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Eigenvalues, and instabilities of solitary waves

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Contents

We study a type of 'eigenvalue' problem for systems of linear ordinary differential equations with asymptotically constant coefficients by using the analytic function $D(\lambda)$ introduced by J. W. Evans (1975) in his study of the stability of nerve impulses. We develop a general theory of $D(\lambda)$ that clarifies the role of the essential spectrum in applications. New formulae for derivatives of $D(\lambda)$ are used to study linear exponential instabilities of solitary waves for generalizations of: (1) the Korteweg–de Vries equation (KdV); (2) the Benjamin–Bona–Mahoney equation (BBM); and (3) the regularized Boussinesq equation.

A pair of real eigenvalues exists, indicating a non-oscillatory instability, when the 'momentum' of the wave (a time-invariant functional associated with the hamiltonian structure of the equation) is a decreasing function of wave speed. Also

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we explain the mechanism of the transition to instability. Unexpectedly, these transitions are unlike typical transitions to instability in finite-dimensional hamiltonian systems. Instead they can be understood in terms of the motion of poles of the resolvent formula extended to a multi-sheeted Riemann surface. Finally, for a generalization of the KdV–Burgers equation (a model for bores), we show that a conjectured transition to instability does *not* involve real eigenvalues emerging from the origin, suggesting an oscillatory type of instability.

0. Introduction and outline

When studying the linear stability properties of nonlinear waves with inhomogeneous spatial structure, one usually encounters non-self-adjoint eigenvalue problems for variable coefficient operators. Rather few systematic techniques are available to study such problems. This stands in contrast to spectral problems derived from linearization about constant states, plane waves or other solutions with very special structure, which can be handled by standard eigenfunction expansion methods (e.g. Fourier transform or other classical eigenfunction expansions).

Here, we develop a method which can yield linear instability criteria in a certain general class of eigenvalue problems for systems of ordinary differential equations with asymptotically constant coefficients. We apply the method to study instabilities of some nonlinear evolution equations that model long-wave propagation in dispersive media. The hamiltonian structures of these equations share many features, and the instability criteria turn out to share a common form.

In particular, we establish criteria for the linear exponential instability of solitary wave solutions of generalized KdV, BBM, and regularized Boussinesq equations. These equations respectively have the forms:

I (generalized) Korteweg–de Vries equation (Korteweg & de Vries 1895; Whitham 1974)

$$\partial_t u+\partial_x f(u)+\partial_x^3 u = 0; \tag{gKdV}$$

II (generalized) Benjamin–Bona–Mahoney equation (Peregrine 1966; Benjamin *et al.* 1972; Whitham 1974)

$$\partial_t u+\partial_x u+\partial_x f(u)-\partial_t \partial_x^2 u = 0; \tag{gBBM}$$

III (generalized, regularized) Boussinesq equation (Whitham 1974)

$$\partial_t^2 u-\partial_x^2 u-\partial_x^2 f(u)-\partial_t^2 \partial_x^2 u = 0. \tag{gBou}$$

We assume throughout that f is a C^1 and convex for $u > 0$, with $f(0) = 0 = f'(0)$, and $f(u)/u \to \infty$ as u increases. (These hypotheses on f can be relaxed considerably. What is used below is the existence of a family of solitary waves for a range of speeds, c.)

Each of the equations above admits solitary wave solutions of the form $u(x,t) = u_c(x-ct)$ (for $c > 0$, $c > 1$ and $c^2 > 1$ respectively), where $u_c(x) \to 0$ as $|x| \to \infty$, at an exponential rate. For example, when the nonlinearity is homogeneous, with $f(u) = u^{p+1}/(p+1)$ for some $p \geqslant 1$, u_c has the form

$$u_c(x) = \alpha \operatorname{sech}^{2/p}(\gamma x), \tag{0.1}$$

for appropriate constants α, γ depending on c, p. We consider the evolution of small perturbations of the solitary wave, writing $u(x,t) = u_c(x-ct)+v(x-ct,t)$. Neglecting terms nonlinear in the perturbation v, we seek a solution of the linearized evolution equation for the perturbation $v(x,t)$ in the form $v = e^{\lambda t}Y(x)$, where $\lambda \in \mathbb{C}$ and Y satisfies the equation below, corresponding to the gKdV, gBBM and gBou equations respectively:

$$(\lambda - c\partial_x)\,Y + \partial_x(f'(u_c)\,Y) + \partial_x^3\,Y = 0, \tag{0.2a}$$

$$(\lambda - c\partial_x)\,(I - \partial_x^2)\,Y + \partial_x\,Y + \partial_x(f'(u_c)\,Y) = 0, \tag{0.2b}$$

$$(\lambda - c\partial_x)^2\,(I - \partial_x^2)\,Y - \partial_x^2\,Y - \partial_x^2(f'(u_c)\,Y) = 0. \tag{0.2c}$$

If this equation admits a square integrable solution for some λ with $\operatorname{Re}\lambda \neq 0$, we call λ an unstable eigenvalue for (0.1) and Y the associated eigenfunction. (By reflection, $-\lambda$ is an eigenvalue if λ is.)

Previous work (Laedke & Spatschek 1984; Weinstein 1985, 1986*a*, *b*; Bona *et al.* 1987; Souganidis & Strauss 1990; see also Grillakis *et al.* 1987, 1990) has shown that for a class of equations including the gKdV and gBBM equations, but not gBou, u_c is nonlinearly stable in H^1 (modulo spatial translations) if

$$\frac{d}{dc}\mathcal{N}[u_c] > 0, \tag{0.3}$$

and unstable if

$$\frac{d}{dc}\mathcal{N}[u_c] < 0, \tag{0.4}$$

where the functional $\mathcal{N}[u]$ is a generalized momentum or impulse functional associated with the translation-invariant hamiltonian structure of the equation, and is independent of time for solutions. For example, for gKdV,

$$\mathcal{N}[u] = \frac{1}{2}\int_{-\infty}^{\infty} u^2\,dx,$$

and (0.3) holds if and only if $p < 4$. The stability proofs rely on establishing that u_c is a local minimizer of a conserved energy functional, subject to the constraint of fixed momentum.

In this paper we show that for all three equations (0.2) above, *when the instability condition* (0.4) *holds, a real unstable eigenvalue exists with* $\lambda > 0$. *This gives rise to a non-oscillatory and exponentially growing solution of the linearized evolution equation. For the gKdV and gBBM equation, we also show that there is at most one eigenvalue with* $\operatorname{Re}\lambda > 0$. These results clarify the mechanisms for the instability proved for gKdV and gBBM in Bona *et al.* (1987) and Souganidis & Strauss (1990); see Laedke & Spatschek (1984) for an alternative approach to studying linear exponential instability. Our result concerning gBou seems to be the first regarding the stability or instability of the solitary waves of this equation. The methods used in the works mentioned above rely on being able to characterize the solitary wave as a critical point of a modified hamiltonian functional, whose second variation has a finite-dimensional negative subspace. But for gBou (Smereka 1992), and many other problems of physical and mathematical interest, the solitary wave (or other nonlinear mode of interest) does not appear to have such a characterization, and the second variation of the appropriate functional is highly indefinite. The method used in the present paper does not require such a characterization.

The conditions (0.3) and (0.4) can be expressed in terms of other functionals which arise in the hamiltonian formalism of I, II, and III. We have

$$\frac{d}{dc}\mathcal{N}[u_c] = -\frac{d^2}{dc^2}(\mathcal{H}[u_c] - c\mathcal{N}[u_c]) = \frac{1}{c}\frac{d}{dc}\mathcal{H}[u_c].$$

Here, $\mathscr{H}$ denotes the hamiltonian energy functional of the system (see §2). These expressions arise in some of the cited articles.

When the nonlinearity has the special form $f(u) = u^{p+1}/(p+1)$, the results of this paper together with previous works may be summarized as follows.

gKdV. If $p > 4$, then u_c is linearly exponentially unstable for all $c > 0$. For $1 < p < 4$, u_c is H^1-orbitally stable (Benjamin 1972; Bona 1975; Weinstein 1986*a*, *b*; Bona *et al.* 1987).

gBBM. For each $p > 4$, there exists a positive number $c_0(p)$, such that solitary waves u_c with $1 < c < c_0(p)$ are linearly exponentially unstable. Furthermore, for each $p > 4$ there is a threshold $\mathscr{N}_0(p) > 0$ such that for any $\mathscr{N} > \mathscr{N}_0(p)$, gBBM has two solitary wave profiles u_{c_1} and u_{c_2} with $1 < c_1 < c_0(p) < c_2$. u_{c_1} is exponentially unstable, while u_{c_2} is H^1-orbitally stable (Weinstein 1987; Souganidis & Strauss 1990). Here,

$$c_0(p) = (p/(4+2p))\,[1+\surd(2+\tfrac{1}{2}p)].$$

gBou. For each $p > 4$, u_c is linearly exponentially unstable if $1 < c^2 < c_0^2(p)$, where

$$c_0^2(p) = 3p/(4+2p).$$

In analogy with gBBM there is a threshold $\mathscr{N}_0(p)$ such that for any $\mathscr{N} > \mathscr{N}_0(p)$, gBou has two solitary wave profiles u_{c_1} and u_{c_2} of speeds c_1 and c_2 with $1 < c_1^2 < c_0^2(p) < c_2^2$. The question of stability of u_{c_2} is open. (Numerical calculations of Smereka (1992) suggest that it is stable, however.)

The method we use to study the existence of eigenvalues for (0.2) is related to the study of eigenvalues in boundary value problems for ordinary differential operators. As $|x| \to \infty$, the coefficients in equations (0.2) converge rapidly to those of the following equations, respectively:

$$(\lambda - c\partial_x)\,Y + \partial_x^3\,Y = 0, \tag{0.5a}$$

$$(\lambda - c\partial_x)\,(I-\partial_x^2)\,Y + \partial_x\,Y = 0, \tag{0.5b}$$

$$(\lambda - c\partial_x)^2\,(I-\partial_x^2)\,Y - \partial_x^2\,Y = 0. \tag{0.5c}$$

For $\operatorname{Re}\lambda > 0$, we will see that these equations have solutions $Y(x) = e^{\mu_j x}$ for $j = 1$ to m ($m = 3$ or 4), where the μ_j, which depend on λ, satisfy

$$\operatorname{Re}\mu_1(\lambda) < 0 < \operatorname{Re}\mu_j(\lambda) \quad \text{for} \quad j > 1. \tag{0.6}$$

Correspondingly, for each equation (0.2) there is a one-dimensional subspace of solutions which decay as $x \to \infty$, and an $(m-1)$-dimensional subspace of solutions which decay to zero as $x \to -\infty$: λ is an eigenvalue when these subspaces meet non-trivially. The angle between these subspaces may be measured by a wronskian-like analytic function $D(\lambda)$, named Evans's function by Alexander, Gardner & Jones (Alexander *et al.* 1990), after J. W. Evans, who pioneered its use in the study of stability of nerve impulses (Evans 1972*a–c*, 1975). One interpretation of the function $D(\lambda)$, much exploited by Yanagida (1985), is that it is like a transmission coefficient, in the sense that for the solution of (0.2) satisfying

$$Y(x) \sim e^{\mu_1 x} \quad \text{as} \quad x \to \infty,$$

we have

$$Y(x) \sim D(\lambda)\,e^{\mu_1 x} \quad \text{as} \quad x \to -\infty.$$

In each equation (0.2), for $\operatorname{Re}\lambda > 0$, if $D(\lambda)$ vanishes, then λ is an eigenvalue, and conversely.

It usually happens naturally that $D(0)=0$ when linearizing about a travelling wave: for $\lambda=0$ the function $Y(x)=\partial_x u_c$ satisfies (0.2). This follows from translation invariance in x. In the normalization we choose, we find $D(\lambda)\rightarrow 1$ as $|\lambda|\rightarrow\infty$ with $\operatorname{Re}\lambda\geqslant 0$, and $D(\lambda)$ is real for real λ. The crux of our method is that we have new integral formulae for derivatives of $D(\lambda)$ (Theorem 1.11), which we can evaluate at $\lambda=0$ and show that

$$D'(0)=0,\quad \operatorname{sgn} D''(0)=\operatorname{sgn}\mathrm{d}\mathcal{N}[u_c]/\mathrm{d}c.$$

Thus, if $\mathrm{d}\mathcal{N}[u_c]/\mathrm{d}c<0$, we find that $D(\lambda)<0$ for small $\lambda>0$, hence $D(\lambda)$ vanishes for some positive λ, yielding the existence of an unstable eigenvalue for (0.2).

A snag which arises in carrying out this program is that the description of decaying solutions of (0.2) changes as λ crosses the imaginary axis. This happens because the essential spectrum of the differential operator in (0.2) is the imaginary axis: for $\operatorname{Re}\lambda=0$, it happens that $\operatorname{Re}\mu_j=0$ (for some $j\neq 1$), and (0.2) has solutions with neutral growth as $|x|\rightarrow\infty$. To study the neighbourhood of $\lambda=0$ and the possibility of eigenvalues embedded in the essential spectrum, we need $D(\lambda)$ to be well defined in a neighbourhood of the imaginary axis. For this purpose, we will find it useful from the beginning to define $D(\lambda)$ in the broader way described below, so its domain is not restricted by the inequalities in (0.6). The domain of definition will be determined by the requirement that

$$\operatorname{Re}\mu_1(\lambda)<\operatorname{Re}\mu_j(\lambda)\quad\text{for}\quad j>1, \tag{0.7}$$

and in each application, we find that $D(\lambda)$ is naturally defined at least for $\operatorname{Re}\lambda>-\epsilon$ for some $\epsilon>0$. (But we warn that under the expanded definition, zeros of $D(\lambda)$ need only correspond to eigenvalues when the inequalities (0.6) hold, which means in our applications that $\operatorname{Re}\lambda>0$.)

In its simplest form, Evans's function $D(\lambda)$, for an m-dimensional system of ordinary differential equations with asymptotically constant coefficients, detects the intersections of a one-dimensional subspace of solutions decaying as $x\rightarrow\infty$, and an $(m-1)$-dimensional subspace of solutions decaying as $x\rightarrow-\infty$. Evans (1975) defined $D(\lambda)$ for eigenvalue problems associated with travelling waves of a class of nerve impulse models, consisting of a single reaction diffusion equation coupled to a set of ODEs. Among other results, he obtained an instability criterion by relating the sign of $D'(0)$ to the geometry of the travelling wave construction: the direction in which stable and unstable manifolds of the rest state cross as the wave speed is varied in the construction. (In our problems, this crossing is degenerate: the wave exists for a range of wave speeds c. Correspondingly $D'(0)=0$.) Jones (1984) showed how zeros of $D(\lambda)$ in the right half-plane could be proved absent by geometric and dynamical systems techniques in a singularly perturbed FitzHugh–Nagumo system. Yanagida (1985) redid this result from a somewhat more analytical point of view. Alexander *et al.* (1990), in studying the stability of travelling waves of fully parabolic systems, gave topological and analytic generalizations of Evans's function for systems with k-dimensional and $(m-k)$-dimensional subspaces of solutions decaying as $x\rightarrow\pm\infty$ respectively. An extension to a higher-order scalar equation was given by Gardner & Jones (1990).

One aspect of Jones's (1984) analysis was that in order for $D(\lambda)$ to be defined in a fixed neighbourhood of the origin, he had to analytically continue $D(\lambda)$ to a region that crosses the essential spectrum (where the dimension of the subspace of decaying solutions changes), by considering a solution decaying at the *maximal* exponential rate. A similar construction is made by Alexander *et al.* (1990) in their application to a fully parabolic FitzHugh–Nagumo system. The point of view taken in the present paper is to expand on Jones's construction, and make the basic definition of $D(\lambda)$ designed to detect the intersection of a one-dimensional subspace of solutions with maximal decay rate as $x \to +\infty$, and an $(m-1)$-dimensional subspace with submaximal growth rate as $x \to -\infty$. This has the advantage that the domain of definition of D is restricted only by (0.7), and zeros of D have a natural interpretation when they satisfy $\operatorname{Re}\lambda \leqslant 0$ in our applications.

The plan of this paper is as follows. In §1 we have thought it useful to develop a general existence theory for Evans's function in the context of a type of 'eigenvalue' problem, for first-order linear $m \times m$ systems with continuous coefficients that decay to constant values in an integrable fashion as $|x| \to \infty$. Aside from assuming analytic dependence on the parameter λ, our basic hypotheses are common in the classical theory of asymptotic behaviour in such systems of ordinary differential equations (see Coddington & Levinson 1955; Coppel 1965). (In particular, these hypotheses are much weaker than the assumptions of smoothness and exponential decay of coefficients that have been exploited in the applications of Evans's function to date (cf. Evans 1975; Jones 1984; Alexander *et al.* 1990). Among other results, we establish new integral formulae for the derivatives of $D(\lambda)$, and specialize the results to systems obtained via the standard reduction of a higher-order scalar ordinary differential equation. We remark that one can study intersections of k-dimensional and $(m-k)$-dimensional subspaces as well, in a manner similar to that developed by Alexander *et al.* (1990) for travelling waves of reaction–diffusion systems; see also Swinton (1992). But the case $k = 1$ is considerably simpler, and suffices to treat the present applications.

(*Note added in proof*: It has been pointed out to us by X.-B. Lin that our integral formula for $D'(\lambda)$ can be regarded as an application of Melnikov's method for analysing the intersection of stable and unstable manifolds (cf. Melnikov 1964; Palmer 1984).

In §2 we apply the theory of Evans's function to establish the criterion (0.4) for the existence of unstable eigenvalues in the equations (0.2). This suggests that when (0.4) is satisfied, the corresponding solitary wave is exponentially unstable. We recognize that our results do not fully establish nonlinear instability in its proper sense for travelling waves: modulo translations, but regard this issue as one outside the scope of this paper. See Grillakis *et al.* (1990), who treat this issue in a manner appropriate to nonlinear Schrödinger or Klein–Gordon equations. Note that even when (0.3) holds and the solitary wave for gKdV is nonlinearly stable modulo translations, the linearized evolution equation, for the perturbation $v(x,t)$, admits solutions that grow linearly in time, associated with perturbations that change the wave speed. (Such a solution is $v(x,t) = t\,\partial_c u_c$.) The eigenfunctions we find corresponding to positive eigenvalues are always (trivially) independent of the eigenfunctions associated with spatial translation and changes in wave speed, so we anticipate that (0.4) does suffice to guarantee exponential instability modulo translations.

