ are those given in Section 3. Since $\mathbf{Y}_{\mathrm{s}}^{-}(\omega,x,0) = \mathbf{Y}_{\mathrm{s}}^{+}(\omega,x,0)$, a routine calculation shows that

$$\partial_{\tilde{\epsilon}}E(\omega,0) = -\partial_{\tilde{\epsilon}}(\mathbf{Y}_{\mathrm{s}}^{-} - \mathbf{Y}_{\mathrm{s}}^{+})(\omega,x,0) \wedge (\mathbf{Y}_{\mathrm{f}}^{-} \wedge \mathbf{Y}_{\mathrm{f}}^{+} \wedge \mathbf{Y}_{\mathrm{s}}^{+})(\omega,x,0).$$

The 3-form $(\mathbf{Y}_{\mathrm{f}}^{-} \wedge \mathbf{Y}_{\mathrm{f}}^{+} \wedge \mathbf{Y}_{\mathrm{s}}^{+})(\omega,x,0)$ is uniformly bounded as $|x| \to \infty$, with

$$\lim_{x\to-\infty}(\mathbf{Y}_{\mathrm{f}}^{-} \wedge \mathbf{Y}_{\mathrm{f}}^{+} \wedge \mathbf{Y}_{\mathrm{s}}^{+})(\omega,x,0) = 2\gamma_{\mathrm{f}}(\omega)\,\mathbf{e}_{123}, \tag{4.2}$$

where $\mathbf{e}_{ijk} = \mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k$. Writing

$$-(\mathbf{Y}_{\mathrm{f}}^{-} \wedge \mathbf{Y}_{\mathrm{f}}^{+} \wedge \mathbf{Y}_{\mathrm{s}}^{+})(\omega,x,0) = a_1(x)\mathbf{e}_{123} + a_2(x)\mathbf{e}_{124} + a_3(x)\mathbf{e}_{134} + a_4(x)\mathbf{e}_{234},$$

this 3-form induces a solution to the adjoint equation, $\mathbf{Z}_s(\omega, x, 0)$, which is given by

$$\mathbf{Z}_s(\omega, x, 0) = [a_4(x), -a_3(x), a_2(x), -a_1(x)]^{\mathrm{T}}$$

[20,41]. In other words,

$$\partial_{\tilde{\epsilon}} E(\omega, 0) = \partial_{\tilde{\epsilon}}(\mathbf{Y}_s^- - \mathbf{Y}_s^+)(\omega, x, 0) \cdot \mathbf{Z}_s(\omega, x, 0). \tag{4.3}$$

Using (4.2) and Lemma 3.3, one can compute explicitly that

$$\mathbf{Z}_s(\omega, x, 0) = 2\gamma_{\mathrm{f}}(\omega) \begin{bmatrix} -\partial_x P^+(\omega, x) \\ P^+(\omega, x) \end{bmatrix},$$

where $P^+(\omega, x)$ is defined in Lemma 3.4. Unfortunately, the evaluation of $\partial_{\tilde{\epsilon}}(\mathbf{Y}_s^- - \mathbf{Y}_s^+)$ is not as straightforward.

We compute $\partial_{\tilde{\epsilon}} \mathbf{Y}_s^+(\omega, x, 0)$. Consider

$$\mathbf{Y}' = M(\omega, x, \tilde{\epsilon})\mathbf{Y},$$

and let $M_0(\omega, \tilde{\epsilon}) = \lim_{|x|\to\infty} M(\omega, x, \tilde{\epsilon})$. Using the definition (3.4) of $\mathbf{Y}_s^+(\omega, x, 0)$ and Lemma 3.4, we have

$$\mathbf{Y}_s^+(\omega, x, 0) = \eta_0 + v_s(x)$$

where $M_0(\omega, 0)\eta_0 = 0$ and $v_s(x)$ decays exponentially. We employ the ansatz

$$\mathbf{Y}_s^+(\omega, x, \tilde{\epsilon}) = \eta(\tilde{\epsilon}) + v_s(x) + w_s^+(x, \tilde{\epsilon}),$$

where $M_0(\omega, \tilde{\epsilon})\eta(\tilde{\epsilon}) = 0$ for all $\tilde{\epsilon}$ and $\eta(\tilde{\epsilon})$ is smooth in $\tilde{\epsilon}$. Such a choice is clearly possible due to Hypothesis 2.11. We denote the evolution operator of (4.1) by $\Phi(x, y)$. Moreover, let P_+^{u} be a projection with kernel given by $Y_{\mathrm{f}}^+(\omega, 0, 0)$. It is then straightforward to show that

$$\partial_{\tilde{\epsilon}} w_s^+(0, 0)|_{\tilde{\epsilon}=0} = \int_{\infty}^{0} P_+^{\mathrm{u}}\Phi(0, x)(\partial_{\tilde{\epsilon}} M(\omega, x, 0)v_s(x) \\ + (M(\omega, x, 0) - M_0(\omega, 0))\partial_{\tilde{\epsilon}}\eta(0) + \partial_{\tilde{\epsilon}}(M(\omega, x, 0) - M_0(\omega, 0))\eta_0)\,\mathrm{d}x,$$

see, for instance, [20,41] for similar calculations. Note that the integral converges since $P_+^{\mathrm{u}}\Phi(0, x)$ grows only linearly, while $v_s(x)$ and $M(\omega, x, 0) - M_0(\omega, 0)$ decays exponentially. Therefore,

$$\partial_{\tilde{\epsilon}} \mathbf{Y}_s^+(\omega, 0, 0) = \partial_{\tilde{\epsilon}}\eta(0) + \int_{\infty}^{0} P_+^{\mathrm{u}}\Phi(0, x)(\partial_{\tilde{\epsilon}} M(\omega, x, 0)v_s(x) \\ + (M(\omega, x, 0) - M_0(\omega, 0))\partial_{\tilde{\epsilon}}\eta(0) + \partial_{\tilde{\epsilon}}(M(\omega, x, 0) - M_0(\omega, 0))\eta_0)\,\mathrm{d}x,$$

and similarly

$$\partial_{\tilde{\epsilon}} \mathbf{Y}_s^-(\omega, 0, 0) = \partial_{\tilde{\epsilon}}\eta(0) + \int_{-\infty}^{0} P_-^{\mathrm{s}}\Phi(0, x)(\partial_{\tilde{\epsilon}} M(\omega, x, 0)v_s(x) \\ + (M(\omega, x, 0) - M_0(\omega, 0))\partial_{\tilde{\epsilon}}\eta(0) + \partial_{\tilde{\epsilon}}(M(\omega, x, 0) - M_0(\omega, 0))\eta_0)\,\mathrm{d}x,$$

where

$$\mathbf{Y}_s^-(\omega, x, 0) = \eta_0 + v_s(x).$$

Using (4.3) and the definition of P_+^{u} and P_-^{s}, we conclude that

$$\partial_{\bar\epsilon} E(\omega,0) = \int_{-\infty}^{\infty} \mathbf{Z}_{\mathrm{s}}(\omega,x,0)\cdot(\partial_{\bar\epsilon} M(\omega,x,0)v_{\mathrm{s}}(x) + (M(\omega,x,0)-M_0(\omega,0))\partial_{\bar\epsilon}\eta(0) + \partial_{\bar\epsilon}(M(\omega,x,0)-M_0(\omega,0))\eta_0)\,\mathrm{d}x. \tag{4.4}$$