In §3 we prove some additional results concerning the spectral analysis of the gKdV, gBBM and gBou solitary waves. We show that any zero of $D(\lambda)$ on the imaginary axis must be an eigenvalue embedded in the essential spectrum of the differential operator associated with (0.2), with corresponding eigenfunction $Y(x)$ that decays exponentially as $|x| \to \infty$. This result depends on a particular symmetry

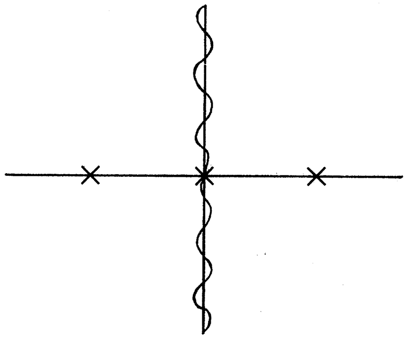

Figure 1. Spectrum of JL when (0.4) holds. Wavy line refers to essential spectrum covering the imaginary axis. The crosses refer to point eigenvalues.

of the eigenvalue equation in these examples, and is not implied by the general theory of Evans's function. We conjecture that no such embedded eigenvalues exist apart from $\lambda = 0$.

Also, we bound the number of unstable eigenvalues for a class of eigenvalue problems in the form

$$JLu = \lambda u, \tag{0.8}$$

where L is a self-adjoint operator with a finite number of negative eigenvalues and J is skew-symmetric. (This result improves a lemma of Grillakis *et al.* (1990).) For the solitary waves of the gKdV and gBBM equations, it follows from this that there is at most one unstable eigenvalue with $\operatorname{Re}\lambda > 0$. When the instability condition (0.4) holds, then the full spectrum of JL is as pictured in figure 1. (When condition (0.4) does not hold, the full spectrum of JL is the imaginary axis.)

In §4 we study the emergence of the unstable eigenvalue when $f(u) = u^{p+1}/(p+1)$ as p and c pass through values at which $\mathrm{d}\mathcal{N}[u_c]/\mathrm{d}c = 0$. Contrary to expectations generated from typical transitions to instability in finite-dimensional hamiltonian systems (cf. Arnold & Avez 1968; Arnold 1978; MacKay 1987), the unstable eigenvalue does not arise from the collision of pure imaginary eigenvalues at the origin. Instead, a zero of $D(\lambda)$ crosses from the left to the right half-plane along the real axis.

Some care in interpretation is required here: Zeros of D in the left half-plane need not be eigenvalues of (0.2), and eigenvalues in the left half-plane need not be zeros of D. Zeros of D in the left half-plane may be interpreted in terms of the analytic continuation of a restricted resolvent formula. They are poles of this resolvent on a second sheet of a Riemann surface that is analytically continued from the right half-plane across the essential spectrum. See §4 for a discussion. This phenomenon is related to the concepts of resonance poles in quantum scattering theory, and Landau damping in the Vlasov–Poisson system (see Reed & Simon 1978; Crawford & Hislop 1989*a*, *b*).

Finally, in §5 we consider a problem for which Evans's function yields the information that any transition to instability is *not* associated with eigenvalues emerging from the origin. Such information is obtained by showing that the order of $\lambda = 0$ as a zero of $D(\lambda)$ remains constant as parameters vary. As a consequence, any instabilities which arise are likely to be associated with a pair of complex conjugate eigenvalues crossing the imaginary axis into the right half-plane. Therefore this is likely to be an oscillatory type of exponential instability.

In particular, we study the stability of travelling waves of the generalized KdV–Burgers equation

$$\partial_t u + \partial_x f(u) + \partial_x^3 u = \alpha \partial_x^2 u. \qquad \text{(gKdVB)}$$

As α varies, this equation admits a family of travelling waves moving with the same speed c, which are monotone decreasing for large α, and for small $\alpha > 0$ may be roughly described as looking like a solitary wave for gKdV followed by an oscillatory tail which slowly decays to a non-zero constant state (Jeffrey & Kakutani 1972; Canosa & Gazdag 1977; Bona & Schonbek 1985; Khodja 1989). The monotone waves are known to be stable (Pego 1985), and stability persists for nearly monotone waves (Khodja 1989). It seems reasonable to conjecture that if the solitary wave for gKdV is exponentially unstable, then so is the corresponding travelling wave for gKdVB for α sufficiently small. While $D(0) = 0$, we prove in §5 that $D'(0) \neq 0$ independent of α for travelling waves of gKdVB. Thus the order of $\lambda = 0$ as a zero of $D(\lambda)$ remains constant. We can conclude that if there is a transition from stability to instability as $\alpha \to 0+$, it cannot occur simply by the emergence of a zero of $D(\lambda)$ from $\lambda = 0$. The actual mechanism of such a transition must be more complicated, and remains unknown. It could very likely involve a complex conjugate pair of eigenvalues crossing the imaginary axis. This could lead to a Hopf bifurcation to time periodic solutions.

1. A class of eigenvalue problems on $\mathbb{R}$

We shall study a type of 'eigenvalue' problem for linear systems of the following form, together with their associated transposed systems:

$$\mathrm{d}y/\mathrm{d}x = A(x, \lambda)\, y, \qquad (1.1)$$

$$\mathrm{d}z/\mathrm{d}x = -zA(x, \lambda). \qquad (1.2)$$

Here $y(x)$ is considered to be a column vector with m complex components, and $z(x)$ is a row vector. λ is a complex parameter.

(a) *Hypotheses*

Assume $\Omega \subset \mathbb{C}$ is a simply connected domain. We make the following hypotheses on the matrix-valued function A:

H1 (*Domain and smoothness*). *We assume* $A: \mathbb{R} \times \Omega \to \mathbb{C}^{m\times m}$ *is continuous, and analytic in* λ *for each fixed* x.

H2 (*Limits at infinity*). *We assume* $\lim_{x\to\pm\infty} A(x, \lambda) = A^{\pm\infty}(\lambda)$ *exists for* $\lambda \in \Omega$, *and that the limit is attained uniformly on compact subsets of* Ω.

Below, we frequently suppress the $\pm$ indication, while recognizing that the limits at $\pm\infty$ may be different. A statement made concerning $A^{\infty}(\lambda)$ or related quantities will represent two similar statements concerning $A^{+\infty}(\lambda)$ and $A^{-\infty}(\lambda)$. For example, H2 implies that A^{∞} is (i.e. both $A^{+\infty}$ and $A^{-\infty}$ are) analytic on Ω.

H3 (*Lowest asymptotic eigenvalue is simple*). *We assume that for* $\lambda \in \Omega$, $A^{\infty}(\lambda)$ *has a unique eigenvalue of smallest real part, which is simple. We denote it by* $\mu = \mu(\lambda)$. *We denote by* $\mu_*(\lambda)$ *the smallest real part of any other eigenvalue of* $A^{\infty}(\lambda)$. *Thus*

$$\operatorname{Re}\mu(\lambda) < \mu_*(\lambda) \equiv \min\{\operatorname{Re}\nu \mid \nu \neq \mu \text{ and } \nu \in \sigma(A^{\infty}(\lambda))\}.$$

Corresponding to the eigenvalue $\mu(\lambda)$, we require an analytic choice of normalized right and left eigenvectors of $A^{\infty}(\lambda)$, satisfying

$$(A^{\infty} - \mu)\, v(\lambda) = 0, \quad w(\lambda)\,(A^{\infty} - \mu) = 0, \quad w \cdot v = 1. \qquad (1.3)$$

Such eigenvectors can always be chosen, by a result of Kato (1982). For later reference, we describe the strategy of the proof. Since $\mu(\lambda)$ is always simple, the spectral projection

$$P(\lambda) = \int_{\Gamma} (A^{\infty} - \nu I)^{-1} \, \mathrm{d}\nu \tag{1.4}$$

exists and has one-dimensional range, where Γ is any contour enclosing $\mu(\lambda)$ and excluding the other eigenvalues of $A^{\infty}(\lambda)$. P is analytic in Ω. Fix $\lambda_0 \in \Omega$ and choose v_0, w_0, so (1.3) holds for $\lambda = \lambda_0$.

Now Kato proves:

Lemma 1.1. *There exists a 'transformation function' $U: \Omega \to \mathbb{C}^{m \times m}$ which is analytic, such that for $\lambda \in \Omega$,* (i) *$U(\lambda)$ is invertible,* (ii) *$U(\lambda) P(\lambda_0) = P(\lambda) U(\lambda)$.*

The transformation function is obtained by solving

$$U' = [P', P] U, \quad U(\lambda_0) = I,$$

where $[A, B] = AB - BA$. The desired eigenvectors are now given by

$$v(\lambda) = U(\lambda) v_0, \quad w(\lambda) = w_0 U(\lambda)^{-1}.$$

We also require that $A(x, \lambda)$ approach its asymptotic values sufficiently rapidly as $x \to \pm\infty$. The precise rate of approach required will sometimes vary, so we define the *deviator*

$$R(x, \lambda) = \begin{cases} A(x, \lambda) - A^{+\infty}(\lambda), & x > 0, \\ A(x, \lambda) - A^{-\infty}(\lambda), & x < 0. \end{cases} \tag{1.5}$$

H4. *We assume that the integral*

$$\int_{-\infty}^{\infty} \|R(x, \lambda)\| \, \mathrm{d}x$$

converges for all $\lambda \in \Omega$, uniformly on compact subsets.

(*b*) *Asymptotic behaviour of solutions*

It is well known that if the deviator R decays sufficiently rapidly as $x \to \pm\infty$, then solutions of (1.1), (1.2) behave like solutions of the constant coefficient systems

$$\mathrm{d}y/\mathrm{d}x = A^{\infty}(\lambda) y, \tag{1.6}$$

$$\mathrm{d}z/\mathrm{d}x = -z A^{\infty}(\lambda). \tag{1.7}$$

A particular solution of (1.6) for $A^{\infty} = A^{+\infty}$ is $y = v \, \mathrm{e}^{\mu x}$, where $v = v^+$, $\mu = \mu^+$. In general, we shall see that (1.1) has a one-dimensional subspace of solutions which are $O(\mathrm{e}^{\mu x})$ as $x \to +\infty$ (with $\mu = \mu^+$), and an $(m-1)$-dimensional subspace of solutions which are $o(\mathrm{e}^{\mu x})$ as $x \to -\infty$ (with $\mu = \mu^-$). The *'eigenvalue' problem* we shall consider is to characterize those values of λ such that these subspaces intersect non-trivially, so that some non-zero solution of (1.1) has 'maximal decay' $O(\mathrm{e}^{\mu x})$ as $x \to +\infty$, but 'submaximal growth' as $x \to -\infty$. Roughly speaking, to specify such a solution, one should impose one 'boundary condition' as $x \to -\infty$, and $m-1$ 'boundary conditions' as $x \to \infty$.

To proceed, we study when solutions of (1.1) behave like the solution $y = v \, \mathrm{e}^{\mu x}$ of (1.6), and similarly when solutions of (1.2) behave like the solution $z = w \, \mathrm{e}^{-\mu x}$ of (1.7).

Proposition 1.2. *There exist unique solutions* $\zeta^+(x,\lambda)$ *of* (1.1) *and* $\eta^-(x,\lambda)$ *of* (1.2) *which satisfy*

$$e^{-\mu^+x}\zeta^+(x,\lambda)\to v^+(\lambda) \quad \text{as} \quad x\to+\infty, \tag{1.8}$$

$$e^{\mu^-x}\eta^-(x,\lambda)\to w^-(\lambda) \quad \text{as} \quad x\to-\infty, \tag{1.9}$$

ζ^+ *and* η^- *are analytic in* λ *for* $\lambda\in\Omega$, *and the limits above occur uniformly on compact subsets of* Ω. *Any solution of* (1.1) *with* $y(x)=O(e^{\mu^+x})$ *as* $x\to+\infty$ *is a constant multiple of* ζ^+. *Any solution of* (1.2) *with* $z(x)=O(e^{-\mu^-x})$ *as* $x\to-\infty$ *is a constant multiple of* η^-.

Proof. The method of proof is standard (see Coppel 1965; Coddington & Levinson 1955), but there are some nuances in obtaining the global analytic dependence on λ from our hypotheses. We consider the existence of $\zeta^+(x,\lambda)$, the treatment of $\eta^-(x,\lambda)$ is similar. Let $B(\lambda)=A^{+\infty}(\lambda)-\mu^+(\lambda)I$. Under the change of variable $v(x)=\exp(-\mu^+x)\,y(x)$, solutions of (1.1) correspond to solution of

$$dv/dx=[B(\lambda)+R(x,\lambda)]\,v. \tag{1.10}$$

By H3, $B(\lambda)$ has one simple eigenvalue $\nu=0$ and $m-1$ eigenvalues with positive real part. Hence $\|e^{Bx}\|\leqslant C_1(\lambda)$ for all $x\leqslant 0$, where C_1 is bounded on compact subsets of Ω.

Given x_0, we may now define a linear operator $\mathscr{F}=\mathscr{F}(\lambda)$ on the space $C([x_0,\infty])$ of bounded continuous functions on $[x_0,\infty)$ by

$$\mathscr{F}v(x)=-\int_x^\infty e^{(x-s)B(\lambda)}R(s,\lambda)\,v(s)\,ds.$$

By H4, if Ω_1 is a compact subset of Ω, there exists x_0 sufficiently large so that

$$\theta=\sup_{\lambda\in\Omega_1}C_1(\lambda)\int_{x_0}^\infty\|R(s,\lambda)\|\,ds<1.$$

It follows that

$$\sup_{x\geqslant x_0}|\mathscr{F}v(x)|\leqslant\theta\sup_{x\geqslant x_0}|v(x)|,$$

i.e. $\mathscr{F}$ is a contraction on $C([x_0,\infty))$, uniformly for $\lambda\in\Omega_1$. As Coppel (1965) argues, given any bounded continuous function $\tilde{v}(x)$, the integral equation

$$v=\tilde{v}+\mathscr{F}v \tag{1.11}$$

has a unique bounded continuous solution, and one has

$$d(v-\tilde{v})/dx=B(\lambda)\,(v-\tilde{v})+R(x,\lambda)\,v.$$

In particular, if $\tilde{v}$ is chosen to be a bounded solution of $d\tilde{v}/dx=B\tilde{v}$, v given by (1.11) is a C^1 solution of (1.10). Conversely, to any bounded C^1 solution of (1.10) the function $\tilde{v}$ defined by

$$\tilde{v}\equiv v-\mathscr{F}v$$

is a bounded solution of $d\tilde{v}/dx=B\tilde{v}$. To summarize, there is a bicontinuous correspondence between bounded solutions of (1.10) and bounded solutions of $d\tilde{v}/dx=B(\lambda)\,\tilde{v}$. Clearly, any bounded solution of the latter has the form

$$\tilde{v}(x)=cv^+(\lambda)$$

for some $c\in\mathbb{C}$.

To obtain ζ^+, put $\tilde{v}(x,\lambda) = v^+(\lambda)$, then

$$\zeta^+(x,\lambda) = e^{\mu^+ x} v(x,\lambda) = e^{\mu^+ x}(I-\mathscr{F})^{-1} v^+(\lambda),$$

for $x \geqslant x_0$. For $x < x_0$, $\zeta^+(x,\lambda)$ is extended as a solution of (1.1). It is clear that $\zeta^+(x,\lambda)$ is independent of x_0, so it is defined for all $\lambda \in \Omega$. Since $v(x,\lambda)$ can be obtained by iterations which yield a sequence of analytic functions of λ that converge uniformly for $\lambda \in \Omega_1$, analyticity follows. That the limit in (1.8) is uniform on compact sets follows also, from the fact that $\theta = o(1)$ as $x_0 \to \infty$ above. □

The use of the transposed equation (1.2) in what follows relies on the following fundamental facts.

Lemma 1.3. *If $y(x)$ satisfies (1.1) and $z(x)$ satisfies (1.2), then $z\cdot y$ is independent of x.*

Proof. $\mathrm{d}/\mathrm{d}x(z\cdot y) = (-zA)y + z(Ay) = 0$.

Proposition 1.4. *If $y(x)$ satisfies (1.1), and $z(x)$ satisfies (1.2), then we have*

$$\lim_{x\to-\infty} e^{-\mu^- x} y(x) = (\eta^-\cdot y)v^-, \quad \lim_{x\to+\infty} e^{\mu^+ x} z(x) = (z\cdot\zeta^+)w^+. \tag{1.12}$$

Moreover, if $y(x,\lambda)$ and $z(x,\lambda)$ are solutions of (1.1) and (1.2) which are analytic on Ω in λ for each x, then the limits above are achieved uniformly in λ on compact subsets of Ω.

Proof. We establish the first limit, the second is similar. Fix λ, let $B = A^{-\infty}(\lambda) - \mu^-(\lambda)I$, and $R(x) = R(x,\lambda)$. Make the changes of variables $v(x) = e^{-\mu x}y(x)$, $w(x) = e^{\mu x}\eta^-(x)$. Then $\eta^-\cdot y = w\cdot v$ and

$$\frac{\mathrm{d}v}{\mathrm{d}x} = (B+R(x))v, \quad \frac{\mathrm{d}w}{\mathrm{d}x} = -w(B+R(x)). \tag{1.13}$$

We know that $\lim_{x\to-\infty} w(x) = w^-$ by Proposition 1.2. B has a simple eigenvalue $\nu = 0$ with eigenvector v^-, and the other eigenvalues have positive real part. It follows that every solution of $\mathrm{d}\tilde{v}/\mathrm{d}x = B\tilde{v}$ is bounded as $x \to -\infty$ and satisfies $\lim_{x\to-\infty}\tilde{v}(x) = cv^-$ for some c. Because of H4, there is a 1–1 correspondence between bounded solutions $v(x)$ in (1.19) and $\tilde{v}(x)$, see (Coppel 1965); the proof is similar to that of Proposition 1.2 and $v(x)$ satisfies an equation of the form (1.11). One has $v(x) - \tilde{v}(x) \to 0$ as $x \to -\infty$. It follows that for some c, $\lim_{x\to-\infty} v(x) = cv^-$. But $\eta^-\cdot y = w\cdot v \to cw^-\cdot v^- = c$, concluding the proof.

To prove the assertion that the limit is achieved uniformly on compact sets for an analytic family of solutions $y(x,\lambda)$, we argue as follows. For this family the corresponding limit $\tilde{v} = (w\cdot v)v^-(\lambda)$ is analytic, hence bounded on compact subsets of Ω. From the 1–1 correspondence between v and $\tilde{v}$ it is easy to show that $v(x,\lambda) = e^{-\mu x}y(x,\lambda)$ is uniformly bounded in λ on compact sets, uniformly for x sufficiently negative. Uniform convergence as $x \to -\infty$ now follows. □

To characterize precisely the growth rate for solutions of (1.1) which grow at a *submaximal rate* as $x \to +\infty$, consider that such solutions of the constant coefficient system $\mathrm{d}\tilde{y}/\mathrm{d}x = A^{\infty}(\lambda)\tilde{y}$ must satisfy $y(x) = O(e^{\mu_* x}|x|^r)$ as $x \to -\infty$ for some integer $r \geqslant 0$.

Definition. We define $r = r(\lambda)$ as the minimal such integer. It is the smallest integer such that $\ker(A^{\infty}-\nu)^r = \ker(A^{\infty}-\nu)^{r+1}$ for any eigenvalue ν of $A^{\infty}(\lambda)$ with

$\operatorname{Re}\nu = \mu_*(\lambda)$. (Typically $A^\infty(\lambda)$ will have a unique simple eigenvalue of second smallest real part; in that case $r(\lambda) = 0$.)

Proposition 1.5. *There exist rank $(m-1)$ matrices $\zeta^-(x,\lambda)$ and $\eta^+(x,\lambda)$, of size $m\times(m-1)$ and $(m-1)\times m$ respectively, which are analytic in $\lambda\in\Omega$ for each x, whose columns (resp. rows) satisfy (1.1) (resp. (1.2)) and which satisfy, for each $\lambda\in\Omega$,*

$$\zeta^-(x,\lambda) = O(a(x)\,\mathrm{e}^{\mu_*^- x}) \quad \text{as} \quad x\to-\infty,$$
$$\eta^+(x,\lambda) = O(a(x)\,\mathrm{e}^{-\mu_*^+ x}) \quad \text{as} \quad x\to+\infty,$$

where we may take either
(i) $a(x) = \mathrm{e}^{\delta|x|}$ *for any* $\delta\in(0,\mu_*-\operatorname{Re}\mu)$, *or*
(ii) $a(x) = |x|^{r(\lambda)}$, *provided we assume that*

$$\int_{-\infty}^{\infty} |x|^r\,\|R(x,\lambda)\|\,\mathrm{d}x \text{ converges.} \tag{1.14}$$

Moreover, any solution of (1.1) with $y(x) = O(a(x)\,\mathrm{e}^{\mu_*^- x})$ as $x\to-\infty$ has the form $y(x) = \zeta^-(x,\lambda)\,c$ for some constant column vector $c\in\mathbb{C}^{m-1}$. Any solution of (1.2) with $z(x) = O(a(x)\,\mathrm{e}^{-\mu_*^- x})$ as $x\to+\infty$ has the form $z(x) = c\eta^+(x,\lambda)$, for some constant row vector $c\in\mathbb{C}^{m-1}$.

Proof. It suffices to prove the existence of $\zeta^-(x,\lambda)$; the treatment of η^+ is similar. From classical results (see Dunkel 1902; Coddington & Levinson 1955; Coppel 1965) it is straightforward to show that for any fixed $\lambda\in\Omega$, (1.1) has $m-1$ linearly independent solutions satisfying

$$y(x) = O(a(x)\,\mathrm{e}^{\mu_*^- x}) \quad \text{as} \quad x\to-\infty \tag{1.15}$$

for $a(x)$ as in the statement of the Proposition. Our goal is to show that a basis of such solutions can be chosen which is globally analytic in λ on Ω.

First, observe that if a solution of (1.1) satisfies (1.15), then $\eta^-\cdot y\equiv 0$, as a consequence of Proposition 1.4, and the fact that $\mathrm{e}^{-\mu^- x}\,\mathrm{e}^{\mu_*^- x}\,a(x)\to 0$ as $x\to-\infty$. The converse also holds: If $\eta^-\cdot y = 0$, then y satisfies (1.15). This is because the set of solutions y satisfying $\eta^-\cdot y = 0$ is exactly $m-1$ dimensional.

Now fix x_0. To prove the Proposition, it suffices to construct an $m\times(m-1)$ matrix $V(\lambda)$, analytic on Ω, whose columns span the space orthogonal to $\eta^-(x_0,\lambda)$, so that $\eta^-(x_0,\lambda)\cdot V(\lambda) = 0$. We will then let $\zeta^-(x,\lambda)$ be the solution of

$$\mathrm{d}\zeta^-/\mathrm{d}x = A(x,\lambda)\,\zeta^-, \quad \zeta^-(x_0,\lambda) = V(\lambda).$$

This $\zeta^-(x,\lambda)$ will yield the desired conclusions.

To prove that a globally analytic $V(\lambda)$ exists as above, we use the idea of Kato's transformation. Recall that $\eta^-(x_0,\lambda)$ is analytic in λ on Ω, by Proposition 1.2. For $\lambda\in\Omega$ we may define an analytic projection onto the space orthogonal to $\eta=\eta^-$ by

$$\tilde{P}(\lambda) = I - \eta^{\mathrm{t}}\eta/\eta\eta^{\mathrm{t}}$$

(recall η^- is a row vector). For $\lambda_0\in\Omega$ given, choose a basis of $m-1$ vectors for the space orthogonal to $\eta^-(x_0,\lambda_0)$, and form an $m\times(m-1)$ matrix V_0 with these vectors as columns. As in Lemma 1.1, there exists an analytic transformation function $U:\Omega\to\mathbb{C}^{m\times m}$ such that $U(\lambda)$ is invertible and $U(\lambda)\tilde{P}(\lambda_0) = \tilde{P}(\lambda)\,U(\lambda)$, for all $\lambda\in\Omega$. Put $V(\lambda) = U(\lambda)\,V_0$. Then

$$V(\lambda) = U(\lambda)\,\tilde{P}(\lambda_0)\,V_0 = \tilde{P}(\lambda)\,U(\lambda)\,V_0$$

so $\eta^-(x_0,\lambda)\,V(\lambda) = 0$ as desired. This proves the Proposition. □

We remark that the proof implies that

$$\eta^- \cdot \zeta^-(\lambda) \equiv 0 \quad \text{and} \quad \eta^+ \cdot \zeta^+(\lambda) \equiv 0. \tag{1.16}$$

We summarize the various characterizations of the asymptotic behaviour of solutions of (1.1), (1.2) in the following result

Proposition 1.6. *Let $\lambda \in \Omega$ and suppose that $y(x)$ is a solution of* (1.1) *and $z(x)$ is a solution of* (1.2). *Then in parts* 1–4 *below, statements* (*a*)–(*d*) *are equivalent.*

1. (*a*) $y(x) = o(\mathrm{e}^{\mu x})$ *as* $x \to -\infty$;
   (*b*) $\eta^- \cdot y = 0$;
   (*c*) $y(x) = \zeta^-(x, \lambda)\, c$ *for some constant vector* $c \in \mathbb{C}^{m-1}$;
   (*d*) $y(x) = O(a(x)\, \mathrm{e}^{\mu_* x})$ *as* $x \to -\infty$, *where we may take either*
      (i) $a(x) = \mathrm{e}^{\epsilon|x|}$ *for* $0 < \epsilon < \mu_* - \operatorname{Re}\mu$, *or*
      (ii) $a(x) = |x|^r$, $r = r(\lambda)$ *as defined previously, provided we assume* (1.14) *holds.*
2. (*a*) $y(x) = O(\mathrm{e}^{\mu x})$ *as* $x \to +\infty$;
   (*b*) $\eta^+ \cdot y = 0$;
   (*c*) $y(x) = c\zeta^+(x, \lambda)$ *for some constant* c;
   (*d*) $y(x) = o(b(x)\, \mathrm{e}^{\mu_* x})$ *as* $x \to +\infty$, *where we may take either*
      (i) $b(x) = 1/a(x)$, $a(x)$ *from part* 1, *or*
      (ii) $b(x) = 1$, *provided we assume* (1.14) *holds.*
3. (*a*) $z(x) = O(\mathrm{e}^{-\mu x})$ *as* $x \to -\infty$;
   (*b*) $z \cdot \zeta^- = 0$;
   (*c*) $z(x) = c\eta^-(x, \lambda)$ *for some constant* c;
   (*d*) $z(x) = o(b(x)\, \mathrm{e}^{-\mu_* x})$, *where $b(x)$ is as in part* 2.
4. (*a*) $z(x) = O(\mathrm{e}^{-\mu x})$ *as* $x \to +\infty$;
   (*b*) $z \cdot \zeta^+ = 0$;
   (*c*) $z(x) = c\eta^+(x, \lambda)$ *for some constant vector* $c \in \mathbb{C}^{m-1}$;
   (*d*) $z(x) = O(a(x)\, \mathrm{e}^{-\mu_* x})$, *where $a(x)$ is as in part* 1.

Proof of Proposition 1.6. We prove parts 1 and 2; parts 3 and 4 are similar. Consider part 1. Proposition 1.4 implies that 1(*a*) is equivalent to 1(*b*). Since $\eta^- \cdot \zeta^- = 0$ and the $m-1$ columns of ζ^- are independent, it is clear that 1(*b*) is equivalent to 1(*c*). 1(*c*) implies 1(*d*) by Proposition 1.5, and 1(*d*) implies 1(*a*). Thus all statements in part 1 are equivalent.

Now consider part 2. By Proposition 1.2, 2(*c*) and 2(*a*) are equivalent. Since the $m-1$ rows of η^+ are independent and $\eta^+ \cdot \zeta^+ = 0$, it follows that 2(*b*) and 2(*c*) are equivalent. 2(*a*) implies 2(*d*). We claim that 2(*d*) implies 2(*b*), which will finish the proof. Consider case (i), and assume $b(x) = 1/a(x)$. By Proposition 1.5 we have $\eta^+ \cdot y = O(a(x)\, \mathrm{e}^{-\mu_* x})\, o(\mathrm{e}^{\mu_* x}/a(x))$, hence $\eta^+ \cdot y = 0$.

Finally, consider case (ii), $b(x) = 1$. Let r_* be the sum of the multiplicities of eigenvalues of $A^{+\infty}(\lambda)$ with real parts equal to μ_*. By standard methods (Coppel 1965; Coddington & Levinson 1955), one can show that for $\epsilon > 0$ sufficiently small, any solution of (1.1) with $y(x) = O(\mathrm{e}^{\mu_* x}\, \mathrm{e}^{\epsilon x})$ as $x \to +\infty$ must lie in a unique subspace of solutions of dimension $r_* + 1$. Consider now the constant coefficient system (1.6) at $+\infty$. For this system, one can find a set of solutions $y_1(x), \ldots, y_{r_*}(x)$ such that for each $j = 1, \ldots, r_*$, there is an integer k_j, a vector c_j and an eigenvalue ν_j of $A^{+\infty}(\lambda)$ with $\operatorname{Re} v_j = \mu_*$ with $\tilde{y}_j(x) = x^{k_j}\, \mathrm{e}^{\nu_j x}(c_j + O(1/x))$ as $x \to +\infty$, and such that the functions $x^{k_j}\, \mathrm{e}^{\nu_j x}\, c_j$, $j = 1, \ldots, r_*$, are linearly independent.

Now we invoke a result originally proved by Dunkel (1902) (see Theorem 4 on p. 92 of Coppel (1965)): If (1.14) holds, then (1.1) has solutions $\tilde{y}_1(x), \ldots, \tilde{y}_{r_*}(x)$ such that

$$y_j(x) - \tilde{y}_j(x) = o(x^{k_j}\,\mathrm{e}^{\mu_* x}) \quad \text{as} \quad x \to \infty.$$

It follows that $y_1(x), \ldots, y_{r_*}(x)$ are linearly independent, and also that *no* linear combination $y(x)$ of these solutions can satisfy $y(x) = o(\mathrm{e}^{\mu_* x})$. But any solution for which $y = o(\mathrm{e}^{\mu_* x})$ must be a linear combination of the $r_* + 1$ independent solutions $\zeta^+(x,\lambda)$, $y_1(x), \ldots, y_{r_*}(x)$. So such a solution must be a multiple of ζ^+, hence $\eta^+ \cdot y = 0$. This finishes the proof. □

(*c*) *Definition of 'eigenvalues' and* $D(\lambda)$

We can now state precisely what is the 'eigenvalue' problem we are considering. We seek those values of λ such that (1.1) admits a nontrivial solution $y(x)$ satisfying:

$$y(x) = o(\mathrm{e}^{\mu x}) \quad \text{as} \quad x \to -\infty, \quad y(x) = O(\mathrm{e}^{\mu x}) \quad \text{as} \quad x \to \infty. \tag{1.17}$$

By Proposition 1.6, these conditions are equivalent to requiring

$$\eta^- \cdot y = 0 \quad \text{and} \quad \eta^+ \cdot y = 0. \tag{1.18}$$

These conditions may be interpreted as boundary conditions on $\mathbb{R}$: the condition $\eta^- \cdot y = 0$ represents one constraint asymptotically as $x \to -\infty$, and $\eta^+ \cdot y = 0$ represents $m-1$ constraints asymptotically as $x \to +\infty$. If $\operatorname{Re}\mu < 0 < \mu_*$, as in the case when $\operatorname{Re}\lambda > 0$ for the applications we consider in §2, then (1.17) is equivalent to the requirement that $y(x)$ is exponentially decaying (or merely bounded) as $|x| \to \infty$

Definition 1.7. If $\lambda \in \Omega$ is such that a non-zero solution $y(x)$ exists satisfying both (1.1) and (1.17), we say that λ is an 'eigenvalue' for the problem (1.1), (1.17).

We place quotes around the term 'eigenvalue' to emphasize, when it comes to the applications, the slight difference between the notion described here and the more usual notion. By Proposition 1.6, λ is an 'eigenvalue' if and only if the m solutions of (1.1) represented by $\zeta^+(x,\lambda)$ and the $m-1$ columns of $\zeta^-(x,\lambda)$ are linearly dependent, i.e. if and only if the wronskian $\det[\zeta^-, \zeta^+] = 0$. For our purposes, a more useful characterization of 'eigenvalues' is the following, due to J. W. Evans (1975)

Definition 1.8. We define Evans' function $D(\lambda)$ for $\lambda \in \Omega$ by

$$D(\lambda) = \eta^-(x,\lambda) \cdot \zeta^+(x,\lambda).$$

Theorem 1.9.

(*a*) *$D(\lambda)$ is analytic for $\lambda \in \Omega$.*

(*b*) *λ is an 'eigenvalue' of problem* (1.1), (1.17) *if and only if $D(\lambda) = 0$.*

We remark also that λ is an 'eigenvalue' of (1.1), (1.17) if and only if equation (1.2) admits a non-zero solution $z(x)$ such that

$$z(x) = O(\mathrm{e}^{-\mu x}) \quad \text{as} \quad x \to -\infty, \; z(x) = o(\mathrm{e}^{-\mu x}) \quad \text{as} \quad x \to +\infty, \tag{1.19}$$

or equivalently,

$$z \cdot \zeta^- = 0 \quad \text{and} \quad z \cdot \zeta^+ = 0. \tag{1.20}$$

Proof of Theorem 1.9. We study under what condition $\zeta^+(x,\lambda)$ is an eigenfunction. By construction, $\zeta^+(x) = O(\mathrm{e}^{\mu x})$ as $x \to +\infty$. Choosing $y = \zeta^+$ in part 1 of Proposition 1.6, we get $\zeta^+(x) = o(\mathrm{e}^{\mu x})$ as $x \to -\infty$ if and only if $D(\lambda) = \eta^- \cdot \zeta^+ = 0$. □

When $A(x,\lambda)$ is real for real λ, one may expect that $D(\lambda)$ is real for real λ. Recalling that the domain Ω is simply connected, it is easy to establish the following.

Proposition 1.10. *Suppose that whenever* $\lambda \in \Omega$ *is real,* $A(x,\lambda)$ *is real for all* x. *Then whenever* λ *and* $\bar\lambda$ *lie in* Ω, *we have* $A(x,\bar\lambda) = \overline{A(x,\lambda)}$, $\zeta^{\pm}(x,\bar\lambda) = \overline{\zeta^{\pm}(x,\lambda)}$, $\eta^{\pm}(x,\bar\lambda) = \overline{\eta^{\pm}(x,\lambda)}$, *and* $D(\bar\lambda) = \overline{D(\lambda)}$.

It follows that if λ and $\bar\lambda$ lie in Ω, λ is an 'eigenvalue' for (1.1), (1.17) if and only if $\bar\lambda$ is.

(*d*) *Derivatives of* $D(\lambda)$

The following new formulae for the derivatives of $D(\lambda)$ are the basis for the main results of the paper.

Theorem 1.11. *Put*

$$\mu(x,\lambda) = \begin{cases} \mu^-(\lambda) & \text{for} \quad x<0, \\ \mu^+(\lambda) & \text{for} \quad x>0. \end{cases}$$

Then for all $\lambda \in \Omega$,

$$D'(\lambda) = -\int_{-\infty}^{\infty} \eta^-(x,\lambda)\frac{\partial}{\partial\lambda}[A(x,\lambda)-\mu(x,\lambda)I]\,\zeta^+(x,\lambda)\,\mathrm{d}x + D(\lambda)\,[\mathrm{d}w^-/\mathrm{d}\lambda\cdot v^- + w^+\cdot \mathrm{d}v^+/\mathrm{d}\lambda]. \tag{1.21}$$

In particular, the integral exists as an improper integral. Also, higher derivatives $\partial_\lambda^k D(\lambda)$ *are given by formal differentiation of* (1.21).

The formula (1.21) simplifies in important special cases. Namely, if $D(\lambda_0) = 0$, then

$$D'(\lambda_0) = -\int_{-\infty}^{\infty} \eta^-(x,\lambda_0)\frac{\partial A}{\partial\lambda}(x,\lambda_0)\,\zeta^+(x,\lambda_0)\,\mathrm{d}x. \tag{1.22}$$

Alternatively, if $A^{-\infty}(\lambda) \equiv A^{+\infty}(\lambda)$, then since $w^- = w^+$, $v^- = v^+$ and $w^{\pm}\cdot v^{\pm} \equiv 1$, the second term in (1.21) vanishes.