In the final step, we bring this expression into the form shown in Theorem 4.2. The key is to use the identity

$$\int_{-\infty}^{\infty} \mathbf{Z}_{\mathrm{s}}(\omega,x,0)\cdot(M(\omega,x,0)-M_0(\omega,0))\partial_{\bar\epsilon}\eta(0)\,\mathrm{d}x = \int_{-\infty}^{\infty} \mathbf{Z}_{\mathrm{s}}(\omega,x,0)\cdot\partial_{\bar\epsilon} M_0(\omega,0)\eta_0\,\mathrm{d}x, \tag{4.5}$$

which shows that the integral on the right-hand side converges and proves the expression in Theorem 4.2. The identity (4.5) follows by transposing $(M(\omega,x,0)-M_0(\omega,0))$, using that $\mathbf{Z}_{\mathrm{s}}(\omega,x,0)$ satisfies the adjoint equation and integrating once by parts. During this process, it can be easily checked that the integrals still converge. We omit the details.

5. Bifurcations from the essential spectrum near $\lambda=\omega$ for concrete perturbations of the cubic NLS

The approach summarized in Theorems 4.2 and 4.4 allows us to determine whether eigenvalues bifurcate from the essential spectrum of the cubic NLS under perturbations. Here, we exploit these results and apply them to several concrete perturbations of the NLS.

Note that the upcoming eigenvalue problems are formulated as in Section 3, so that a wave is unstable if $\mathrm{Im}\,\lambda<0$, see Remark 3.1.

5.1. *Evaluation at $\lambda=\omega$: CQNLS*

For the CQNLS,

$$\mathrm{i}\phi_t+\phi_{xx}-\omega\phi+4|\phi|^2\phi+3\alpha|\phi|^4\phi=0,$$

the solitary-wave solution is given by

$$\Phi^2(x,\omega,\alpha)=\frac{\omega}{1+\sqrt{1+\alpha\omega}\cosh(2\sqrt{\omega}x)} \tag{5.1}$$

(see [37]).

Following the formulation in Section 3, for the eigenvalue problem we get the matrix

$$M(\lambda,x,\alpha)=\begin{bmatrix}0&0&1&0\\0&0&0&1\\\omega+\lambda-g(x,\alpha)&-h(x,\alpha)&0&0\\-h(x,\alpha)&\omega-\lambda-g(x,\alpha)&0&0\end{bmatrix},$$

where

$$g(x,\alpha) = 8\Phi^2 + 9\alpha\Phi^4, \qquad h(x,\alpha) = 4\Phi^2 + 6\alpha\Phi^4.$$

Here, the wave is unstable if $\operatorname{Im}\lambda < 0$, see Remark 3.1. Theorem 2.21 shows that the Evans function is differentiable at $\lambda = \omega$. By Theorem 4.2, we therefore know that

$$\partial_\alpha E(\omega, 0) = \int_{-\infty}^{\infty} \partial_\alpha M(\omega, x)\mathbf{Y}_s^+(\omega, x) \cdot \mathbf{Z}_s(\omega, x)\,\mathrm{d}x \tag{5.2}$$

with

$$\mathbf{Z}_s(\omega, x, 0) = 2\sqrt{2\omega}\begin{bmatrix} -\partial_x P^+(\omega, x) \\ P^+(\omega, x) \end{bmatrix}, \quad \mathbf{Y}_s^+(\omega, x, 0) = \begin{bmatrix} P^+(\omega, x) \\ \partial_x P^+(\omega, x) \end{bmatrix}, \tag{5.3}$$

and

$$P^+(\omega, x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{2}{\omega}\Phi^2(x, \omega, 0)\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{5.4}$$

Using the fact that

$$\partial_\alpha \Phi^2 = -\tfrac{1}{2}\Phi^2(\omega - \Phi^2),$$

which can be readily verified using the representation given in (5.1), it is easy to show that

$$\partial_\alpha g = -4\omega\Phi^2 + 13\Phi^4, \qquad \partial_\alpha h = -2\omega\Phi^2 + 8\Phi^4.$$

Since

$$\partial_\alpha M(\omega, x) = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \partial_\alpha g(x) & \partial_\alpha h(x) & 0 & 0 \\ \partial_\alpha h(x) & \partial_\alpha g(x) & 0 & 0 \end{bmatrix},$$

a tedious calculation then shows that

$$\partial_\alpha M\mathbf{Y}_s^+ \cdot \mathbf{Z}_s = -2\sqrt{2\omega}\left(\frac{168}{\omega^2}\Phi^8 - \frac{132}{\omega}\Phi^6 + 37\Phi^4 - 4\omega\Phi^2\right).$$

Thus, upon using (5.2) and integrating,

$$\partial_\alpha E(\omega, 0) = -\frac{2\sqrt{2}}{3}\omega^2. \tag{5.5}$$

Now set $\beta = \alpha\omega$. Eq. (5.5) can then be rewritten as

$$\partial_\beta E(\omega, 0) = -\frac{2\sqrt{2}}{3}\omega.$$

As a consequence of Theorem 4.4, it can be seen that if $\beta < 0$, then $E(\lambda, \beta) \neq 0$ for $\lambda \in U$, while if $\beta > 0$, then $E(\lambda^*, \beta) = 0$, where

$$\lambda^* = \omega\left(1 - \frac{1}{36}\beta^2\right) + \mathrm{o}(\beta^2) \in \mathbb{R}. \tag{5.6}$$

Thus, if $\beta > 0$, an eigenvalue moves out of the continuous spectrum. Note that $\lambda^* \in \mathbb{R}$ due to the symmetries of the eigenvalue problem. Indeed, λ is an eigenvalue if, and only if, $-\lambda$ is, see Section 3. Since we are in the region where the Evans function has not been extended artificially, any eigenvalue corresponds to a zero of $E(\lambda, \alpha)$. Thus, since there is precisely one eigenvalue bifurcating, it must be real. The following lemma has now been proved.

Lemma 5.1. Let $\beta = \alpha\omega$. If $0 < \beta \ll 1$, then one and only one eigenvalue moves out of the continuous spectrum, with that eigenvalue being real and its location given by (5.6). Furthermore, λ^* is the only zero of the Evans function in the half-plane $\mathrm{Re}\,\lambda > 0$. If $0 < -\beta \ll 1$, then the Evans function is nonzero for all λ such that $\mathrm{Re}\,\lambda > 0$.