Proof. Let $(\ )_\lambda = \partial/\partial\lambda$ denote differentiation with respect to λ. With $\mu = \mu(x,\lambda)$, define

$$w(x,\lambda) = \mathrm{e}^{\mu x}\,\eta^-(x,\lambda) \quad \text{and} \quad v(x,\lambda) = \mathrm{e}^{-\mu x}\,\zeta^+(x,\lambda).$$

Then $D(\lambda) = w\cdot v$ and $D'(\lambda) = w_\lambda v + w v_\lambda$ are independent of x. Since $\mathrm{d}v/\mathrm{d}x = (A-\mu)\,v$, and $\mathrm{d}w/\mathrm{d}x = -w(A-\mu)$, we have

$$\mathrm{d}v_\lambda/\mathrm{d}x = (A-\mu)\,v_\lambda + (A_\lambda - \mu_\lambda)\,v, \tag{1.23}$$

$$\mathrm{d}w_\lambda/\mathrm{d}x = -w_\lambda(A-\mu) - w(A_\lambda-\mu_\lambda). \tag{1.24}$$

It is easy to verify that

$$\frac{\mathrm{d}}{\mathrm{d}x}(wv_\lambda) = w(A_\lambda-\mu_\lambda)\,v = -\frac{\mathrm{d}}{\mathrm{d}x}(w_\lambda v) \tag{1.25}$$

hence for any $R, S > 0$ we have

$$wv_\lambda(0,\lambda) = wv_\lambda(S,\lambda) - \int_0^S w(A_\lambda-\mu_\lambda)\,v\,\mathrm{d}x,$$

$$w_\lambda v(0,\lambda) = w_\lambda v(-R,\lambda) - \int_{-R}^0 w(A_\lambda-\mu_\lambda)\,v\,\mathrm{d}x,$$

hence

$$D'(\lambda) = -\int_{-R}^{S} w(A_\lambda-\mu_\lambda)\,v\,\mathrm{d}x + w_\lambda v(-R,\lambda) + wv_\lambda(S,\lambda). \tag{1.26}$$

From Propositions 1.2 and 1.5 follows

$$v(x,\lambda)\to v^{+}(\lambda),\quad w(x,\lambda)\to D(\lambda)\,w^{+}(\lambda) \tag{1.27}$$

as $x\to+\infty$, and

$$v(x,\lambda)\to D(\lambda)\,v^{-}(\lambda),\quad w(x,\lambda)\to w^{-}(\lambda) \tag{1.28}$$

as $x\to-\infty$. These limits are attained uniformly for λ in compact subsets of Ω, hence the limit of derivatives with respect to λ converge to derivatives of the limits. In particular we have

$$w_\lambda(x,\lambda)\to w_\lambda^{-}(\lambda)\quad\text{as}\quad x\to-\infty,\quad v_\lambda(x,\lambda)\to v_\lambda^{+}(\lambda)\quad\text{as}\quad x\to+\infty.$$

The formula (1.21). now follows by taking $R,S\to\infty$ in (1.26), since $w(A_\lambda-\mu_\lambda)\,v=\eta^{-}(A_\lambda-\mu_\lambda)\,\zeta^{+}$. Formulae for higher derivatives follow by differentiating (1.26)–(1.28) with respect to λ and again taking $R,S\to\infty$. □

Remark 1.12. Several complex parameters

In our applications, it is sometimes useful to consider problem (1.1) as depending on several parameters. It will be convenient to assume analytic dependence in each of these. To extend the above results to this case, let $s\geqslant 1$ be an integer, and suppose $\Omega=\Omega_1\times\ldots\times\Omega_s\subset\mathbb{C}^s$ where $\Omega_j\subset\mathbb{C}$ is a simply connected domain for $j=1,\ldots,s$. Write $\lambda=(\lambda_1,\ldots,\lambda_s)$ for $\lambda\in\Omega$. If we make the hypotheses H1–H4, then all the results 1.1–1.9 and 1.11 are valid, *mutatis mutandi*. In Theorem 1.11, derivatives with respect to λ refer to gradients with respect to $(\lambda_1,\ldots,\lambda_s)$.

(*e*) *Higher-order scalar equations*

An important special case of the foregoing is the case when $A(x,\lambda)$ arises in the standard reduction of an mth order ordinary differential equation to a first-order system. Let ∂_x denote differentiation with respect to x, and suppose our ordinary differential equation is

$$\mathscr{A}(x,\lambda)\,Y=\partial_x^m\,Y+\sum_{j=0}^{m-1}a_j(x,\lambda)\,\partial_x^j\,Y=0. \tag{1.29}$$

Put $y_j=\partial_x^{j-1}\,Y$, $j=1,\ldots,m$. Then (1.29) is equivalent to (1.1), $\mathrm{d}y/\mathrm{d}x=A(x,\lambda)\,Y$, where

$$A(x,\lambda)=\begin{bmatrix}0 & 1 & 0 & \ldots & 0\\ 0 & 0 & 1 & \ldots & 0\\ \vdots & & & & 0\\ 0 & 0 & & \ldots & 1\\ -a_0 & -a_1 & & \ldots & -a_{m-1}\end{bmatrix}. \tag{1.30}$$

The associated transposed system (1.2), $\mathrm{d}z/\mathrm{d}x=-zA(x,\lambda)$, may be related to the transposed equation

$$\mathscr{A}^{\mathrm{t}}(x,\lambda)\,Z=(-\partial_x)^m\,Z+\sum_{j=0}^{m-1}(-\partial_x)^j(a_j(x,\lambda)\,Z)=0 \tag{1.31}$$

by the relations

$$z_m=Z,\quad z_j=-\partial_x\,z_{j+1}+a_j\,Z,\quad j=m-1,\ldots,1. \tag{1.32}$$

Computing the z_j amounts to computing $\mathscr{A}^{\mathrm{t}}Z$ in (1.31) by considering it as a polynomial in $-\partial_x$ and applying Horner's rule for nested multiplication of polynomials: We have

$$\mathscr{A}^{t}(x,\lambda)\,Z = z_0, \quad \text{where} \quad z_0 = -\partial_x z_1 + a_0 Z. \tag{1.33}$$

Making the hypotheses H1–H4, write $a_j^{\pm\infty}(\lambda) = \lim_{x\to\pm\infty} a_j(x,\lambda)$. The eigenvalue $\mu(\lambda)$ of $A^{\infty}(\lambda)$ must satisfy

$$\mathscr{P}(\mu) \equiv \mu^m + \sum_{j=0}^{m-1} a_j^{\infty}(\lambda)\,\mu^j = 0.$$

The components of any right eigenvector v with $(A^{\infty}-\mu)\,v = 0$ satisfy $v_j = \mu^{j-1}v_1$, $j = 2, \dots, m$, where $v_1 \neq 0$ is arbitrary. The components of any left eigenvector w with $w(A^{\infty}-\mu) = 0$ satisfy $\quad w_{j-1} = \mu w_j + a_{j-1}^{\infty}\, w_m, \quad j = m, \dots, 2.$

Computing the w_j when $w_m = 1$ amounts to computing $\mathscr{P}(\mu)$ by using Horner's rule. We have $0 = \mathscr{P}(\mu) = \mu w_1 + a_0^{\infty}\, w_m$; in fact

$$w_j = \left(\sum_{k=j}^{m} a_k^{\infty}\, \mu^{k-j}\right) w_m,$$

where $a_m^{\infty} = 1$. One may verify easily that $w\cdot v = \mathscr{P}'(\mu)\, w_m\, v_1$; since μ is a simple eigenvalue, $\mathscr{P}'(\mu) \neq 0$. We assume that a normalization of v and w is chosen so that $w\cdot v = 1$.

To obtain expressions for Evans's function $D(\lambda)$ in this context, suppose that $\zeta^{+}(x,\lambda)$, $\eta^{-}(x,\lambda)$ are given by Proposition 1.2 and put

$$Y^{+}(x,\lambda) = \zeta_1^{+}(x,\lambda), \quad Z^{-}(x,\lambda) = \eta_m^{-}(x,\lambda).$$

Then $D(\lambda) = \eta^{-}\cdot\zeta^{+}$ can be written as

$$D(\lambda) = \sum_{j=0}^{m-1} (\mathscr{A}_j(x,\lambda)\,Z^{-})\,(\partial_x^j\, Y^{+}), \tag{1.34}$$

where the operators $\mathscr{A}_j$ are given by

$$\mathscr{A}_m = 1, \quad \mathscr{A}_j = -\partial_x\cdot\mathscr{A}_{j+1} + a_j \quad \text{for} \quad j = m-1, \dots, 0.$$

Equation (1.21) may be written as

$$D'(\lambda) = \int_{-\infty}^{\infty}\left[Z^{-}(x,\lambda)\frac{\partial\mathscr{A}}{\partial\lambda}\,Y^{+}(x,\lambda) + \frac{\partial\mu}{\partial\lambda}(x,\lambda)\,D(\lambda)\right]\mathrm{d}x$$
$$+D(\lambda)\,[(\mathrm{d}w^{-}/\mathrm{d}\lambda)\cdot v^{-} + w^{+}\cdot \mathrm{d}v^{+}/\mathrm{d}\lambda], \tag{1.35}$$

where
$$\frac{\partial\mathscr{A}}{\partial\lambda}\,Y \equiv \sum_{j=0}^{m-1} \frac{\partial a_j}{\partial\lambda}(x,\lambda)\,\partial_x^j\, Y(x,\lambda).$$

For systems arising from higher-order scalar equations of the form (1.29), the criteria in (1.17) describing when λ is an 'eigenvalue' require bounds on the derivatives $\partial_x^j Y$ for $j = 0, \dots, m-1$. Such bounds are automatically equivalent to the bound on Y itself (see Coppel 1965). From the result in Coppel (1965) it is easy to prove the following.

Proposition 1.13. *Suppose* $Y(x)$ *satisfies* (1.29) *and* $y_j = \partial_x^{j-1} Y$, $j = 1, \dots, m$. *Then for any real number* ν *we have*

$$Y(x) = O(\mathrm{e}^{\nu x}) \quad \textit{if and only if} \quad y(x) = O(\mathrm{e}^{\nu x}) \quad \textit{as} \quad x\to\infty \ (\textit{or } -\infty)$$

and $\quad Y(x) = o(\mathrm{e}^{\nu x}) \quad$ *if and only if* $\quad y(x) = o(\mathrm{e}^{\nu x}) \quad$ *as* $\quad x\to\infty$ (*or* $-\infty$).

Therefore, the criteria (1.17) for λ to be an 'eigenvalue' are equivalent to the requirements

$$Y(x) = o(\mathrm{e}^{\mu x}) \quad \text{as} \quad x \to -\infty, \quad Y(x) = O(\mathrm{e}^{\mu x}) \quad \text{as} \quad x \to +\infty. \tag{1.36}$$

By using Propositions 1.6 and 1.13, these requirements are also respectively equivalent to requiring that

$$Y(x) = O(a(x)\,\mathrm{e}^{\mu_* x}) \quad \text{as} \quad x \to -\infty, \quad Y(x) = o(b(x)\,\mathrm{e}^{\mu_* x}) \quad \text{as} \quad x \to +\infty, \tag{1.37}$$

where $a(x) = \mathrm{e}^{\epsilon|x|}$ for $0 < \epsilon < \mu_* - \mathrm{Re}\,\mu$, and $b(x) = 1$ or $1/a(x)$ as in part 2 of Proposition 1.6.

(*f*) *Equivalent forms of $D(\lambda)$ and the resolvent formula*

Recall that Proposition 1.6 implies that λ is an 'eigenvalue' for (1.1), (1.17) if and only if there exists $c \in \mathbb{C}^{m-1}$ such that $\zeta^+(x,\lambda) = \zeta^-(x,\lambda)\,c$. By using Proposition 1.6 it is then easy to show the following.

Proposition 1.14. *For $\lambda \in \Omega$ the following are equivalent:*

$$(a)\ D(\lambda) = \eta^-\zeta^+ = 0,\ (b)\ \det[\eta^+\,\zeta^-] = 0,\ (c)\ \det[\zeta^+\,\zeta^-] = 0,\ (d)\ \det\begin{bmatrix}\eta^-\\ \eta^+\end{bmatrix} = 0.$$

From this result, we see that any of the four quantities in parts (*a–d*) can be used to detect 'eigenvalues'. A stronger result is in fact true: At an eigenvalue λ, the order of vanishing of these four quantities is the same.

Proposition 1.15. *There exist analytic functions $\beta_1(\lambda)$, $\beta_2(\lambda)$, and $\beta_3(\lambda)$ defined on Ω, which have no zeros in Ω, such that*

$$D(\lambda) = \eta^-\zeta^+ = \beta_1 \det[\eta^+\zeta^-] = \beta_2 \det[\zeta^+\,\zeta^-] = \beta_3 \det\begin{bmatrix}\eta^-\\ \eta^+\end{bmatrix}.$$

Proof. First, we claim there exists a solution $\tilde{\zeta}(x,\lambda)$ of (1.1), and $\tilde{\eta}(x,\lambda)$ of (1.2), which are analytic for $\lambda \in \Omega$ and have the property that

$$\det[\tilde{\zeta},\zeta^-] \neq 0, \quad \det\begin{bmatrix}\tilde{\eta}\\ \eta^+\end{bmatrix} \neq 0.$$

To construct $\tilde{\zeta}$, we may simply fix x_0 and solve the initial value problem consisting of (1.1) with $\tilde{\zeta}(x_0,\lambda)$ given by the cross product of the $m-1$ columns of $\zeta^-(x_0,\lambda)$. $\tilde{\eta}$ is constructed in a similar way.

Now we claim that $\eta^-\tilde{\zeta} \neq 0$, and $\tilde{\eta}\zeta^+ \neq 0$, for all $\lambda \in \Omega$. This follows from Proposition 1.6. Namely, if $\eta^-\tilde{\zeta} = 0$ then $\tilde{\zeta} = \zeta^- c$, for some $c \in \mathbb{C}^{m-1}$, a contradiction. Now we may write

$$\begin{bmatrix}\eta^-\\ \eta^+\end{bmatrix}[\zeta^+\zeta^-] = \begin{bmatrix}D(\lambda) & 0\\ 0 & \eta^+\zeta^-\end{bmatrix},$$

$$\begin{bmatrix}\eta^-\\ \eta^+\end{bmatrix}[\tilde{\zeta}\zeta^-] = \begin{bmatrix}\eta^-\tilde{\zeta} & 0\\ * & \eta^+\zeta^-\end{bmatrix},$$

$$\begin{bmatrix}\tilde{\eta}\\ \eta^+\end{bmatrix}[\zeta^+\zeta^-] = \begin{bmatrix}\tilde{\eta}\zeta^+ & *\\ 0 & \eta^+\zeta^-\end{bmatrix}.$$

Taking determinants, the Proposition may now be easily established. □

Next, we develop a resolvent formula for (1.1)–(1.17). Let $f: \mathbb{R} \to \mathbb{C}^m$ be continuous with compact support. We seek y which satisfies

$$(\mathrm{d}/\mathrm{d}x - A(x,\lambda))\, y = f(x) \quad -\infty < x < \infty \tag{1.38}$$

together with the asymptotic boundary conditions in (1.17). It is straightforward to verify that if λ is not an 'eigenvalue' for (1.1)–(1.17), then there is a unique solution given by the resolvent formula

$$y(x) = \mathscr{R}(\lambda) f(x) = \zeta^+(x,\lambda)\, D(\lambda)^{-1} \int_{-\infty}^{x} \eta^-(s,\lambda) f(s)\,\mathrm{d}s \\ + \zeta^-(x,\lambda)\,(\eta^+\zeta^-)^{-1} \int_{+\infty}^{x} \eta^+(s,\lambda) f(s)\,\mathrm{d}s. \tag{1.39}$$

Alternatively, we may write

$$\mathscr{R}(\lambda) f(x) = D(\lambda)^{-1} \int_{-\infty}^{\infty} K(x,s,\lambda) f(s)\,\mathrm{d}s, \tag{1.40}$$

where

$$K(x,s,\lambda) = \begin{cases} \zeta^+(x,\lambda)\,\eta^-(s,\lambda) & x \geqslant s, \\ -D(\lambda)\,(\eta^+\zeta^-)^{-1}(\lambda)\,\zeta^-(x,\lambda)\,\eta^+(s,\lambda) & x < s. \end{cases} \tag{1.41}$$

As a corollary of Proposition 1.15, for each real x, s, K is analytic in Ω with only removable singularities. Thus we see that Evans's function $D(\lambda)$ accounts for any singularities of the resolvent in (1.39)–(1.40):

Proposition 1.16. *Suppose f is continuous with compact support. Then for any real x, the function $\lambda \mapsto D(\lambda)\mathscr{R}(\lambda) f(x)$ is analytic in Ω with only removable singularities at most.*

The resolvent formula (1.39) can be used to define a bounded operator in a variety of function spaces, but we will not pursue this issue in this paper.

(g) *Behaviour for large λ of $D(\lambda)$*

Assume it is possible to take $|\lambda| \to \infty$ in Ω. We wish to study the behaviour of $D(\lambda)$ as $|\lambda| \to \infty$. Our results here are of limited generality, but suffices to cover the examples we discuss in this paper. The basic approach we take is to diagonalize $A^{\infty}(\lambda)$ and use perturbation arguments.

Proposition 1.17. *Assume that $A^{\infty}(\lambda)$ is diagonalizable for large λ, and that $V(\lambda)$ (V_+ or V_- as appropriate) is a matrix of right eigenvectors whose first column is $v^{\pm}(\lambda)$. Let $W = V^{-1}$ and let*

$$F(x,\lambda) = \begin{cases} W_+ R(x,\lambda)\, V_+ & \text{for} \quad x > 0, \\ W_- R(x,\lambda)\, V_- & \text{for} \quad x < 0. \end{cases}$$

Suppose that as $|\lambda| \to \infty$ in Ω we have:

$$\int_{-\infty}^{\infty} |F(x,\lambda)|\,\mathrm{d}x \leqslant C \quad \textit{independent of } \lambda, \tag{1.42}$$

$$\int_{|x|>x_0} |F(x,\lambda)|\,\mathrm{d}x \to 0 \quad \textit{as} \quad x_0 \to \infty \quad \textit{uniformly in } \lambda, \tag{1.43}$$

and
$$\int_{-\infty}^{\infty} |F(x,\lambda)\, e_1|\, \mathrm{d}x \rightarrow 0, \tag{1.44}$$

where $e_1 = (1, 0, \ldots, 0)^{\mathrm{t}}$. *Then it follows that as* $|\lambda| \rightarrow \infty$ *in* Ω *we have*

$$W_+(\lambda)\, \zeta^+(0,\lambda) = W_+ v^+ + o(1) = e_1 + o(1) \tag{1.45}$$

and $\eta^-(0,\lambda)\, V_-(\lambda)$ *is bounded, with*

$$\eta^-(0,\lambda)\, V_-(\lambda)\, e_1 = 1 + o(1). \tag{1.46}$$

Corollary 1.18. *Suppose* $A^{+\infty} = A^{-\infty}$ *and the hypotheses of Proposition* 1.17 *are satisfied with* $V_+ = V_-$. *Then*

$$D(\lambda) \rightarrow 1 \quad \textit{as} \quad |\lambda| \rightarrow \infty \quad \textit{in} \quad \Omega.$$

Proof. $D(\lambda) = \eta^-(0,\lambda)\, \zeta^+(0,\lambda) = \eta^-(0,\lambda)\, V(\lambda)\, W(\lambda)\, \zeta^+(0,\lambda) = 1 + o(1)$ as $|\lambda| \rightarrow \infty$ in Ω. □

Remark. It is not universally true that $D(\lambda) \rightarrow 1$ as $\lambda \rightarrow \infty$, even for equations not unlike our applications. For example consider

$$\partial_x^2 u + 2a(x)\, \partial_x u + (a(x)^2 - \lambda)\, u = 0,$$

where $a(x) = \alpha$ for $0 \leqslant x \leqslant 1, 0$ otherwise. It is straightforward to compute explicitly that

$$D(\lambda) = \mathrm{e}^{\alpha}(1 + (\mathrm{e}^{2\mu} - 1)\, \alpha^2/4\mu^2) \rightarrow \mathrm{e}^{\alpha}$$

as $\lambda \rightarrow \infty$, where $\mu = -\sqrt{\lambda}$.