Remark 5.2. Eq. (5.6) agrees with the results of Afanasjev et al. [1] and Pelinovsky et al. [37] in the case that $\alpha = 1$.

5.2. Evaluation at $\lambda = \omega$: PFNLS

The PFNLS is given by

$$\mathrm{i}\phi_t + (\partial_x^2 - \omega)\phi + 4|\phi|^2\phi + \mathrm{i}\epsilon(\gamma\phi - \mu\phi^*) = 0, \tag{5.7}$$

where $\epsilon \geq 0$ is not necessarily small. By setting $\phi \to \phi\mathrm{e}^{-\mathrm{i}\theta}$, where

$$\cos 2\theta = \frac{\gamma}{\mu},$$

Eq. (5.7) can be rewritten as

$$\mathrm{i}\phi_t + (\partial_x^2 - \omega)\phi + 4|\phi|^2\phi + \mathrm{i}\epsilon(\gamma\phi - \mu\phi^*\mathrm{e}^{-\mathrm{i}2\theta}) = 0. \tag{5.8}$$

The solitary-wave solution is given by

$$\Phi(x, \omega, \epsilon) = \sqrt{\frac{\omega + \epsilon\mu\sin 2\theta}{2}}\,\mathrm{sech}(\sqrt{\omega + \epsilon\mu\sin 2\theta}\,x).$$

It is known that if $\mu\sin 2\theta < 0$, then the wave is unstable [6].

As a system, Eq. (5.8) can be written as

$$\begin{aligned} \mathrm{i}\phi_t + (\partial_x^2 - \omega)\phi + 4\phi^2\psi + \mathrm{i}\epsilon(\gamma\phi - \mu\psi\mathrm{e}^{-\mathrm{i}2\theta}) &= 0, \\ -\mathrm{i}\psi_t + (\partial_x^2 - \omega)\psi + 4\phi\psi^2 - \mathrm{i}\epsilon(\gamma\psi - \mu\phi\mathrm{e}^{+\mathrm{i}2\theta}) &= 0, \end{aligned}$$

where $\psi = \phi^*$. Linearization yields the system

$$\mathrm{i}P_t + LP + \mathrm{i}\epsilon\gamma P = 0,$$

where

$$L = (\partial_x^2 - \omega)\sigma_3 + 4\Phi^2(2\sigma_3 + \mathrm{i}\sigma_2) - \mathrm{i}\epsilon\mu\cos 2\theta\,\sigma_1 + \mathrm{i}\epsilon\mu\sin 2\theta\,\sigma_2.$$

Here the σ_i are the Pauli spin matrices, i.e.,

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

By setting $P(x, t) \to P(x)\mathrm{e}^{\rho t}$, one then gets the linear eigenvalue problem

$$(L + \mathrm{i}(\rho + \epsilon\gamma))P = 0.$$

Setting

$$\rho = \mathrm{i}\lambda - \epsilon\gamma,$$

we then get the eigenvalue problem

$$(L-\lambda)P = 0. \tag{5.9}$$

Remark 5.3. The eigenvalue problem admits a symmetry: λ is an eigenvalue if, and only if, $-\lambda$ is.

Letting $\mathbf{Y} = [P, Q]^{\mathrm{T}}$, where $Q = P'$, the eigenvalue equation can be rewritten as the first-order system

$$\mathbf{Y}' = M(\lambda, x, \epsilon)\mathbf{Y},$$

where

$$M(\lambda, x, \epsilon) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega + \lambda - 8\Phi^2 & -4\Phi^2 + \epsilon\mu\sin 2\theta + \mathrm{i}\epsilon\gamma & 0 & 0 \\ -4\Phi^2 + \epsilon\mu\sin 2\theta - \mathrm{i}\epsilon\gamma & \omega - \lambda - 8\Phi^2 & 0 & 0 \end{pmatrix}.$$

We want to apply Theorem 4.2 and calculate the derivative of the Evans function $E(\lambda, \epsilon)$ with respect to ϵ. In order to verify Hypothesis 2.11 in Section 2.3, we have to show that the eigenvalues of the limiting matrix

$$M_0(\lambda, \epsilon) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega + \lambda & \epsilon\mu\sin 2\theta + \mathrm{i}\epsilon\gamma & 0 & 0 \\ \epsilon\mu\sin 2\theta - \mathrm{i}\epsilon\gamma & \omega - \lambda & 0 & 0 \end{pmatrix}$$

are independent of ϵ. It is easy to check that they are independent of ϵ after replacing λ by

$$\lambda = \sqrt{\tilde{\lambda}^2 - \epsilon^2(\mu^2\sin^2 2\theta + \gamma^2)} \tag{5.10}$$

for $\tilde{\lambda} \in U$. This transformations accounts for the fact that the essential spectrum, which is located on the real axis, moves towards zero as ϵ increases. Note that we have

$$\tilde{E}(\tilde{\lambda}, \epsilon) = E\left(\sqrt{\tilde{\lambda}^2 - \epsilon^2(\mu^2\sin^2 2\theta + \gamma^2)}, \epsilon\right)$$

for the new Evans function $\tilde{E}(\tilde{\lambda}, \epsilon)$, and that $\tilde{E}(\tilde{\lambda}, \epsilon)$ is differentiable in ϵ.

By Theorem 4.2 we have that

$$\partial_\epsilon \tilde{E}(\omega, 0) = \int_{-\infty}^{\infty} \partial_\epsilon M(\omega, x)\mathbf{Y}_{\mathrm{s}}^{+}(\omega, x) \cdot \mathbf{Z}_{\mathrm{s}}(\omega, x)\,\mathrm{d}x.$$

Upon substituting $\mathbf{Y}_{\mathrm{s}}^{+}$ and $\mathbf{Z}_{\mathrm{s}}$ from (5.3) and (5.4), a routine, yet tedious, calculation shows that

$$\partial_\epsilon M\mathbf{Y}_{\mathrm{s}}^{+} \cdot \mathbf{Z}_{\mathrm{s}} = 16\sqrt{2\omega}\left(-1 + \frac{6}{\omega}\Phi^2 - \frac{12}{\omega^2}\Phi^4\right)\partial_\epsilon(\Phi^2) + 8\sqrt{\frac{2}{\omega}}\,\Phi^2\left(-1 + \frac{2}{\omega}\Phi^2\right)\mu\sin 2\theta.$$