Proof of Proposition 1.17. Similar to the proof of Proposition 1.2, put $B(\lambda) = W(A^{\infty} - \mu I)\, V$ and let $v(x) = -e_1 + \exp(-\mu x)\, Wy(x)$. Then since $Be_1 = 0$, solutions of (1.1) correspond to solutions of

$$\mathrm{d}v/\mathrm{d}x = B(\lambda)\, v + F(x,\lambda)\, (v + e_1).$$

The particular choice $y(x) = \zeta^+(x,\lambda)$ yields $v(x)$, which satisfies $v(x) \rightarrow 0$ as $x \rightarrow \infty$ and

$$v(x) = -\int_x^{\infty} \mathrm{e}^{(x-s)\,B} F(s,\lambda)\, (v(s) + e_1)\, \mathrm{d}s.$$

Now, $B(\lambda)$ is diagonal with $\mathrm{Re}\,(B_{kk}) \geqslant 0$ for $1 \leqslant k \leqslant m$. Using hypotheses (1.43), (1.44), we may choose x_0 sufficiently large, independently of λ so that for $x > x_0$,

$$\begin{aligned} |v(x)| &\leqslant \sup_{s \geqslant x_0} |v(s)| \int_{x_0}^{\infty} |F(x,\lambda)|\, \mathrm{d}s + \int_{x_0}^{\infty} |F(s,\lambda)\, e_1|\, \mathrm{d}s \\ &\leqslant \tfrac{1}{2} \sup_{s \geqslant x_0} |v(s)| + \int_{x_0}^{\infty} |F(s,\lambda)\, e_1|\, \mathrm{d}s. \end{aligned}$$

It follows that

$$\sup_{x \geqslant x_0} |v(x)| \leqslant 2 \int_{x_0}^{\infty} |F(s,\lambda)\, e_1|\, \mathrm{d}s = o(1) \quad \text{as} \quad |\lambda| \rightarrow \infty.$$

Now $v(x)$ also satisfies

$$v(x) = \mathrm{e}^{(x-x_0)\,B}\, v(x_0) + \int_x^{x_0} \mathrm{e}^{(x-s)\,B} F(s,\lambda)\, (v(s) + e_1)\, \mathrm{d}s,$$

from which we obtain the estimate, for $0 \leqslant x \leqslant x_0$,

$$|v(x)| \leqslant C \int_x^{x_0} |F(s,\lambda)|\,|v(s)|\,\mathrm{d}s + C\left(|v(x_0)| + \int_x^{x_0} |F(s,\lambda)\,e_1|\,\mathrm{d}s\right).$$

Applying Gronwall's inequality we obtain

$$\begin{aligned}|v(x)| &\leqslant C \exp\left(C\int_x^{x_0} |F(s,\lambda)|\,\mathrm{d}s\right)\left(|v(x_0)| + \int_x^{x_0} |F(s,\lambda)\,e_1|\,\mathrm{d}s\right)\\ &= o(1) \quad \text{as} \quad |\lambda| \to \infty \quad \text{in } \Omega,\end{aligned}$$

by using the estimate on $v(x_0)$ above and hypotheses (1.44). This proves (1.45).

To establish (1.46), put $w(x) = -e_1^{\mathrm{t}} + \exp(\mu x)\,z(x)\,V$. Then solutions of (1.2) correspond to solutions of

$$\mathrm{d}w/\mathrm{d}x + wB(\lambda) + (w + e_1^{\mathrm{t}})\,F(x,\lambda) = 0.$$

The particular choice $z(x) = \eta^{-}(x,\lambda)$ yields $w(x)$ which satisfies $w(x) \to 0$ as $x \to -\infty$ and

$$w(x) = -\int_{-\infty}^{x} (w(s) + e_1^{\mathrm{t}})\,F(s,\lambda)\,\mathrm{e}^{-(x-s)B}\,\mathrm{d}s.$$

For x_0 sufficiently negative we have, for $x < x_0$,

$$|w(x)| \leqslant (C + \sup_{s \leqslant x_0} |w(s)|) \int_{-\infty}^{x_0} |F(s,\lambda)|\,\mathrm{d}s \leqslant \tfrac{1}{2}(C + \sup_{s \leqslant x_0} |w(s)|)$$

so that $\sup_{s \leqslant x_0} |w(s)| \leqslant C$. Furthermore, $w_1(x) = w(x)\,e_1$ satisfies (since $Be_1 = 0$)

$$|w_1(x)| \leqslant (C + \sup_{s \leqslant x_0} |w(s)|) \int_{-\infty}^{x_0} |F(s,\lambda)\,e_1|\,\mathrm{d}s = o(1) \quad \text{as} \quad |\lambda| \to \infty.$$

In a fashion similar to the argument for (1.45), applying Gronwall's inequality for $x_0 \leqslant x \leqslant 0$ to estimate $|w(x)|$ and $|w_1(x)|$, we find that

$$|w(x)| \leqslant C \quad \text{for} \quad x \leqslant 0, \quad |w_1(x)| \leqslant C \int_{-\infty}^{0} |F(s,\lambda)\,e_1|\,\mathrm{d}s = o(1)$$

as $|\lambda| \to \infty$ in Ω. This establishes (1.46) and finishes the proof of Proposition 1.17. □

Let us consider the typical application of the results above to systems obtained from higher-order scalar ODEs by the standard reduction. Let

$$\mathscr{P}(\nu) = \nu^m + \sum_{j=0}^{m-1} a_j^{\infty}(\lambda)\,\nu^j$$

denote the characteristic polynomial of A^{∞}. The m distinct eigenvalues of A^{∞} correspond to simple roots of $\mathscr{P}(\nu) = 0$, labelled $\nu_1, \ldots, \nu_m$. The matrix of eigenvectors may be taken as $V_{jk} = \nu_k^{j-1}$ assuming $\nu_1 = \mu$. The components of the deviator $R(x,\lambda)$ are

$$R_{jk}(x,\lambda) = \begin{cases} 0 & \text{if } j < m, \\ -a_{k-1}(x,\lambda) + a_{k-1}^{\infty}(\lambda) & \text{if } j = m, \end{cases}$$

where $k = 1, 2, \ldots, n$.

Put $\rho_k(x,\lambda) = -a_k(x,\lambda)+a_k^\infty$ for $k = 1, \ldots, m$. Since the last column of $W = V^{-1}$ is given by $W_{jm} = \mathscr{P}'(\nu_j)^{-1}$, we find that

$$F_{jk}(x,\lambda) = \mathscr{P}'(\nu_j)^{-1} \sum_{i=0}^{m-1} \rho_i(x,\lambda)\,\nu_k^i.$$

In this situation, sufficient conditions for the hypotheses of Proposition 1.17 to be satisfied are given below. In the applications studied in this paper, $\rho_i(x,\lambda)$ does not depend on λ so the conditions (1.47) and (1.48) hold automatically.

Corollary 1.19. *Suppose that for some integer i_0, $\rho_i \equiv 0$ if $i > i_0$, and that for some $C>0$, as $\lambda \to \infty$ we have*

$$\int_{-\infty}^{\infty} |\rho_i(x,\lambda)|\,\mathrm{d}x \leqslant C \quad \textit{independent of } \lambda \tag{1.47}$$

and

$$\int_{|x|>x_0} |\rho_i(x,\lambda)|\,\mathrm{d}x \to 0 \quad \textit{as} \quad x_0 \to \infty \quad \textit{uniformly in } \lambda. \tag{1.48}$$

Suppose also that for $j, k = 1, \ldots, m$, $0 \leqslant i \leqslant i_0$ we have

$$|\mathscr{P}'(\nu_j(\lambda))^{-1}\,\nu_k^i| \leqslant C \tag{1.49}$$

and (recall $\mu = \nu_1$)

$$|\mathscr{P}'(\nu_j(\lambda))^{-1}\,\mu^i| = o(1) \quad \textit{as} \quad \lambda \to \infty. \tag{1.50}$$

Then the hypotheses (hence the conclusions) of Proposition 1.17 hold.

To verify the conditions (1.49) and (1.50) in applications, we have found it convenient to use a perturbation argument based on the following lemma.

Lemma 1.20. *Assume that analytic functions $\tilde{\mathscr{P}}(\nu)$ and $\mathscr{Q}(\nu)$, depending on a parameter λ, are given, and that $\mathscr{P}(\nu) = \tilde{\mathscr{P}}(\nu)+\mathscr{Q}(\nu)$. Assume that as $|\lambda| \to \infty$, there is a (simple) zero $\tilde{\nu} = \tilde{\nu}(\lambda)$ of $\tilde{\mathscr{P}}$, a positive function $\rho(\lambda) \to 0$ as $|\lambda| \to \infty$ and a constant $\rho_0 > 1$ such that for $|\nu-\tilde{\nu}| \leqslant \rho$*

$$\tilde{\mathscr{P}}'(\nu) = \tilde{\mathscr{P}}'(\tilde{\nu})\,(1+o(1)) \quad \textit{and} \quad \mathscr{Q}(\nu) = \mathscr{Q}(\tilde{\nu})\,(1+o(1)) \tag{1.51}$$

as $|\lambda| \to \infty$, and $\rho \geqslant \rho_0 |\mathscr{Q}(\tilde{\nu})/\tilde{\mathscr{P}}'(\tilde{\nu})|$. Then for $|\lambda|$ sufficiently large, $\mathscr{P}$ has exactly one root $\nu_0 = \nu_0(\lambda)$ satisfying $|\nu_0 - \tilde{\nu}| \leqslant \rho$.

Proof. The conclusion follows from Rouché's theorem if we can prove $|\tilde{\mathscr{P}}(\nu)| > |\mathscr{Q}(\nu)|$ on the circle $|\nu-\tilde{\nu}| = \rho(\lambda)$ with $|\lambda|$ sufficiently large. By using Taylor's theorem and (1.51), for $|\nu-\tilde{\nu}| = \rho$ we have

$$\tilde{\mathscr{P}}(\nu) = \int_0^1 \tilde{\mathscr{P}}'(\tilde{\nu}+\tau(\nu-\tilde{\nu}))\,\mathrm{d}\tau(\nu-\tilde{\nu}) = \tilde{\mathscr{P}}'(\tilde{\nu})\,(\nu-\tilde{\nu})\,(1+o(1))$$

so for $|\lambda|$ large,

$$|\tilde{\mathscr{P}}(\nu)| > \rho|\tilde{\mathscr{P}}'(\tilde{\nu})|\,(1+o(1)) > \rho_0|\mathscr{Q}(\nu)|\,(1+o(1)) > |\mathscr{Q}(\nu)|. \qquad \square$$

(h) Zeros of higher-order and Jordan chains

If λ is a zero of $D(\lambda)$ of order greater than one, a number of derivatives of $\zeta^+(x,\lambda)$ in λ also enjoy submaximal growth as $x \to -\infty$. Such a fact has been used in applications to identify the order of vanishing of a zero of $D(\lambda)$ with its algebraic multiplicity as an eigenvalue of the original problem, by using the derivatives of ζ^+ to construct a Jordan chain (Evans 1975; Alexander *et al.* 1990). Here we prove only the following result, which is needed in §4.

Proposition 1.21. *Suppose that* $\lambda \in \Omega$ *is a zero of* $D(\lambda)$ *of order* $k+1$, $k \geqslant 1$, *so* $0 = D(\lambda) = \ldots = D^{(k)}(\lambda) \neq D^{(k+1)}(\lambda)$. *Then we have, for* $0 \leqslant j \leqslant k$ *and any* $\epsilon > 0$,

$$\partial_\lambda^j \zeta^+(x, \lambda) = o(\mathrm{e}^{\mu_*^- x}\, \mathrm{e}^{\epsilon|x|}) \quad as \quad x \to -\infty \tag{1.52}$$

and

$$D^{(k+1)}(\lambda) = \lim_{x \to -\infty} \eta^-(x, \lambda) \cdot \partial_\lambda^{k+1} \zeta^+(x, \lambda). \tag{1.53}$$

Similarly,

$$\partial_\lambda^j \eta^-(x, \lambda) = o(\mathrm{e}^{\mu_*^+ x}\, \mathrm{e}^{\epsilon|x|}) \quad as \quad x \to +\infty. \tag{1.54}$$

Proof. We omit the proof of (1.54); the treatment of η^- is similar to that of ζ^+. We prove (1.52) by induction on k: Assume it holds for $j \leqslant k-1$. The function $y(x) = \partial_\lambda^k \zeta^+(x, \lambda)$ satisfies the equation

$$\mathrm{d}y/\mathrm{d}x = A(x, \lambda)\, y + f(x), \tag{1.55}$$

where $f(x) = \partial_\lambda^k (A\zeta^+) - A\, \partial_\lambda^k \zeta^+$. By the induction hypothesis and H2, for any $\epsilon > 0$,

$$f(x) = o(\mathrm{e}^{\mu_*^- x}\, \mathrm{e}^{\epsilon|x|}) \quad \text{as} \quad x \to -\infty. \tag{1.56}$$

Lemma 1.22. *Assume* (1.56). *Then equation* (1.55) *has some solution* $\tilde{y}(x)$ *satisfying*

$$\tilde{y}(x) = o(\mathrm{e}^{\mu_*^- x}\, \mathrm{e}^{\epsilon|x|}) \quad as \quad x \to -\infty$$

for every $\epsilon > 0$.

This lemma may be proved by a standard method by using the variation of constants formula and the contraction mapping principle, as in the proof of Proposition 1.5. Or, since λ is fixed, Theorem 11 of Coppel (1965) may be applied to give the result almost immediately.

Now, $y(x) = \partial_\lambda^k \zeta^+ - \tilde{y}(x)$ is a solution of the homogeneous equation (1.1), so $\eta^- \cdot y$ is a constant. Observe that

$$\eta^- \cdot \tilde{y} = O(\mathrm{e}^{-\mu^- x}\, \mathrm{e}^{\mu_*^- x}\, \mathrm{e}^{\epsilon|x|}) \to 0$$

as $x \to -\infty$. Using Proposition 1.6, we conclude that the following is true:

Lemma 1.23. *Assume* (1.52) *holds for* $j \leqslant k-1$. *Then it holds for* $j = k$ *if and only if*

$$\lim_{x \to -\infty} \eta^-\, \partial_\lambda^k \zeta^+(x, \lambda) = 0. \tag{1.57}$$

Now we shall show that (1.57) holds under the hypotheses of the Proposition. First, observe that for all $j \geqslant 0$,

$$\partial_\lambda^j \eta^-(x, \lambda) = o(\mathrm{e}^{-\mu^- x}\, \mathrm{e}^{\epsilon|x|}) \quad \text{as} \quad x \to -\infty$$

for every $\epsilon > 0$. This follows from (1.9). Then

$$D^{(k)}(\lambda) = \partial_\lambda^k (\eta^- \zeta^+) = \eta^-\, \partial_\lambda^k \zeta^+ + o(\mathrm{e}^{-\mu^- x}\, \mathrm{e}^{\epsilon|x|})\, o(\mathrm{e}^{\mu_*^-}\, \mathrm{e}^{\epsilon|x|}) \quad as \quad x \to -\infty,$$

so since $D^{(k)}(\lambda) = 0$, (1.57) follows, proving (1.52). This calculation also proves (1.53), finishing the proof of the Proposition. □

2. Instability of solitary waves

(a) *Outline of strategy*

In this section we apply the results of §1 to derive a hamiltonian criterion for the linearized exponential instability of solitary waves of gKdV, gBBM and gBou. For each of these equations, the instability criterion is

$$\frac{\mathrm{d}}{\mathrm{d}c}\mathscr{N}[u_c] < 0, \tag{2.1}$$

where $\mathscr{N}$ is an appropriate *momentum* or *impulse* functional which is constant in time for solutions of the nonlinear equation. Our discussion of how this criterion is made precise and proved will be broken down into parts corresponding to the calculations which are performed for each particular example.

1. The momentum functional $\mathscr{N}$ is obtained from a translation invariant hamiltonian $\mathscr{H}$, and the solitary wave profile $u_c(x)$ is characterized variationally as a stationary point for $\mathscr{H} - c\mathscr{N}$. Namely, u_c satisfies

$$\delta_u(\mathscr{H}[u_c] - c\mathscr{N}[u_c]) = 0. \tag{2.2}$$

In each example, this equation is a second-order ODE which admits a homoclinic loop with vertex 0 in the (u, u') phase plane, and it is easy to check that u_c and its derivatives approach zero at an exponential rate as $|x| \to \infty$.

2. When the linearized evolution equation for small perturbations in the form $v(x-ct, t) = u(x, t) - u_c(x-ct)$ is considered, and separated solutions are sought in the form $v(x, t) = \mathrm{e}^{\lambda t}\, Y(x)$, the equation for Y takes the form in (1.29),

$$\mathscr{A}(x, \lambda)\, Y = 0. \tag{2.3}$$

This yields a first-order system $\mathrm{d}y/\mathrm{d}x = A(x, \lambda)\, y$ via the standard reduction.

In this section, we call λ *an eigenvalue of* (2.3) *if* (2.3) admits a square integrable solution Y. We must relate this notion to the definition of 'eigenvalue' used in §1 and the criteria in (1.36), (1.37). We show that in each example, the theory in §1 may be applied for λ in a domain Ω that contains the closed right half-plane $\operatorname{Re}\lambda \geqslant 0$. Recalling the definitions in H3, we will show more precisely that

$$\operatorname{Re}\mu(\lambda) < 0 < \mu_*(\lambda) \quad \text{for} \quad \operatorname{Re}\lambda > 0, \tag{2.4}$$

$$\operatorname{Re}\mu(\lambda) < 0 = \mu_*(\lambda) \quad \text{for} \quad \operatorname{Re}\lambda = 0. \tag{2.5}$$

Provided $\operatorname{Re}\lambda > 0$, therefore, Propositions 1.6 and 1.13 imply that λ is an eigenvalue of (2.3) if and only if λ is an 'eigenvalue' in the sense of §1, i.e. $D(\lambda) = 0$. In this case the estimates in (1.36), (1.37) imply that $Y(x)$ decays to zero at an exponential rate as $|x| \to \infty$. (However, if $D(\lambda) = 0$ but $\operatorname{Re}\lambda < 0$, the estimate in (1.37) does not imply that $Y(x)$ is bounded as $x \to -\infty$, because it turns out that $\mu_*(\lambda) < 0$. Also, if $\operatorname{Re}\lambda = 0$ (2.3) may admit a bounded solution even if $D(\lambda) \neq 0$.)