Since

$$\partial_\epsilon(\Phi^2) = \frac{\mu\sin 2\theta}{\omega}\left(\Phi^2 + \frac{1}{2}x\partial_x(\Phi^2)\right),$$

and

$$\int_{-\infty}^{\infty} x\Phi^{2k}(x)\partial_x(\Phi^2(x))\,\mathrm{d}x = \frac{1}{k+1}\int_{-\infty}^{\infty} x\partial_x(\Phi^{2(1+k)}(x))\,\mathrm{d}x = -\frac{1}{k+1}\int_{-\infty}^{\infty} \Phi^{2(1+k)}(x)\,\mathrm{d}x,$$

upon integrating we see that

$$\partial_\epsilon \tilde{E}(\omega, 0) = -8\sqrt{2}\mu \sin 2\theta.$$

As a consequence of Theorem 4.4, we see that if $\mu \sin 2\theta < 0$, then $\tilde{E}(\tilde{\lambda}, \epsilon) \neq 0$ for $\tilde{\lambda}$ near ω, while if $\mu \sin 2\theta > 0$, then $\tilde{E}(\tilde{\lambda}^*, \epsilon) = 0$, where

$$\tilde{\lambda}^* = \omega\left(1 - \frac{8}{\omega^2}\mu^2 \sin^2(2\theta)\epsilon^2\right) + \mathrm{o}(\epsilon^2).$$

Going back to the original variable λ given in (5.10), we have $E(\lambda^*, \epsilon) = 0$, where

$$\lambda^* = \omega\left(1 - \frac{17\mu^2 - 16\gamma^2}{2\omega^2}\epsilon^2\right) + \mathrm{o}(\epsilon^2). \tag{5.11}$$

In the above equation, the relation $\mu \sin 2\theta = \pm\sqrt{\mu^2 - \gamma^2}$ was used. Note that $\lambda^* \in \mathbb{R}$ on account of the symmetries of (5.9) mentioned in Remark 5.3. Summarizing the above discussion, we have the following lemma.

Lemma 5.4. Let $0 < \epsilon \ll 1$. If $\mu \sin 2\theta < 0$, then the Evans function is nonzero for all λ such that $\mathrm{Re}\,\lambda > \mathrm{O}(\epsilon) > 0$. If $\mu \sin 2\theta > 0$, then one and only one eigenvalue moves out of the continuous spectrum. This eigenvalue is real and given by (5.11). Furthermore, λ^* is the only zero of the extended Evans function in the half-plane $\mathrm{Re}\,\lambda > \mathrm{O}(\epsilon) > 0$.

Remark 5.5. The Evans function will have four discrete zeros which are of $\mathrm{O}(\epsilon)$ (see Section 7).

5.3. Evaluation at $\lambda = \omega$: PFCQNLS

Consider the PFCQNLS

$$\mathrm{i}\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi + 3\alpha|\phi|^4\phi + \mathrm{i}\epsilon(\gamma\phi - \mu\phi^*) = 0.$$

The Evans function will be given by $E(\lambda, \beta, \epsilon)$, where $\beta = \alpha\omega$. As a consequence of the results of the previous sections, we know that after changing variables according to (5.10)

$$\partial_\beta E(\omega, 0, 0) = -\frac{2\sqrt{2}}{3}\omega, \qquad \partial_\epsilon E(\omega, 0, 0) = -8\sqrt{2}\mu \sin 2\theta.$$

Therefore, as a result of Theorem 4.4, we get the following lemma.

Lemma 5.6. Let $0 < \epsilon$, $|\alpha\omega| \ll 1$. If

$$\alpha < -\frac{12\mu \sin 2\theta}{\omega^2}\epsilon,$$

then the Evans function is nonzero for all λ such that $\mathrm{Re}\,\lambda > \mathrm{O}(\epsilon) > 0$, and hence no eigenvalues bifurcate out of the continuous spectrum. Otherwise, one eigenvalue bifurcates out of the continuous spectrum.

From a physical viewpoint, this means that a nonlinearly saturated refraction index (represented by a negative α) prevents an eigenvalue from popping out of the essential spectrum. As explained in Section 1, this may stabilize the pulse for larger values of ϵ, since Hopf bifurcations are inhibited.

6. The cubic–quintic nonlinear Schrödinger equation

The PCQNLS is given by

$$\mathrm{i}\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi + 3\alpha|\phi|^4\phi = \mathrm{i}\epsilon(d_1\phi_{xx} + d_2\phi + d_3|\phi|^2\phi + d_4|\phi|^4\phi),$$

where $\epsilon > 0$ is small and the other parameters are real and of $\mathrm{O}(1)$. In this section, we will investigate the stability of the solitary wave $\Phi(x, \omega, \epsilon)$, where

$$\Phi^2(x, \omega, 0) = \frac{\omega}{1 + \sqrt{1+\alpha\omega}\,\cosh(2\sqrt{\omega}\,x)}.$$

The wave $\Phi(x, \omega, \epsilon)$ is a smooth perturbation of $\Phi(x, \omega, 0)$ [44]. In [19], it is shown that in order for the wave to persist, it must be true that $d_3 = d_3^*$, where

$$d_3^* = d_1 - C_{d_2}d_2 - C_{d_4}(d_4 - \alpha d_1) + \mathrm{O}(\epsilon),$$

and the constants are given by

$$C_{d_2} = \frac{3}{\omega}\left(1 + \frac{4}{15}\beta + \mathrm{O}(\beta^2)\right), \quad C_{d_4} = \frac{2}{5}\omega\left(1 - \frac{9}{35}\beta + \mathrm{O}(\beta^2)\right),$$

where $\beta = \alpha\omega$. For small β and ϵ, one can rewrite the existence condition to get that the wave persists for $\omega = \omega_\pm$, where

$$\omega_\pm = \frac{5}{4|d_4|}\left(d_3 - d_1 \pm \sqrt{(d_3 - d_1)^2 - \frac{24}{5}d_2d_4}\right).$$

In the above expression it is assumed that d_2 and d_4 are negative, $d_3 > d_1 > 0$, and $(d_3 - d_1)^2 > \frac{24}{5}d_2d_4$.

When locating the eigenvalues, it is first necessary to locate those eigenvalues near the origin. This study was undertaken in [19], and the following result was derived.

Lemma 6.1. Consider the PCQNLS. Set $\beta = \alpha\omega$, and assume that $0 < \epsilon, |\beta| \ll 1$. When $\omega = \omega_-$ there is one positive real eigenvalue, and one negative real eigenvalue, both of which are $\mathrm{O}(\epsilon)$. If $\omega = \omega_+$, then there are two negative real eigenvalues which are $\mathrm{O}(\epsilon)$. Except for the double eigenvalue at zero, there are no other eigenvalues of $\mathrm{O}(\epsilon)$.

Remark 6.2. The condition $d_1 > 0$ means that the PCQNLS is a well-posed PDE. The condition $d_2 < 0$ means that the solution $\phi = 0$ is stable for the PCQNLS.