Our strategy for proving (2.4) and (2.5) breaks into two parts. As a preliminary step, we have the following obvious observation.

Lemma 2.1. *The number of eigenvalues (counting multiplicity) of $A^{\infty}(\lambda)$ having negative real part is constant as λ varies in any connected component of the complement of the closed set*

$$S_e = \{\lambda \in \mathbb{C} \mid A^{\infty}(\lambda) \quad \textit{has some purely imaginary eigenvalue}\}.$$

3. The first step in proving (2.4) and (2.5) is to show:

Proposition 2.2 *S_e coincides with the imaginary axis. Moreover, as λ varies on the imaginary axis, the number of purely imaginary eigenvalues of $A^{\infty}(\lambda)$ is constant.*

4. The second step is to study the eigenvalues of $A^{\infty}(\lambda)$ for $|\lambda|$ large and show that (2.4) holds for $|\lambda|$ large. Combined with Proposition 2.2, this establishes (2.4) and (2.5), and it turns out in each example that we may take Ω in the form

$$\Omega = \{\lambda \mid \operatorname{Re}\lambda > -\epsilon\} \tag{2.6}$$

for some $\epsilon > 0$. At this point, it is also easy to verify that $D(\lambda) \to 1$ as $|\lambda| \to \infty$, $\lambda \in \Omega$, by using Lemma 1.20 to verify the hypotheses of Corollary 1.19.

5. Next, we proceed to follow the strategy outlined in the introduction to show that when the instability criterion (2.1) holds, then $D(\lambda) = 0$ for some $\lambda > 0$. Namely, we compute Y^+, Z^-, Y_λ^+ and Z_λ^- at $\lambda = 0$ (cf. §1*e*) explicitly in terms of the solitary wave $u_c(x)$ and its derivatives with respect to x and c.

6. Then the derivatives of D are evaluated at $\lambda = 0$ by using (1.35) and we prove that

$$D(0) = 0, \quad D'(0) = 0, \quad \operatorname{sgn} D''(0) = \operatorname{sgn}\frac{\mathrm{d}}{\mathrm{d}c}\mathscr{N}[u_c]. \tag{2.7}$$

It follows that whenever the instability criterion (2.1) holds, then $D(\lambda) < 0$ for small $\lambda > 0$. Since $D(\lambda) \to 1$ as $\lambda \to \infty$, we have $D(\lambda) = 0$ for some $\lambda > 0$ and the existence of an unstable eigenvalue has been established.

7. For the explicit nonlinearity $f(u) = u^{p+1}/(p+1)$, it turns out that the solitary waves always have the form $u_c(x) = \alpha\,\mathrm{sech}^{2/p}(\gamma x)$. We compute the instability criterion explicitly for each example. For this purpose it is useful to define

$$I(r) = \int_{-\infty}^{\infty} \mathrm{sech}^r x\,\mathrm{d}x$$

$$k(p) = \int_{-\infty}^{\infty} \mathrm{sech}^{4/p} x \tanh^2 x\,\mathrm{d}x \bigg/ \int_{-\infty}^{\infty} \mathrm{sech}^{4/p} x\,\mathrm{d}x.$$

As in Weinstein (1987) from Gradshetyn & Ryzhik (1980) follow the identities

$$I(r) = \Gamma(\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}r)/\Gamma(\tfrac{1}{2}(r+1)), \quad k(p) = p/(4+p).$$

(*b*) *The generalized KdV equation*

$$\partial_t u + \partial_x f(u) + \partial_x^3 u = 0. \tag{gKdV}$$

(i) *Hamiltonian structure*

The hamiltonian is

$$\mathscr{H}[u] = \int_{-\infty}^{\infty} (-\tfrac{1}{2}(\partial_x u)^2 + F(u))\,\mathrm{d}x, \tag{2.8}$$

where $F(z) = \int_0^z f(s)\,\mathrm{d}s$. The momentum is

$$\mathscr{N}[u] = \frac{1}{2}\int_{-\infty}^{\infty} u^2\,\mathrm{d}x. \tag{2.9}$$

Equation (2.2) for the solitary wave is

$$-\partial_x^2 u_c + cu_c - f(u_c) = 0. \tag{2.10}$$

For $c > 0$, this equation has a unique positive exponentially decaying solution which is even in x. For $f(u) = u^{p+1}/(p+1)$, explicitly

$$u_c(x) = [\tfrac{1}{2}c(p+2)(p+1)]^{1/p}\operatorname{sech}^{2/p}(\tfrac{1}{2}xp\sqrt{c}).$$

(ii) *Linear evolution and the eigenvalue equation*

Changing to a moving frame $x' = x-ct$, $t' = t$, the linearized equation for small perturbations of $u_c(x')$ is (dropping the prime on the new variables)

$$\partial_t v = \partial_x L_c v, \tag{2.11}$$

where $L_c = -\partial_x^2 + c - f'(u_c)$. The eigenvalue equation (2.3) takes the form

$$\partial_x L_c Y = \lambda Y, \tag{2.12}$$

or

$$\partial_x^3 Y + (f'(u_c)-c)\,\partial_x Y + (\lambda + \partial_x f'(u_c))\,Y = 0. \tag{2.13}$$

Therefore, $a_0^\infty = \lambda$, $a_1^\infty = -c$ and $a_2^\infty = 0$.

(iii) *Imaginary asymptotic eigenvalues*

A number ν is an eigenvalue of $A^\infty(\lambda)$ if and only if

$$\mathscr{P}(\nu) = \nu^3 - c\nu + \lambda = 0. \tag{2.14}$$

Clearly λ must be purely imaginary if ν is, and indeed

$$S_e = \{\lambda \mid \lambda = \mathrm{i}(c\tau + \tau^3) \quad \text{for some real } \tau\}.$$

So S_e coincides with the imaginary axis. Since $\tau \rightarrow c\tau + \tau^3$ is monotone increasing, there is *one* imaginary eigenvalue of $A^\infty(\lambda)$, $\nu = \mathrm{i}\tau$, for any imaginary λ. This proves Proposition 2.2 in this case.

(iv) *Asymptotic eigenvalues for large* λ

We seek to apply Lemma 1.20. We may take

$$\tilde{\mathscr{P}}(\nu) = \nu^3 + \lambda, \quad \mathscr{Q}(\nu) = -c\nu.$$

Then $\tilde{\mathscr{P}}'(\nu) = 3\nu^2$, the roots $\tilde{\nu}$ of $\tilde{\mathscr{P}}$ are the cube roots of $-\lambda$, and for $|\nu - \tilde{\nu}| = o(1)$ we have

$$\mathscr{Q}(\nu) = -c\tilde{\nu}(1+o(1)), \quad \tilde{\mathscr{P}}'(\nu) = 3\tilde{\nu}^2(1+o(1)), \quad |\mathscr{Q}(\tilde{\nu})/\tilde{\mathscr{P}}'(\tilde{\nu})| = c/3|\lambda|^{\frac{1}{3}}.$$

Taking $\rho(\lambda) = \rho_0 c/3|\lambda|^{\frac{1}{3}}$ for any $\rho_0 > 1$, the hypotheses of Lemma 1.20 are satisfied, so the roots of $\mathscr{P}(\nu) = 0$ are given by

$$\nu = (-\lambda)^{\frac{1}{3}} + O(|\lambda|^{-\frac{1}{3}}). \tag{2.15}$$

From this and part (iii), (2.4) and (2.5) follow, and it is easy to see that Ω may be taken in the form (2.6).

To apply Corollary 1.19, we may take $i_0 = 1$. To verify (1.49) and (1.50) it suffices to observe that for any labelling ν_1, ν_2, ν_3 of these roots we have

$$|\nu_k/\mathscr{P}'(\nu_j)| = |\lambda|^{\frac{1}{3}}/3|\lambda|^{\frac{2}{3}}(1+o(1)) = o(1)$$

as $|\lambda| \rightarrow \infty$. By Corollary 1.18, it follows that $D(\lambda) \rightarrow 1$ as $|\lambda| \rightarrow \infty$ in Ω.

(v) *Generalized eigenfunctions at* $\lambda = 0$

According to the requirements laid down in §1*e*, for $\lambda = 0$, $Y^+ = \zeta_1^+$ and $Z^- = \eta_m^-$ are the unique solutions of

$$\partial_x L_c Y^+(x,\lambda) = 0, \quad L_c \partial_x Z^-(x,\lambda) = 0, \tag{2.16}$$

such that

$$\left.\begin{aligned} Y^+(x,\lambda)\,\mathrm{e}^{-\mu x} &\to v_1 \quad \text{as} \quad x\to+\infty,\\ Z^-(x,\lambda)\,\mathrm{e}^{\mu x} &\to w_m \quad \text{as} \quad x\to-\infty,\end{aligned}\right\} \tag{2.17}$$

where $v_1 w_m \mathscr{P}'(\mu) = 1$, $\mathscr{P}$ being the monic polynomial in (1.33*a*).

Differentiating (2.12) and the associated transposed equation, for $\lambda = 0$, we find that the derivatives, Y_λ^+ and Z_λ^-, are solutions of

$$\partial_x L_c Y_\lambda^+ = Y^+, \quad L_c \partial_x Z_\lambda^- = -Z^-, \tag{2.18}$$

which decay exponentially as $x\to\infty$ or $-\infty$ respectively.

For the gKdV equation, for $\lambda = 0$ we have $\mu = -\sqrt{c}$, $\mathscr{P}'(\mu) = 2c$. We may choose $v_1 = 1$, $w_m = 1/2c$.

Differentiating (2.10) in x, we observe

$$L_c \partial_x u_c = 0, \quad \partial_x L_c \partial_x u_c = 0. \tag{2.19}$$

Differentiating (2.10) in c, we find $L_c \partial_c u_c = -u_c$ and hence

$$\partial_x L_c \partial_c u_c = -\partial_x u_c, \quad L_c \partial_x \int_{-\infty}^{x} \partial_c u_c = -u_c. \tag{2.20}$$

From (2.10) one can show that there exists $\beta > 0$ (depending on c) such that

$$(u_c(x), \partial_x u_c(x))\,\mathrm{e}^{-\mu x} \to \beta(1,\mu) \quad \text{as} \quad x\to\infty.$$

By the uniqueness of Y^+, we find that at $\lambda = 0$, $Y^+ = (\mu\beta)^{-1}\partial_x u_c$. It follows from (2.20) that $\partial_x L_c(Y_\lambda^+ + (\mu\beta)^{-1}\partial_c u_c) = 0$; hence for some β_1 independent of x we have $Y_\lambda^+ = -(\mu\beta)^{-1}\partial_c u_c + \beta_1 \partial_x u_c$. We determine Z^- and Z_λ^- in a similar way, summarizing the results as follows: For $\lambda = 0$,

$$\left.\begin{aligned} &Y^+ = (\mu\beta)^{-1}\partial_x u_c, \quad Z^- = (2c\beta)^{-1} u_c,\\ &Y_\lambda^+ = -(\mu\beta)^{-1}\partial_c u_c + \beta_1 \partial_x u_c, \quad Z_\lambda^- = \frac{1}{2c\beta}\int_{-\infty}^{x} \partial_c u_c + \beta_2 u_c.\end{aligned}\right\} \tag{2.21}$$

(vi) *Derivatives of $D(\lambda)$ at $\lambda = 0$*

Here, we use (1.34) and (1.35) and (2.13). At $\lambda = 0$ we have $D(0) = 0$. Using (1.35) and (2.13) we have $\partial\mathscr{A}/\partial\lambda = \partial a_0/\partial\lambda = 1$ and therefore

$$D'(0) = \int_{-\infty}^{\infty} Z^-(x,0)\,Y^+(x,0)\,\mathrm{d}x = \frac{1}{2\mu c\beta^2}\int_{-\infty}^{\infty} u_c \partial_x u_c\,\mathrm{d}x = 0. \tag{2.22}$$

By Theorem 1.11, equation (1.35) may be differentiated in λ. Using that $D'(0) = 0$ and $\partial^2\mathscr{A}/\partial\lambda^2 = 0$ we find that

$$\begin{aligned} D''(0) &= \int_{-\infty}^{\infty} (Z_\lambda^- Y^+ + Z^- Y_\lambda^+)\,\mathrm{d}x\\ &= \frac{1}{2\mu c\beta^2}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x}\partial_c u_c\right)\partial_x u_c + u_c(-\partial_c u_c)\,\mathrm{d}x\\ &= \frac{1}{\mu c\beta^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(-\frac{1}{2}\int_{-\infty}^{\infty} u_c^2\,\mathrm{d}x\right) = -\frac{1}{\mu c\beta^2}\frac{\mathrm{d}}{\mathrm{d}c}\mathscr{N}[u_c].\end{aligned}$$

Since $\mu = \mu(0) < 0$, we have $\operatorname{sgn} D''(0) = \operatorname{sgn} \mathrm{d}\mathscr{N}[u_c]/\mathrm{d}c$ as claimed.

(vii) *Power nonlinearity*

For $f(u) = u^{p+1}/(p+1)$, $u_c(x) = \alpha\,\mathrm{sech}^{2/p}(\gamma x)$ where

$$\alpha = [\tfrac{1}{2}c(p+1)(p+2)]^{1/p}, \quad \gamma = \tfrac{1}{2}p\sqrt{c}.$$

Therefore the momentum is given by

$$\mathcal{N}[u_c] = \frac{1}{2}\int_{-\infty}^{\infty} u_c^2\,\mathrm{d}x = \tfrac{1}{2}\alpha^2 \int_{-\infty}^{\infty} \mathrm{sech}^{2\beta}(\gamma x)\,\mathrm{d}x = \frac{\alpha^2}{2\gamma} I\left(\frac{4}{p}\right),$$

where $I(r)$ is defined in §2*a*(vii). To determine the sign of $\mathrm{d}\mathcal{N}[u_c]/\mathrm{d}c$, we compute

$$\mathcal{N}[u_c]^{-1}\frac{\mathrm{d}}{\mathrm{d}c}\mathcal{N}[u_c] = \frac{\mathrm{d}}{\mathrm{d}c}(2\ln\alpha - \ln\gamma) = \frac{2}{pc} - \frac{1}{2c}.$$

Hence

$$\mathrm{d}\mathcal{N}[u_c]/\mathrm{d}c > 0 \quad \text{for} \quad p < 4 \quad \text{(stability)},$$

$$\mathrm{d}\mathcal{N}[u_c]/\mathrm{d}c < 0 \quad \text{for} \quad p > 4 \quad \text{(instability)}.$$

(*c*) *The generalized BBM equation*

$$\partial_t u + \partial_x u + \partial_x f(u) - \partial_x^2 \partial_t u = 0. \tag{gBBM}$$

(i) *Hamiltonian structure*

The hamiltonian is

$$\mathcal{H}[u] = \int_{-\infty}^{\infty} (\tfrac{1}{2}u^2 + F(u))\,\mathrm{d}x, \tag{2.23}$$

where $F(z) = \int_0^z f(s)\,\mathrm{d}s$. The momentum is

$$\mathcal{N}[u] = \frac{1}{2}\int_{-\infty}^{\infty} (u^2 + (\partial_x u)^2)\,\mathrm{d}x. \tag{2.24}$$

Equation (2.2) for the solitary wave is

$$-c\partial_x^2 u_c + (c-1)u_c - f(u_c) = 0. \tag{2.25}$$

For $c > 1$, this equation has a unique solution which is positive, exponentially decaying, and even in x. For $f(u) = u^{p+1}/(p+1)$, it is explicitly given by

$$u_c(x) = [\tfrac{1}{2}(c-1)(p+2)(p+1)]^{1/p}\,\mathrm{sech}^{2/p}(\tfrac{1}{2}xp\sqrt{((c-1)/c)}).$$

(ii) *Linear evolution and eigenvalue equation*

The linearized equation for perturbations in the moving frame is

$$(I - \partial_x^2)\partial_t v = \partial_x(-c\partial_x^2 + (c-1) - f'(u_c))\,v.$$

The eigenvalue equation (2.3) takes the form

$$\partial_x L_c Y = \lambda(I - \partial_x^2) Y, \tag{2.26}$$

where here $L_c = -c\partial_x^2 + c - 1 - f'(u_c)$, or

$$\partial_x^3 Y - (\lambda/c)\partial_x^2 Y - (1/c)(c-1-f'(u_c))\partial_x Y + (1/c)(\lambda + \partial_x f'(u_c))Y = 0. \tag{2.27}$$

Therefore, $a_0^\infty = \lambda/c$, $a_1^\infty = (c-1)/c$, and $a_2^\infty = -\lambda/c$.

(iii) *Imaginary asymptotic eigenvalues*

A number ν is an eigenvalue of $A^{\infty}(\lambda)$ if and only if

$$c\mathscr{P}(\nu) = (\lambda - c\nu)(1-\nu^2)+\nu = 0. \tag{2.28}$$

λ must be purely imaginary if ν is, and we have

$$S_e = \{\lambda \mid \lambda = \mathrm{i}c\tau - \mathrm{i}\tau/(1+\tau^2) \quad \text{for some real } \tau\}.$$

So S_e is the imaginary axis. Since $c>1$, the map $\tau \to c\tau - \tau(1+\tau^2)^{-1}$ is monotone increasing, so there is exactly *one* imaginary eigenvalue of $A^{\infty}(\lambda)$, $\nu = \mathrm{i}\tau$ for any imaginary λ. This proves Proposition 2.2 for gBBM.

(iv) *Asymptotic eigenvalues for large* λ

To apply Lemma 1.20, we take

$$\tilde{\mathscr{P}}(\nu) = (\lambda - c\nu)(1-\nu^2), \quad \mathscr{Q}(\nu) = \nu.$$

Then $\tilde{\mathscr{P}}'(\nu) = c(\nu^2-1)+2\nu(c\nu-\lambda)$ and the roots of $\tilde{\mathscr{P}}$ are $\tilde{\nu}_1 = -1$, $\tilde{\nu}_2 = 1$, and $\tilde{\nu}_3 = \lambda/c$. The corresponding values of $\tilde{\mathscr{P}}'(\tilde{\nu}_j)$ are $2(\lambda+c)$, $2(c-\lambda)$, and $(\lambda^2-c^2)/c$. The condition in (1.51) can easily be verified for $|\nu - \tilde{\nu}_j| = o(1)$, and in each case we may take $\rho(\lambda) = O(|\lambda|^{-1})$, so the roots of $\mathscr{P}$ satisfy

$$\mu = \nu_1 = -1+O(|\lambda|^{-1}), \quad \nu_2 = 1+O(|\lambda|^{-1}), \quad \nu_3 = \lambda/c + O(|\lambda|^{-1}). \tag{2.29}$$

Now (2.4) and (2.5) follow directly, and Ω may be taken in the form (2.6).

To apply Corollary 1.19, we may take $i_0 = 1$. We have $|\mathscr{P}'(\nu_j)^{-1}| = O(|\lambda|^{-1})$ for $j = 1, 2$, and $O(|\lambda|^{-2})$ for $j=3$. Hence the conditions (1.49), (1.50) hold. So by Corollary 1.18, it follows that $D(\lambda) \to 1$ as $|\lambda| \to \infty$ in Ω.

(v) *Generalized eigenfunctions at* $\lambda = 0$

As in §2*b*(v), Y^+ and Z^- are described by (2.16), (2.17) for $\lambda = 0$, with L_c as in (2.26). Differentiating (2.26) we find that for $\lambda = 0$,

$$\partial_x L_c Y_\lambda^+ = (I-\partial_x^2)Y^+, \quad L_c \partial_x Z_\lambda^- = -(I-\partial_x^2)Z^-, \tag{2.30}$$

while Y_λ^+ and Z_λ^- decay exponentially as $x\to\infty$ or $-\infty$ respectively.

In the present case, for $\lambda = 0$ we have $\mu = -\surd((c-1)/c)$, and $\mathscr{P}'(\mu) = 2(c-1)$. We may choose $v_1 = 1$, $w_m = (2(c-1))^{-1}$. By differentiating (2.25) we find

$$\left.\begin{aligned} &L_c \partial_x u_c = 0, && \partial_x L_c \partial_x u_c = 0, \\ &L_c \partial_x \int_{-\infty}^{x} \partial_c u_c = -(I-\partial_x^2)u_c, && \partial_x L_c \partial_c u_c = -(I-\partial_x^2)\partial_x u_c, \end{aligned}\right\} \tag{2.31}$$

and there exists $\beta > 0$ such that

$$(u_c, \partial_x u_c)\,\mathrm{e}^{-\mu x} \to \beta(1,\mu) \quad \text{as} \quad x\to\infty.$$

Using the fact that Y^+ and Z^- are the unique exponentially decaying solutions of (2.16) up to multiplication by constants (Proposition 1.2), we determine that

$$\left.\begin{aligned} &Y^+ = (\mu\beta)^{-1}\partial_x u_c, && Z^- = (2(c-1)\beta)^{-1} u_c, \\ &Y_\lambda^+ = -(\mu\beta)^{-1}\partial_c u_c + \beta_1 \partial_x u_c, && Z_\lambda^- = \frac{1}{2(c-1)\beta}\int_{-\infty}^{x} \partial_c u_c + \beta_2 u_c. \end{aligned}\right\} \tag{2.32}$$

(vi) *Derivatives of* $D(\lambda)$

At $\lambda = 0$ we have $D(0) = 0$. Using (1.35) and (2.27) we have

$$cD'(0) = \int_{-\infty}^{\infty} Z^{-}(I-\partial_x^2)\,Y^{+}\,\mathrm{d}x = \int_{-\infty}^{\infty} u_c(I-\partial_x^2)\,\partial_x u_c\,\mathrm{d}x = 0.$$

Differentiating (1.35) and using $D'(0) = 0$, $\partial^2\mathscr{A}/\partial\lambda^2 = 0$, we find

$$\begin{aligned} cD''(0) &= \int_{-\infty}^{\infty} [Z_\lambda^{-}(I-\partial_x^2)\,Y^{+} + Z^{-}(I-\partial_x^2)\,Y_\lambda^{+}]\,\mathrm{d}x \\ &= \frac{1}{2\mu(c-1)\,\beta^2}\int_{-\infty}^{\infty}\left[\left(\int_{-\infty}^{x}\partial_c u_c\right)(I-\partial_x^2)\,\partial_x u_c + u_c(I-\partial_x^2)\,(-\partial_c u_c)\right]\mathrm{d}x \\ &= \frac{1}{\mu(c-1)\,\beta^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(-\frac{1}{2}\int_{-\infty}^{\infty}[u_c^2 + (\partial_x u_c)^2]\,\mathrm{d}x\right) = -\frac{1}{\mu(c-1)\,\beta^2}\frac{\mathrm{d}}{\mathrm{d}c}\mathscr{N}[u_c]. \end{aligned}$$

Since $\mu = \mu(0) < 0$, we have shown $\operatorname{sgn} D''(0) = \operatorname{sgn}\,\mathrm{d}\mathscr{N}[u_c]/\mathrm{d}c$.

(vii) *Power nonlinearity*

For $f(u) = u^{p+1}/(p+1)$, $u_c(x) = \alpha\,\mathrm{sech}^{2/p}(\gamma x)$, where

$$\alpha = [\tfrac{1}{2}(c-1)\,(p+2)\,(p+1)]^{1/p}, \quad \gamma = \tfrac{1}{2}p((c-1)/c)^{\frac{1}{2}}.$$

Hence

$$\begin{aligned} \mathscr{N}[u_c] &= \frac{1}{2}\int_{-\infty}^{\infty}(u_c^2 + (\partial_x u_c)^2)\,\mathrm{d}x = \frac{\alpha^2}{2\gamma}\int_{-\infty}^{\infty}(\mathrm{sech}^{2\beta}y + \beta^2\gamma^2\,\mathrm{sech}^{2\beta}y\tanh^2 y)\,\mathrm{d}y \\ &= I\left(\frac{4}{p}\right)\frac{\alpha^2}{2\gamma}\left[1 + \frac{c-1}{c}k(p)\right] \end{aligned}$$

in the notation of §2a(vii). A computation gives that

$$\frac{\mathrm{d}}{\mathrm{d}c}\mathscr{N}[u_c]\frac{p(c-1)\,c^2\gamma}{\alpha^2 I(4/p)} = (1+k)\,c^2 - 2kc - \tfrac{1}{4}kp,$$

from which it follows that for $c > 1$, $\mathrm{d}\mathscr{N}[u_c]/\mathrm{d}c < 0$ if and only if $c < c_0(p)$, where

$$c_0(p) = (p/(4+2p))\,[1 + \surd(2+\tfrac{1}{2}p)].$$

For $p \leqslant 4$, we have $\mathrm{d}\mathscr{N}[u_c]/\mathrm{d}c > 0$ for all $c > 1$. When $p > 4$, for $1 < c < c_0(p)$, the instability condition (2.1) holds. Furthermore, let $p > 4$. Then there is a threshold $\mathscr{N}_*(p) > 0$, such that for any $\mathscr{N} > \mathscr{N}_*(p)$, there are two solitary waves u_{c_1} and u_{c_2} with speeds $1 < c_1 < c_0(p) < c_2$. The results of this paper imply the exponential instability of u_{c_1}, the slower wave.

(d) *The generalized regularized Boussinesq equation*

$$\partial_t^2 u - \partial_x^2 u - \partial_x^2 f(u) - \partial_x^2\partial_t^2 u = 0. \tag{gBou}$$

(i) *Hamiltonian structure*

The hamiltonian is

$$\mathscr{H}[u] = \int_{-\infty}^{\infty}[\tfrac{1}{2}v^2 + \tfrac{1}{2}(\partial_t u)^2 + \tfrac{1}{2}u^2 + F(u)]\,\mathrm{d}x \tag{2.33}$$

with the constraint that $\partial_x v = \partial_t u$, and $F(u) = \int_0^z f(s)\,\mathrm{d}s$. The momentum is

$$\mathcal{N}[u] = \int_{-\infty}^{\infty} (uv + \partial_x u \partial_x v)\,\mathrm{d}x. \tag{2.34}$$

Equation (2.2) for the solitary wave eventually implies that $v_c = cu_c$ and

$$-c^2\partial_x^2 u_c + (c^2-1)\,u_c - f(u_c) = 0. \tag{2.35}$$

This is the same as (2.25) after replacing c^2 by c. For $c^2 > 1$, (2.35) has a unique solution which is positive, exponentially decaying and even in x. (We thank Peter Smereka for showing us this hamiltonian structure.)

(ii) *Linear evolution and eigenvalue equations*

The linearized equation for the evolution of small perturbations of u_c in the moving frame is

$$(\partial_t - c\partial_x)^2 (I-\partial_x^2)\,v = \partial_x^2(v + f'(u_c)\,v). \tag{2.36}$$

The eigenvalue problem (2.3) becomes

$$(\lambda - c\partial_x)^2 (I-\partial_x^2)\,Y - \partial_x^2(Y + f'(u_c)\,Y) = 0, \tag{2.37}$$

which may also be written as

$$-\partial_x^2 L_c Y = (\lambda^2 - 2c\lambda\partial_x)(I-\partial_x^2)\,Y, \tag{2.38}$$

where $L_c = -c^2\partial_x^2 + c^2 - 1 - f'(u_c)$, or

$$\partial_x^4 Y - (2/c)\,\lambda\partial_x^3 Y + [1 - c^2 + \lambda^2 + f'(u_c)]\,c^{-2}\,\partial_x^2 Y + (2/c^2)\,[c\lambda + \partial_x f'(u_c)]\,\partial_x Y + c^{-2}[\partial_x^2 f'(u_c) - \lambda^2]\,Y = 0. \tag{2.38$'$}$$

Thus $a_0^\infty = -\lambda^2/c^2$, $a_1^\infty = 2\lambda/c$, $a_2^\infty = (c^2-1+\lambda^2)/c^2$, $a_3^\infty = -2\lambda/c$.

(iii) *Imaginary asymptotic eigenvalue*

A number ν is an eigenvalue of $A^\infty(\lambda)$ if and only if

$$-c^2\mathscr{P}(\nu) = (\lambda - c\nu)^2 (1-\nu^2) - \nu^2 = 0.$$

Again, λ must be purely imaginary if ν is, and we have

$$S_e = \{\lambda \mid \lambda = \mathrm{i}c\tau \pm \mathrm{i}\sqrt{(\tau^2/(1+\tau^2))} \quad \text{for some real } \tau\}.$$

So S_e is the imaginary axis. Since $c^2 > 1$, both maps $\tau \to c\tau \pm \sqrt{(\tau^2/(1+\tau^2))}$ are monotone, increasing or decreasing according to the sign of c. So there are exactly two imaginary eigenvalues of $A^\infty(\lambda)$, $\nu = \mathrm{i}\tau$ for any purely imaginary λ. This proves Proposition 2.2 in this case.

(iv) *Asymptotic eigenvalues for large* λ

To apply Lemma 1.20, we take

$$\tilde{\mathscr{P}}(\nu) = (\nu^2-1)\,((\lambda - c\nu)^2 + 1), \quad \mathscr{Q}(\nu) = 1.$$

Then $\tilde{\mathscr{P}}'(\nu) = 2\nu((\lambda-c\nu)^2+1) - 2c(\nu^2-1)(\lambda - c\nu)$ and the roots of $\tilde{\mathscr{P}}$ are $\tilde{\nu}_1 = -1$, $\tilde{\nu}_2 = 1$, $\tilde{\nu}_3 = (\lambda + \mathrm{i})/c$, and $\tilde{\nu}_4 = (\lambda - \mathrm{i})/c$. The corresponding values of $\tilde{\mathscr{P}}'(\tilde{\nu}_j)$ are

$$-2((\lambda+c)^2+1), \quad 2((\lambda-c)^2+1), \quad 2\mathrm{i}c(\tilde{\nu}_3^2-1), \quad \text{and} \quad -2\mathrm{i}c(\tilde{\nu}_4^2-1).$$

It is easy to check that (1.51) holds for $|\nu-\tilde{\nu}| = o(1)$, and in each case we may take $\rho(\lambda) = O(|\lambda|^{-2})$. Thus we find that for $|\lambda|$ large, the roots of $\mathscr{P}$ satisfy

$$\left.\begin{aligned} \mu = \nu_1 = -1+O(|\lambda|^{-2}), \quad \nu_2 = 1+O(|\lambda|^{-2}), \\ \nu_3 = (\lambda+\mathrm{i})/c+O(|\lambda|^{-2}) \quad \nu_4 = (\lambda-\mathrm{i})/c+O(|\lambda|^{-2}). \end{aligned}\right\} \tag{2.39}$$

Now (2.4) and (2.5) follow, and one may see that Ω may be taken in the form (2.6).

To apply Corollary 1.19, we may take $i_0 = 2$. We have $|\mathscr{P}'(\nu_j)^{-1}| = O(|\lambda|^{-2})$ for all j, so one can check easily that conditions (1.49) and (1.50) hold. By Corollary 1.18 it follows that $D(\lambda)\to 1$ as $|\lambda|\to\infty$ in Ω.

(v) *Generalized eigenfunctions at* $\lambda = 0$

According to the discussion in §1*e*, for $\lambda = 0$, Y^+ and Z^- are the unique solutions of

$$-\partial_x^2 L_c Y^+ = 0, \quad -L_c\partial_x^2 Z^- = 0, \tag{2.40}$$

such that (2.17) holds. Differentiating (2.38) and the transposed equation for Z^-, we find that for $\lambda = 0$,

$$-\partial_x^2 L_c Y_\lambda^+ = -2c\partial_x(I-\partial_x^2)Y^+, \quad -L_c\partial_x^2 Z_\lambda^- = 2c\partial_x(I-\partial_x^2)Z^-, \tag{2.41}$$

while Y_λ^+ and Z_λ^- again decay exponentially as $x\to+\infty$ or $-\infty$ respectively.

For gBou, at $\lambda = 0$ we have $\mu = -\surd((c^2-1)/c^2)$, and $\mathscr{P}'(\mu) = 2\mu^3$. We may choose $v_1 = 1$, $w_m = 1/(2\mu^3)$. By differentiating (2.35) we find

$$L_c\partial_x u_c = 0 \quad\text{and}\quad L_c\partial_c u_c = -2c(I-\partial_x^2)u_c,$$

so that

$$\left.\begin{aligned} -\partial_x^2 L_c\partial_x u_c = 0, \quad -L_c\partial_x^2\int_{-\infty}^x u_c = 0, \\ -\partial_x^2 L_c\partial_c u_c = -2c\,\partial_x(I-\partial_x^2)(-\partial_x u_c), \\ -L_c\partial_x^2\int_{-\infty}^x\int_{-\infty}^s u_c = 2c\partial_x(I-\partial_x^2)\int_{-\infty}^x u_c. \end{aligned}\right\} \tag{2.42}$$

From (2.35) also follows that there exists $\beta > 0$ such that

$$(u_c, \partial_x u_c)\,\mathrm{e}^{-\mu x}\to\beta(1,\mu) \quad\text{as}\quad x\to\infty.$$

Using that Y^+ and Z^- are the unique exponentially decaying solutions of (2.40) up to multiplication by constants, we determine that

$$\left.\begin{aligned} Y^+ &= (\mu\beta)^{-1}\partial_x u_c, & Z^- &= -\frac{1}{2\beta\mu^2}\int_{-\infty}^x u_c, \\ Y_\lambda^+ &= -(\mu\beta)^{-1}\partial_c u_c+\beta_1\partial_x u_c, & Z_\lambda^- &= -\frac{1}{2\beta\mu^2}\int_{-\infty}^x\int_{-\infty}^s \partial_c u_c+\beta_2\int_{-\infty}^x u_c. \end{aligned}\right\} \tag{2.43}$$

(vi) *Derivatives of* $D(\lambda)$

At $\lambda = 0$ we have $D(0) = 0$. Using (1.35) and (2.37) we have $-c^2\partial\mathscr{A}/\partial\lambda = 2(\lambda-c\,\partial_x)(I-\partial_x^2)$ so

$$-c^2D'(0) = -2c\int_{-\infty}^{\infty} Z^-\partial_x(I-\partial_x^2)Y^+ = 2c\int_{-\infty}^{\infty} u_c(I-\partial_x^2)\partial_x u_c = 0,$$

after an integration by parts. Differentiating (1.35) and using $D'(0) = 0$, and $-c^2\,\partial^2\mathscr{A}/\partial\lambda^2 = 2(I-\partial_x^2)$, we find

$$-c^2D''(0) = 2c\int_{-\infty}^{\infty} \partial_x Z_\lambda^-(I-\partial_x^2)\,Y^+ + \partial_x Z^-(I-\partial_x^2)\,Y_\lambda^+ + 2\int_{-\infty}^{\infty} Z^-(I-\partial_x^2)\,Y^+.$$

Using (2.43) and after some integrations by parts, we find

$$\begin{aligned}2\mu^3\beta^2c^2D''(0) &= 2c\int_{-\infty}^{\infty}\left(\int^x \partial_c u_c\right)(I-\partial_x^2)\,\partial_x u_c + u_c(I-\partial_x^2)\,(-\partial_c u_c)\\ &\quad + 2\int_{-\infty}^{\infty}\left(\int^x u_c\right)(I-\partial_x^2)\,\partial_x u_c\\ &= -2\frac{\mathrm{d}}{\mathrm{d}c}\left[c\int_{-\infty}^{\infty} u_c^2 + (\partial_x u_c)^2\right] = -2\frac{\mathrm{d}}{\mathrm{d}c}\mathscr{N}[u_c].\end{aligned}$$

Since $\mu = \mu(0) < 0$, we have $\operatorname{sgn} D''(0) = \operatorname{sgn} \mathrm{d}\mathscr{N}[u_c]/\mathrm{d}c$, as desired.