Remark 6.3. A more detailed discussion is given in [19] for the circumstance that $-1 < \beta \leq 0$ is not necessarily small.

Remark 6.4. One should consult Kodama et al. [25] for a formal calculation when $\alpha = \mathrm{O}(\epsilon)$.

Proof of Theorem 1.3. For the rest of the discussion, assume that $\omega = \omega_+$, so that there are no unstable eigenvalues near zero. In order to determine the stability of the wave, it is then only necessary to locate all eigenvalues which are close to the edge of the essential spectrum. We will again formulate the eigenvalue problem as in Section 3, so that unstable eigenvalues have $\mathrm{Im}\,\lambda < 0$ (see Remark 3.1).

Since $d_1 > 0$ and $d_2 < 0$, when $\epsilon > 0$ the continuous spectrum is contained in the left half-plane and bounded away from the imaginary axis. It is then straightforward to verify Hypothesis 2.11 in Section 2.3 after using the transformation

$$\lambda = (1 + \mathrm{i}\epsilon d_1)\tilde{\lambda} - \mathrm{i}\epsilon d_2 - \mathrm{i}\epsilon d_1 \omega, \tag{6.1}$$

as in Section 5.2. On account of Lemmas 2.10 and 2.12, the Evans function can then be extended continuously for $\epsilon \geq 0$ and all λ with $\mathrm{Re}\,\lambda \geq 0$, and it is differentiable in ϵ at $\lambda = \omega$.

First, we evaluate the Evans function $E(\tilde{\lambda}, \alpha, \epsilon)$ along lines $(\alpha, \epsilon) = (\alpha, -\delta\alpha)$ for small $\delta \geq 0$ and negative $\alpha < 0$. Along that curve, the relevant term is then given by

$$(E_\alpha(\omega, 0, 0) + \delta E_\epsilon(\omega, 0, 0))\alpha. \tag{6.2}$$

Eigenvalues cannot bifurcate from the essential spectrum whenever this expression is in the sector of the complex plane given in Theorem 4.4(i). On account of the discussion in Section 5.1, we see that $\alpha E_\alpha(\omega, 0, 0)$ is in the aforementioned sector. Therefore, expression (6.2) is also in the sector provided $\delta < \delta_0$ for some sufficiently small $\delta_0 > 0$. Thus, for $\alpha < 0$ and $0 \leq \epsilon < \frac{1}{2}\delta_0|\alpha|$ no eigenvalues pop out of the essential spectrum, and the pulse is stable.

Second, let α be such that $|\alpha| \leq K\epsilon^\gamma$ for some $\gamma > \frac{1}{2}$, where $K > 0$ is some fixed constant and $\epsilon > 0$. If an eigenvalue $\tilde{\lambda}_*$ pops out of the essential spectrum, it satisfies the estimate

$$|\tilde{\lambda}_* - \omega| \leq C(|\alpha|^2 + \epsilon^2) \leq C(\epsilon^{2\gamma} + \epsilon^2)$$

for some possibly different constants $C > 0$. Indeed, this is a consequence of Theorem 4.4 and Remark 4.5. Therefore, upon inspecting (6.1), we see that the eigenvalue satisfies

$$\lambda_* = (1 + \mathrm{i}\epsilon d_1)\tilde{\lambda}_* - \mathrm{i}\epsilon d_2 - \mathrm{i}\epsilon d_1\omega = \omega - \mathrm{i}\epsilon d_2 + \mathrm{O}(\epsilon^{2\gamma} + \epsilon^2).$$

Since $2\gamma > 1$ and $d_2 < 0$, we have $\mathrm{Im}\,\lambda_* > 0$, which corresponds to a stable eigenvalue (see Remark 3.1). Hence, even if an eigenvalue pops out, it will not induce an instability. Theorem 1.3 as well as Remark 1.4 have now been proved. □

Now that the primary pulse for the PCQNLS has been shown to be stable, it is natural to inquire as to the existence and stability of multiple-pulse solutions. The existence question has been partially answered in [21]. There the existence of N-pulses which are evenly spaced has been shown; see also [34]. Sandstede [41] has developed a program to study the stability of the N-pulse solutions in the case that $\partial_\lambda E(0) \neq 0$. In order to determine the stability of the multiple-pulse solutions for the PCQNLS, these ideas must be extended to cover the case that $\partial_\lambda E(0) = 0$, but $\partial_\lambda^2 E(0) \neq 0$. This extension is possible and will be the focus of a future paper.

7. The parametrically forced nonlinear Schrödinger equation

In this section, we study the parametrically forced nonlinear Schrödinger equation (PFNLS) given by

$$\mathrm{i}\phi_t + (\partial_x^2 - \omega)\phi + 4|\phi|^2\phi + \mathrm{i}\epsilon(\gamma\phi - \mu\phi^*) = 0, \tag{7.1}$$

where $\omega > 0$ and $\epsilon \geq 0$. Initially, no size restriction on the size of ϵ will be made. By setting $\phi \to \phi \mathrm{e}^{-\mathrm{i}\theta}$, where

$$\cos 2\theta = \frac{\gamma}{\mu}, \tag{7.2}$$

Eq. (7.1) can be rewritten as

$$\mathrm{i}\phi_t + (\partial_x^2 - \omega)\phi + 4|\phi|^2\phi + \mathrm{i}\epsilon(\gamma\phi - \mu\phi^* \mathrm{e}^{-\mathrm{i}2\theta}) = 0. \tag{7.3}$$

The solitary-wave solution is given by

$$\Phi(x, \omega, \epsilon) = \sqrt{\frac{\beta}{2}}\,\mathrm{sech}(\sqrt{\beta}\,x), \tag{7.4}$$

where

$$\beta = \omega + \epsilon\mu \sin 2\theta. \tag{7.5}$$

Note that if θ satisfies (7.2), so does $\theta + \pi$. Thus the sign of the sine term in (7.5) can be chosen positive or negative as we wish.

It is known that if $\mu \sin 2\theta < 0$, then the wave Φ is unstable [6]. We will show that the wave is stable for all $\epsilon > 0$ sufficiently small if $\mu \sin 2\theta > 0$. Of interest is then the existence of multiple pulses resembling N copies of the stable primary wave Φ. Using results from Sandstede et al. [42], we prove that stable N-pulses exist provided a small dissipative term is added to the (7.1):

$$\mathrm{i}\phi_t + (\partial_x^2 - \omega)\phi + 4|\phi|^2\phi + \mathrm{i}\epsilon(\gamma\phi - \mu\phi^*) = \mathrm{i}\delta\partial_x^2\phi, \tag{7.6}$$

$0 < \delta \ll \epsilon$. The dissipative term models spectral filtering of the signals in the optical fiber.

7.1. Stability of Φ

We consider equation (7.3)

$$\mathrm{i}\phi_t + \phi_{xx} - \omega\phi + 4|\phi|^2\phi + \mathrm{i}\epsilon(\gamma\phi - \mu\phi^* \mathrm{e}^{-\mathrm{i}2\theta}) = 0$$

and investigate the stability of the primary solitary-wave

$$\Phi(x, \omega, \epsilon) = \sqrt{\frac{\beta}{2}}\,\mathrm{sech}(\sqrt{\beta}\,x)$$

with $\beta = \omega + \epsilon\mu \sin 2\theta$ and $\mu \sin 2\theta > 0$.

Theorem 7.1. Let $\gamma > 0$, $\mu \neq 0$, and $\omega > 0$. Assume that θ is chosen such that $\mu \sin 2\theta > 0$. The solitary wave Φ given in (7.4) is then orbitally exponentially stable with respect to Eq. (7.3) for all $\epsilon > 0$ sufficiently small.

Proof. First, we determine the spectrum of the linearization of (7.3) around the wave Φ for small $\epsilon > 0$. It is convenient to write Eq. (7.3) as a system by writing down the equations for the real and imaginary part of ϕ. Setting $\phi = u + \mathrm{i}v$, we obtain

$$\begin{aligned} u_t &= -(v_{xx} - (2\omega - \beta)v + 4(u^2 + v^2)v) \\ v_t &= u_{xx} - \beta u + 4(u^2 + v^2)u - 2\epsilon\gamma v. \end{aligned} \tag{7.7}$$

The eigenvalue problem of the linearization of (7.7) about the wave Φ reads

$$LP = \rho P,$$

where

$$L = \begin{pmatrix} 0 & -L_- \\ L_+ & -2\epsilon\gamma \end{pmatrix},$$

and

$$L_- = \partial_x^2 + 4\Phi^2 - (2\omega - \beta), \quad L_+ = \partial_x^2 + 12\Phi^2 - \beta.$$

This eigenvalue problem has been considered in Section 5.2. By Lemma 5.4, the spectrum outside a small neighborhood of zero is contained in the line $\mathrm{Re}\,\rho = -\epsilon\gamma$. Therefore, it suffices to consider eigenvalues near zero.

For that purpose, we rescale $y := \sqrt{\beta}x$ and denote the resulting operators again by $L_\pm$. We then have the equivalent eigenvalue problem

$$\begin{pmatrix} 0 & -L_- \\ L_+ & -\dfrac{2\epsilon\gamma}{\beta} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \frac{1}{\beta}\rho \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \tag{7.8}$$

with

$$L_- = \partial_y^2 + 2\,\mathrm{sech}^2 y - q^2, \quad L_+ = \partial_y^2 + 6\,\mathrm{sech}^2 y - 1,$$

and

$$q^2 = \frac{2\omega - \beta}{\beta} = \frac{\omega - \epsilon\mu \sin 2\theta}{\omega + \epsilon\mu \sin 2\theta} < 1. \tag{7.9}$$

Note that this transformation is actually meaningful as long as $\omega > \epsilon\mu \sin 2\theta$. The eigenvalue problem (7.8) can be written as the fourth-order equation

$$L_- L_+ P_1 = -\frac{\rho(\rho + 2\epsilon\gamma)}{\beta^2} P_1. \tag{7.10}$$

In passing, we note that the spectrum is symmetric with respect to the axis $\mathrm{Re}\,\rho = -\epsilon\gamma$, i.e., $\rho - 2\epsilon\gamma$ is an eigenvalue whenever ρ is.

It has been shown by Kutz and Kath [27] (see also [4]) that zero and $\nu_*(\epsilon) = \mathrm{O}(\epsilon) > 0$ are all of the eigenvalues of the equation

$$L_- L_+ P_1 = \nu P_1 \tag{7.11}$$

inside a small neighborhood of zero for $\epsilon > 0$ small. Therefore, the eigenvalues of (7.10) near zero are simple and given by

$$\rho_1 = 0, \quad \rho_2 = -2\epsilon\gamma, \quad \rho_{3,4} = -\epsilon\gamma \pm \sqrt{\epsilon^2\gamma^2 - \beta^2\nu_*(\epsilon)}.$$

In particular, since $\nu_*(\epsilon) > c\epsilon$ for some $c > 0$, the eigenvalues $\rho_{3,4}$ have nonzero imaginary part with $\mathrm{Re}\,\rho_{3,4} < 0$ (see Fig. 2).

Summarizing the above discussion, the spectrum of the operator L is contained in the left half-plane with the exception of a simple eigenvalue at zero. Unfortunately, however, L will generate only a C^0-semigroup. For these groups, the Spectral Theorem does not hold in general and therefore we cannot conclude asymptotic stability from the knowledge of the spectrum of L alone. However, it follows from a result by Prüß [38, Corollary 4] that if the resolvent $(L - \rho)^{-1}$ is bounded uniformly in the right half-plane outside any small neighborhood of zero as an operator in $L^2(\mathbb{R})$, then the Spectral Theorem holds. In particular, the wave Φ and its translates form an exponentially attracting set in $L^2(\mathbb{R})$.

For the rest of the discussion in this section we will be making the necessary resolvent estimates. Let ρ be such that Re $\rho \geq 0$. Set

$$\tilde{\rho} = \frac{\rho}{\beta}, \quad \tilde{\gamma} = \frac{\rho}{\beta}, \qquad \nu = \tilde{\rho}(\tilde{\rho} + 2\epsilon\tilde{\gamma}).$$

In the following, we will omit the tilde. In order to estimate the resolvent, we must solve

$$\begin{pmatrix} -\rho & -L_- \\ L_+ & -(\rho + 2\epsilon\gamma) \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix},$$

that is, $(L - \rho)P = G$, where $G_i \in L^2(\mathbb{R})$. Since $0 < q^2 < 1$, the operator L_- is invertible [4, Section 2]; therefore, we can solve the first equation for P_2 to get

$$P_2 = -L_-^{-1}(\rho P_1 + G_1), \tag{7.12}$$

and substitute this result into the second equation to get

$$(L_+ + \nu L_-^{-1})P_1 = G_2 - (\rho + 2\epsilon\gamma)L_-^{-1}G_1. \tag{7.13}$$

In solving Eqs. (7.12) and (7.13) it is sufficient to consider the case that $|\rho|$ is large, since the resolvent is bounded in bounded sets. Define the fourth-order operator

$$A = L_+ L_-,$$

and note that $A^* = L_- L_+$. We know from the results above that the fourth-order operators $A + \nu$ and $A^* + \nu$ are invertible for any large $|\rho|$ with Re $\rho \geq 0$. Therefore, we can solve Eqs. (7.12) and (7.13) to get

$$\begin{aligned} P_1 &= -(\rho + 2\epsilon\gamma)(A^* + \nu)^{-1}G_1 + L_-(A + \nu)^{-1}G_2, \\ P_2 &= -L_+(A^* + \nu)^{-1}G_1 - \rho(A + \nu)^{-1}G_2. \end{aligned} \tag{7.14}$$

We shall obtain estimates for $P = (P_1, P_2)$ in terms of $G = (G_1, G_2)$ when $|\rho|$ is large. We claim that for $|\rho|$ large

$$\|(A + \nu)^{-1}\| \leq M/|\rho|, \qquad \|L_-(A + \nu)^{-1}\| \leq M, \tag{7.15}$$

with analogous estimates for the adjoint operators. The constant $M > 0$ may depend on ϵ but not on ρ. Assume for a moment that the claim is true. We then have from Eqs. (7.14) and (7.15) that

$$(|P_1| + |P_2|) \leq (M + 1)(|G_1| + |G_2|)$$

for all ρ with Re $\rho \geq 0$ and $|\rho|$ large.

It remains therefore to prove the above claim, which means that we must estimate the norm of the operator $(A + \nu)^{-1}$. The operators A and A^* are sectorial, so that their resolvent can be estimated in a sector. However, $\nu = \rho(\rho + 2\epsilon\gamma)$ is not contained in any sector near the positive axis, but instead forms a parabola. A priori, it is then not obvious why the estimates (7.15) should be true.

The key is that the operator A is self-adjoint up to terms involving only first-order derivatives. Indeed, it is easy to check that

$$\begin{aligned}Au &= (\partial_y^2 + 6\,\mathrm{sech}^2 y - 1)(\partial_y^2 + 2\,\mathrm{sech}^2 y - q^2)u \\ &= \partial_y^4 u + 4\,\mathrm{sech}^2 y\,\partial_y^2 u + \partial_y^2(4u\,\mathrm{sech}^2 y) - (1+q^2)\partial_y^2 u \\ &\quad + 2(\mathrm{sech}^2 y)_y u_y + (2(\mathrm{sech}^2 y)_{yy} + (2\,\mathrm{sech}^2 y - q^2)(6\,\mathrm{sech}^2 y - 1))u.\end{aligned}$$

In other words, we have

$$Au = Bu + Ru,$$

where

$$\begin{aligned}Bu &= \partial_y^4 u + 4\,\mathrm{sech}^2 y\,\partial_y^2 u + \partial_y^2(4u\,\mathrm{sech}^2 y) - (1+q^2)\partial_y^2 u \\ &\quad + (2(\mathrm{sech}^2 y)_{yy} + (2\,\mathrm{sech}^2 y - q^2)(6\,\mathrm{sech}^2 y - 1))u\end{aligned}$$

is self-adjoint and

$$Ru = 2(\mathrm{sech}^2 y)_y u_y.$$

Note that $RB^{-1/4}$ is a bounded operator.

Using the spectral family associated with B, we see that

$$\|(B+\nu)^{-1}\| \le M/|\rho|, \qquad \|B^{1/4}(B+\nu)^{-1}\| \le M/|\rho|^{1/2}, \qquad \|B^{1/2}(B+\nu)^{-1}\| \le M \tag{7.16}$$

uniformly for Re $\rho \ge 0$ and $|\rho|$ large. We obtain

$$\begin{aligned}(A+\nu)^{-1}u &= (B+R+\nu)^{-1}u \\ &= (B+\nu)^{-1}(\mathrm{id} + R(B+\nu)^{-1})^{-1} \\ &= (B+\nu)^{-1}(\mathrm{id} + RB^{-1/4}B^{1/4}(B+\nu)^{-1})^{-1}.\end{aligned}$$

It follows from (7.16) and the boundedness of $RB^{-1/4}$ that the terms appearing in the above equation are well-defined for all $|\rho|$ sufficiently large. Note that it is crucial that R is only of first order. Otherwise, it would not be clear whether the operator $(\mathrm{id} + R(B+\nu)^{-1})$ is invertible; for instance, for $R = B^{1/2}$ the operator $R(B+\nu)^{-1}$ can only be estimated by a constant. The estimates (7.15) are now an immediate consequence of (7.16), and the proof of Theorem 7.1 is complete. □

Remark 7.2. Since $\nu = 0$ is a simple eigenvalue of (7.11) for all $\epsilon > 0$ (see [4]) and the eigenvalues ρ of (7.8) satisfy $\nu = \rho(\rho + 2\epsilon\gamma)$, we know that if the wave is to become unstable as ϵ increases, it must do so through a Hopf bifurcation.

Remark 7.3. If $\mu \sin 2\theta < 0$, it follows from [27] that the eigenvalue $\nu_*(\epsilon)$ is negative and hence the pulse Φ is unstable for all small ϵ. Thus, one gets another proof of the local instability result presented in [6]. From [4], one can conclude that the wave will never stabilize.

7.2. Existence and stability of multiple pulses

Consider Eq. (7.6)

$$\mathrm{i}\phi_t + (\partial_x^2 - \omega)\phi + 4|\phi|^2\phi + \mathrm{i}\epsilon(\gamma\phi - \mu\phi^*) = \mathrm{i}\delta\partial_x^2\phi$$

for $\delta > 0$ small. The associated steady-state equation reads

$$\phi_{xx} - \omega\phi + 4|\phi|^2\phi + \mathrm{i}\epsilon(\gamma\phi - \mu\phi^*) = \mathrm{i}\delta\phi_{xx}. \tag{7.17}$$

Note that (7.17) is reversible, that is, $\phi(x)$ satisfies (7.17) if, and only if, $\phi(-x)$ does. Since zero is simple eigenvalue of the linearization of (7.1) around Φ, it follows from the results of Vanderbauwhede et al. [45] that the pulse Φ persists for $\delta > 0$. Moreover, since the linearization of (7.6) around the perturbed wave is sectorial, the pulse will be stable for $\delta > 0$ small. Therefore, we have the following corollary of Theorem 7.1.

Corollary 7.4. Eq. (7.6) has a stable solitary-wave solution for all $\delta > 0$ sufficiently small which approaches Φ as $\delta \to 0$.

Consider the steady-state equation (7.17)

$$\phi_{xx} - \omega\phi + 4|\phi|^2\phi + \mathrm{i}\epsilon(\gamma\phi - \mu\phi^*) = \mathrm{i}\delta\phi_{xx}$$

of Eq. (7.6) for $\delta \geq 0$. By Theorem 7.1, Eq. (7.6) admits the stable solitary-wave solution Φ for $\epsilon > 0$, which by Corollary 7.4 persists for $0 \leq \delta \ll \epsilon$. Note that Eq. (7.17) is reversible ($\phi(x)$ is a solution if, and only if, $\phi(-x)$ is) and admits the $\mathbb{Z}_2$-symmetry $\phi \to -\phi$ (ϕ is a solution if, and only if, $-\phi$ is).

We are interested in the existence and stability of multiple solitary waves. These are solutions of (7.17) resembling N widely spaced copies of Φ or $-\Phi$. There are several ways to obtain N-pulses of different shapes, since Φ and $-\Phi$ are concatenated. Denoting Φ and $-\Phi$ by "up" and "down", respectively, we may then consider arbitrary sequences of ups and downs corresponding to whether Φ or $-\Phi$ is used.

It has recently been proved in [42] that stable multiple pulses are expected to occur near so-called orbit-flip bifurcations. This bifurcation is characterized by the property that when $\delta = 0$, the wave Φ is contained in the strong stable manifold of the equilibrium $\phi = 0$, with this no longer being true for $\delta \neq 0$. Now, the eigenvalues of the linearization of (7.17) at $\phi = 0$ for $\delta = 0$ are given by $\pm\sqrt{\omega \pm \epsilon\mu \sin 2\theta}$ with θ given by $\cos 2\theta = \gamma/\mu$. Since we are interested in stable pulses, we assume $\mu \sin 2\theta > 0$. The equilibrium $\phi = 0$ of (7.17) is hyperbolic as long as $0 < \epsilon < \omega/\mu \sin 2\theta$. The stable primary pulse $\Phi(x)$ satisfies (7.17) for $\delta = 0$ and converges to zero exponentially with rate $\sqrt{\omega + \epsilon\mu \sin 2\theta}$ for $\sin\theta > 0$ as $|x| \to \infty$. Thus, it converges with the largest exponential rate possible. Since Φ is contained in the strong stable manifold when $\delta = 0$, an orbit-flip bifurcation is possible.

We have the following theorem concerning existence and stability of multiple solitary waves of (7.17). It is based on an application of Sandstede et al. [42, Theorems 1, 2, and 4].

Theorem 7.5. Fix $\epsilon > 0$ small and $N > 1$, then for any $0 < \delta < \delta(\epsilon, N)$ small, there exists a unique multiple solitary wave of up–down–up–down–$\cdots$ type. These pulses are stable with respect to Eq. (7.6). Any other N-pulse consisting of copies of Φ or $-\Phi$ is unstable.

Remark 7.6. There exist many other N-pulses besides the ones of up–down–up–down–$\cdots$ type, and we refer to [42] for the details.

Proof of Theorem 7.5. As mentioned above, the theorem is an application of results proved in [42]. In particular, we shall verify the hypotheses of Theorems 1 and 2 in that paper. Most of the hypotheses are concerned with the linearization of (7.17) for $\delta = 0$ around the wave Φ. However, this equation can be written as the fourth-order equation studied in [42, Section 4] (see (7.10) with $\rho = 0$ and [42, (4.9)]). Thus, it turns out that most of these hypotheses have already been verified in [42, Theorem 4]. The only assumption which we have to consider here is

Hypothesis (H4)(ii) in [42]. Assumption [42, (H4)(ii)] is used to compute the sign of a certain constant J_2 which determines the bifurcation direction. In fact, $J_2 > 0$ corresponds to the pulses bifurcating for $\delta > 0$.

The constant J_2 arises as follows. Recall that the steady-state equation of (7.6) written as a system for real and imaginary part is given by

$$\delta u_{xx} = -(v_{xx} - (2\omega - \beta)v + 4(u^2 + v^2)v),$$
$$\delta v_{xx} = u_{xx} - \beta u + 4(u^2 + v^2)u - 2\epsilon\gamma v.$$

Let Φ_δ denote the stable primary pulse of (7.6), with $\Phi_0 = \Phi$. We need to calculate the first-order expansion of Φ_δ. Since Φ_δ is smooth, we can substitute Φ_δ into the above equation and take the derivative with respect to δ at $\delta = 0$. The function $(u, v) = (\mathrm{d}/\mathrm{d}\delta)\Phi_\delta|_{\delta=0}$ satisfies

$$\Phi_{xx} = -(v_{xx} - (2\omega - \beta)v + 4\Phi^2 v), \qquad 0 = u_{xx} - \beta u + 12\Phi u - 2\epsilon\gamma v.$$

Solving the second equation for v and substituting the resulting expression into the first equation, we get

$$(\partial_x^2 + 4\Phi^2 - (2\omega - \beta))(\partial_x^2 + 12\Phi^2 - \beta)v = -2\epsilon\gamma\Phi_{xx},$$

i.e., $L_- L_+ v = -2\epsilon\gamma\Phi_{xx}$. It is now clear that the fourth-order equation investigated in [42], that is, the left-hand side of the above equation, and the parametrically forced NLS are related.

Substituting the expression for Φ and rescaling $y = \sqrt{\beta}x$, we obtain

$$(\partial_x^2 + 2\,\mathrm{sech}^2 y - q^2)(\partial_x^2 + 6\,\mathrm{sech}^2 y - 1)v = -\sqrt{\frac{2}{\beta}}\epsilon\gamma(\mathrm{sech}\, y - 2\,\mathrm{sech}^3 y) =: G(y),$$

where $q < 1$ has been defined in (7.9). The crucial point is that the constant J_2 is given by

$$J_2 = \int_{-\infty}^{\infty} G(y)\mathrm{e}^{qy}(q - \tanh y)\,\mathrm{d}y$$
$$= -\sqrt{\frac{2}{\beta}}\epsilon\gamma \int_{-\infty}^{\infty} (\mathrm{sech}\, y - 2\,\mathrm{sech}^3 y)\mathrm{e}^{qy}(q - \tanh y)\,\mathrm{d}y$$

(see [42, Section 4.1]). A straightforward calculation following [42] yields

$$J_2 = 4\sqrt{\frac{2}{\beta}}\epsilon\gamma \int_{-\infty}^{\infty} \mathrm{e}^{y/q}\,\mathrm{sech}^3 y \tanh y\,\mathrm{d}y > 0,$$

which is positive since $q > 0$. This coincides with the sign computed in [42], and hence the multiple pulses bifurcate for $\delta > 0$. The conclusion of the theorem follows now from [42, Theorem 4]. □

Remark 7.7. In fact, we have not used the assumption that $\epsilon > 0$ is small for the existence part of Theorem 7.5. This condition is needed only so that the stability of the multi-bump solutions can be ascertained. In order to have a stable multi-bump solution the primary pulse must be stable, and this has been shown in Theorem 7.1 only for $\epsilon > 0$ small. The assumption that $0 < \epsilon < \omega/\mu \sin 2\theta$ is needed to guarantee that the equilibrium $\phi = 0$ of (7.17) is hyperbolic. Therefore, for all $0 < \epsilon < \omega/\mu \sin 2\theta$, multiple solitary waves of up–down–up–··· type exist for $\delta > 0$ small, and they are stable as long as the primary pulse Φ is stable.

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