(vii) *Power nonlinearity*

For $f(u) = u^{p+1}/(p+1)$, $u_c(x) = \alpha\,\mathrm{sech}^{2/p}(\gamma x)$ where

$$\alpha = [\tfrac{1}{2}(c^2-1)\,(p+2)\,(p+1)]^{1/p}, \quad \gamma = \tfrac{1}{2}p((c^2-1)/c^2)^{\frac{1}{2}}.$$

Hence
$$\mathscr{N}[u_c] = c\int_{-\infty}^{\infty} u_c^2 + (\partial_x u_c)^2 = \frac{c\alpha^2}{2\gamma}I\left(\frac{4}{p}\right)\left[1+\frac{c^2-1}{c^2}k(p)\right],$$

from which we may compute that

$$\frac{\mathrm{d}}{\mathrm{d}c}\mathscr{N}[u_c]\frac{c^2(c^2-1)\,\gamma}{\alpha^2 I(4/p)} = c^4\left(\frac{1+k}{k}\right) - 3c^2.$$

Therefore, the instability criterion (2.1) holds when $1 < c^2 < c_0^2$, where

$$c_0^2 = 3k/(k+1) = 3p/(4+2p).$$

3. Further results that exploit special structure

(*a*) *Bounds on the number of unstable eigenvalues*

As we show below, the eigenvalue problem for gKdV and gBBM can be cast in an abstract form which we now consider. Suppose J is a skew symmetric operator and L is a self-adjoint operator on a real Hilbert space $\boldsymbol{X}$, and consider the eigenvalue problem

$$JLu = \lambda u \tag{0.8}$$

in $\boldsymbol{Z} = \boldsymbol{X} + \mathrm{i}\boldsymbol{X}$. We assume that on the negative real axis $\{\lambda < 0\}$, the spectrum of L consists of a finite number of eigenvalues of finite multiplicity.

Theorem 3.1. *Let J and L be as above and assume that L has exactly k strictly negative eigenvalues, counting multiplicity. Then JL has at most k eigenvalues* (*counting algebraic multiplicity*) *in the right half-plane* $\operatorname{Re}\lambda > 0$. (*The same statement holds for the left half-plane.*)

The proof of Theorem 3.1 rests on two lemmas.

Lemma 3.2. *Let $Y \subset X + \mathrm{i}X$ be an invariant subspace for JL with* $\dim Y = N < \infty$, *such that whenever λ and μ are eigenvalues of $JL|_Y$ we have $\lambda + \bar{\mu} \neq 0$. Then,*

$$\langle Lv, w \rangle = 0 \quad \textit{whenever} \quad v, w \in Y. \tag{3.1}$$

Proof. First, we choose a basis B for Y such that with respect to this basis, $JL|_Y$ is in Jordan canonical form: There exist generalized eigenspaces $M_1, \ldots, M_s$ in Y such that $JL|_{M_j}$ has a single eigenvalue λ_j, and

$$Y = \bigoplus_{j=1}^{s} M_j, \quad \dim M_j = n_j, \quad \sum_{j=1}^{s} n_j = N,$$

and furthermore, with respect to some basis $\{v_0^{(j)}, \ldots, v_{n_j-1}^{(j)}\}$ of M_j, $JL|_{M_j}$ is represented as

$$JL|_{M_j} = \begin{bmatrix} \lambda_j & 1 & 0 & \ldots & 0 \\ 0 & \lambda_j & 1 & \ldots & 0 \\ \vdots & & & \ddots & 0 \\ & & \ddots & & 1 \\ 0 & \ldots & & 0 & \lambda_j \end{bmatrix}.$$

Define the basis B by $B = \{v_k^{(j)} \mid j = 1, \ldots, s, k = 0, \ldots, n_j - 1\}$.

Now, for $f = v_k^{(j)} \in B$, the initial value problem

$$u'(t) = JLu(t), \quad u(0) = f$$

has a solution, which we denote formally by $u(t) = \mathrm{e}^{JLt} f$, given by

$$\mathrm{e}^{JLt} f = \mathrm{e}^{\lambda_j t} \sum_{q=0}^{k} \frac{t^q}{q!} v_{k-q}^{(j)}.$$

For $f, g \in B$, it is easy to check that

$$\langle L\,\mathrm{e}^{JLt} f, \mathrm{e}^{JLt} g \rangle = \langle Lf, g \rangle.$$

For $f = v_k^{(j)}$, $g = v_n^{(m)}$, this yields

$$\langle Lv_k^{(j)}, v_n^{(m)} \rangle = \mathrm{e}^{(\lambda_j + \bar{\lambda}_m) t} \sum_{q=0}^{k} \sum_{q=0}^{n} \frac{t^{q+r}}{q!\,r!} \langle Lv_{k-q}^{(j)}, v_{n-r}^{(m)} \rangle \quad \text{for all} \quad t > 0. \tag{3.2}$$

From this equation, it is easy to use induction to show that $\langle Lf, g \rangle = 0$ for all $f, g \in B$. This implies (3.1).

Next, we observe that, clearly $Y \cap \ker_Z(L) = \{0\}$. Now the following lemma finishes the proof of Theorem 3.1.

Lemma 3.3. *Assume L is as above, and that $Y \subset$* $\mathrm{dom}_Z(L)$ *is a subspace of $Z = X + \mathrm{i}X$ satisfying* (3.1), *with $Y \cap \ker_Z(L) = \{0\}$. Then* $\dim Y \leqslant k$.

Proof. Because of our hypothesis on L, X admits an orthogonal decomposition $X = X_- \oplus X_0 \oplus X_+$ where $X_0 = \ker(L)$, $LX_- = X_-$ and $L(X_+ \cap \mathrm{dom}(L)) \subset X_+$ with

$$\langle Lu, u \rangle < 0 \quad \text{for non-zero} \quad u \in X_-,$$

$$\langle Lu, u \rangle > 0 \quad \text{for non-zero} \quad u \in X_+ \cap \mathrm{dom}(L).$$

We have $\dim X_- = k$. Assuming $N = \dim Y \geqslant k+1$, one can construct a non-zero $u \in Y$ which also lies in the orthogonal complement of X_-:

$$u = u_0 + iv_0 + u_+ + iv_+,$$

where $u_0, v_0 \in \boldsymbol{X}_0$ and $u_+, v_+ \in \boldsymbol{X}_+ \cap \operatorname{dom}(L)$. By (3.1) $\langle Lu, u \rangle = 0$ and therefore $u_+ = v_+ = 0$. Therefore, $u \in \ker_{\boldsymbol{Z}}(L) \cap \boldsymbol{Y} = \{0\}$, by hypothesis. So $u = 0$, a contradiction which proves the result. □

We now apply Theorem 3.1 to gKdV and gBBM to characterize the spectrum.

gKdV. We take $\boldsymbol{X} = L^2(\mathbb{R})$, $J = \partial_x$, and $L = -\partial_x^2 + c - f'(u_c)$. Since $f'(u_c(x)) \to 0$ as $|x| \to \infty$, it is well known that the essential spectrum of L is the interval $[c, \infty)$. Since $Lu_c' = 0$ and u_c' vanishes exactly once (u_c is radially decreasing), it follows from oscillation theory that $\lambda = 0$ is the *second* eigenvalue of L. Hence Theorem 3.1 applies with $k = 1$. We conclude that gKdV has at most one simple eigenvalue with $\operatorname{Re}\lambda > 0$. Note that if λ is an eigenvalue, then $\bar{\lambda}$ and $-\lambda$ are also. Thus any unstable eigenvalue λ must be real ($\lambda > 0$) and is paired with a negative real eigenvalue $-\lambda$.

We also observe that the essential spectrum of JL (spectrum with isolated eigenvalues of finite multiplicity removed) is the imaginary axis. This is easy to verify for the operator $\partial_x(-\partial_x^2 + c)$ by using the Fourier transform; since JL is a relatively compact perturbation, its essential spectrum is the same (Henry 1981).

gBBM. We may multiply equation (2.27) by $(I - \partial_x^2)^{-1}$, to put the eigenvalue problem in the form $JLu = \lambda u$ where $J = (I - \partial_x^2)^{-1}\partial_x$ and $L = -c\partial_x^2 + c - 1 - f'(u_c)$. The Hilbert space is $\boldsymbol{X} = L^2(\mathbb{R})$. It is clear that J is skew symmetric and L is self adjoint. Exactly as for gKdV we find that Theorem 3.1 applies with $k = 1$. The conclusions made above for gKdV all hold here as well. Taken together with the results in §2, we may summarize these results as follows.

Theorem 3.4. *For gKdV and gBBM, let J and L be as above. If $\mathrm{d}\mathcal{N}[u_c]/\mathrm{d}c < 0$, then the spectrum of JL consists of the imaginary axis together with two simple real eigenvalues $\lambda > 0$ and $-\lambda < 0$. If $\mathrm{d}\mathcal{N}[u_c]/\mathrm{d}c \geqslant 0$, then the spectrum of JL is the imaginary axis. For $\mathrm{d}\mathcal{N}[u_c]/\mathrm{d}c < 0$, the spectrum of JL is pictured in figure* 1.

Proof. The only assertion that remains to be verified is that for the case when $\mathrm{d}\mathcal{N}[u_c]/\mathrm{d}c \geqslant 0$: It suffices to show in this case that JL has no non-zero real eigenvalue. The proof is based on part of the stability argument in Weinstein (1985), Bona *et al.* (1987), Grillakis *et al.* (1987). Let w satisfy $LJw = 0$; we take $w = u_c$ for gKdV and $w = (I - \partial_x^2)\, u_c$ for gBBM. In both cases, we have $L\partial_c u_c = -w$ and

$$\langle w, \partial_c u_c \rangle = \mathrm{d}\mathcal{N}[u_c]/\mathrm{d}c = -\langle L\partial_c u_c, \partial_c u_c \rangle.$$

In this situation, Lemma E.1 of Weinstein (1985) (see also Maddocks 1985) yields the following.

Lemma 3.5. *If $\mathrm{d}\mathcal{N}[u_c]/\mathrm{d}c \geqslant 0$, then $\langle Lv, v \rangle \geqslant 0$ for all v such that $\langle v, w \rangle = 0$.*

Now suppose that $JLv = \lambda v$ for some $\lambda > 0$. Then $\langle Lv, v \rangle = 0$ since $\lambda\langle Lv, v \rangle = \langle Lv, JLv \rangle$. Also $\langle v, w \rangle = 0$ since $\lambda \langle v, w \rangle = \langle JLv, w \rangle = \langle v, LJw \rangle$. Now, from Lemma 3.5, we may deduce that $\langle Lv, y \rangle = 0$ for any $y \in \operatorname{dom}(L)$ with $\langle y, w \rangle = 0$, since $\langle L(v + ty), v + ty \rangle$ is minimized at $t = 0$. We may conclude that $Lv = \beta w$ for some constant β. But then

$$LJLv = \beta LJw = 0 = \lambda Lv.$$

Since $Lv \neq 0$ and $\lambda \neq 0$, this is a contradiction. This finishes the proof of the theorem.

Remark. As shown by Smereka (1992), the eigenvalue problem for gBou can formally be written in the form (0.8), $JLu = \lambda u$, where J is skew and L is self-adjoint. But in this case L has continuous spectrum which is unbounded both above and below, so the results of this subsection do not apply. It is exactly the same difficulty which has obstructed previous methods (Benjamin 1972; Bona 1975; Laedke & Spatschek 1984; Weinstein 1986*a*, *b*, 1987; Grillakis *et al.* 1987, 1990; Bona *et al.* 1987; Souganidis & Strauss 1990) from obtaining any result concerning stability or instability for solitary waves of gBou.

(*b*) *Detecting eigenvalues embedded in the essential spectrum*

In each of our applications, when λ lies on the imaginary axis, the constant coefficient system $y' = A^{\infty}(\lambda) y$ admits non-trivial globally bounded solutions, associated with a purely imaginary eigenvalue of $A^{\infty}(\lambda)$. (See the description of the set S_e in §2 parts (iii).) In this situation, consider the consequences of Proposition 1.6, the proof of Proposition 1.3, and the fact that $\operatorname{Re}\mu(\lambda) < 0$ and $\mu_*(\lambda) = 0$, cf. (2.5). We find that if $D(\lambda) = 0$, then the solution $\zeta^+(x, \lambda)$ of (1.1), which decays exponentially as $x \to +\infty$, may be merely bounded as $x \to -\infty$ and fail to decay. In principle, then, zeros of $D(\lambda)$ on the imaginary axis need not correspond to eigenvalues of the linearized evolution equations (2.13), (2.27) or (2.37), embedded in the essential spectrum (considered in the space $L^2(\mathbb{R})$, for example).

However, we find that for the gKdV, gBBM and gBou equations, such zeros do correspond to embedded eigenvalues with exponentially decaying eigenfunctions. This property is associated with the fact that the eigenvalue equations (2.13), (2.27) and (2.38) have the following symmetry: For $\lambda = i\beta$, $\beta \in \mathbb{R}$, if $Y(x)$ denotes a solution, then $\bar{Y}(-x)$ is a solution.

Theorem 3.6. *For the gKdV, gBBM and gBou eigenvalue equations* (2.13), (2.27), *and* (2.38) *suppose λ is purely imaginary. A non-trivial eigenfunction which decays exponentially as $|x| \to \infty$ exists if and only if $D(\lambda) = 0$.*

Proof. We give the proof for gKdV; the treatments of gBBM and gBou are similar. It is only necessary to prove the 'if' part, see Theorem 1.9 and §1*e*. Suppose $D(\lambda) = 0$ with $\lambda = i(c\tau + \tau^3)$ for some real τ, cf. §2*b*(iii). The eigenvalues of $A^{\infty}(\lambda)$ are μ, $i\tau$, and ν where $\operatorname{Re}\mu < 0 < \operatorname{Re}\nu$. As in §2*b*(v) consider $Y^+(x, \lambda) = \zeta_1^+(x, \lambda)$ and $Z^-(x, \lambda) = \eta_m^-(x, \lambda)$, which satisfy

$$\partial_x L_c Y^+ = \lambda Y^+, \quad L_c \partial_x Z^- = -\lambda Z^-, \tag{3.3}$$

$$\left.\begin{aligned} &Y^+(x, \lambda)\, e^{-\mu x} \to 1 && \text{as} \quad x \to +\infty, \\ &Z^-(x, \lambda)\, e^{\mu x} \to 1/(3\mu^2 - c) && \text{as} \quad x \to -\infty. \end{aligned}\right\} \tag{3.4}$$

Our goal is to show that Y^+ decays exponentially as $x \to -\infty$. According to the conventions in §1*e*, we have

$$\zeta^+ = (Y^+, Y^{+\prime}, Y^{+\prime\prime})^t, \quad \eta^- = (-L_c Z^-, -Z^{-\prime}, Z^-)$$

and by Proposition 1.2,

$$\zeta^+(x, \lambda)\, e^{-\mu x} \to (1, \mu, \mu^2) \quad \text{as} \quad x \to +\infty,$$
$$\eta^-(x, \lambda)\, e^{\mu x} \to (\mu^2 - c, \mu, 1)/(3\mu^2 - c) \quad \text{as} \quad x \to -\infty.$$

As a consequence of the hypothesis that $D(\lambda) = 0$, Proposition 1.6 and the fact we may take $r(\lambda) = 0$ in (1.12) of Proposition 1.3, we have that ζ^+ and η^- are bounded, on the whole line. Indeed by invoking results in ch. 4 of Coppel (1965), we find that there exists a bounded solution to $y' = A^\infty(\lambda)\, y$ such that

$$\zeta^+(x,\lambda) - y(x) \to 0 \quad \text{as} \quad x \to -\infty.$$

We may take $y(x) = a\,\mathrm{e}^{\mathrm{i}\tau x}(1, \mathrm{i}\tau, -\tau^2)^{\mathrm{t}}$, for some $a \in \mathbb{C}$, possibly zero. Similarly,

$$\eta^-(x,\lambda) - z(x) \to 0 \quad \text{as} \quad x \to +\infty,$$

where $z(x)$ is a bounded solution of $z' = -zA^\infty(\lambda)$; we may take

$$z(x) = b\,\mathrm{e}^{-\mathrm{i}\tau x}(-\tau^2 - c, \mathrm{i}\tau, 1)/(-3\tau^2 - c)$$

for some $b \in \mathbb{C}$.

Now observe that $Y_*(x) = \overline{Y^+(-x,\lambda)}$ is also a solution of the eigenvalue equation (2.13), since $-\bar{\lambda} = \lambda$ and u_c is an even function of x. Therefore, $\zeta_*(x)$, defined by

$$\zeta_* = (Y_*, Y_*', Y_*'')^{\mathrm{t}} = (\overline{Y^+(-x)}, -\overline{Y^{+\prime}(-x)}, \overline{Y^{+\prime\prime}(-x)})^{\mathrm{t}}$$

satisfies (1.1) with $\zeta_*(x) \to 0$ as $x \to -\infty$ and

$$\zeta_*(x) - y_* \to 0 \quad \text{as} \quad x \to +\infty, \quad \text{with} \quad y_*(x) = \bar{a}\,\mathrm{e}^{\mathrm{i}\tau x}(1, \mathrm{i}\tau, -\tau^2)^{\mathrm{t}}.$$

But now $\eta^-\zeta_* = 0$ (take the limit $x \to -\infty$), which implies $\bar{a}b = 0$. In fact a and b are proportional: It is easy to see that with $Z_*(x) = Z^-(-x)$, $\partial_x Z_*$ satisfies the eigenvalue equation (2.12), and

$$\partial_x Z_*(x)\,\mathrm{e}^{-\mu x} \to \mu/(3\mu^2 - c) \quad \text{as} \quad x \to +\infty.$$

It follows $\partial_x Z_* = \mu(3\mu^2 - c)^{-1}\, Y^+$, from which we infer that a and b are proportional. Hence $a = b = 0$. We conclude that ζ^+ and η^- decay to zero as $|x| \to +\infty$. By results in Coppel (1965) (ch. 4), they must do so at an exponential rate. □

Remarks. (i) As the proof shows, when $D(\lambda) = 0$ the transposed eigenvalue equation in (3.3) also has an exponentially decaying solution as $|x| \to \infty$.

(ii) For gBBM the proof is virtually identical. To implement the above proof for the case of gBou we observe that if $Z^-(x)$ satisfies

$$-L_c\,\partial_x^2 Z^-(x) = (\lambda^2 + 2c\lambda\,\partial_x)(I - \partial_x^2)\,Z^-(x),$$

then, with the notation $Z_*(x) = Z^-(-x)$, $\partial_x^2 Z_*(x)$ satisfies (2.38). Thus, $\partial_x^2 Z_*(x)$ plays the role that $\partial_x Z_*(x)$ played in the cases of gKdV and gBBM. Another point of difference for gBou is that for purely imaginary λ, $A^\infty(\lambda)$ now has two purely imaginary eigenvalues instead of one. This causes no difficulty, however, since by §2*d*(iii) these eigenvalues are always distinct except when $\lambda = 0$.

4. Transitions to instability

In this section we study how instabilities arise as parameters vary in gKdV, gBBM, and gBou. For definiteness we consider the special one parameter family of nonlinearities $f(u) = u^{p+1}/(p+1)$, $p > 0$, and study the stability of solitary waves as a function of the pair $K = (p, c)$, where c is the wave speed.

The results of §2 and in Bona *et al.* (1987), Souganidis & Strauss (1990), Weinstein (1986*a*, *b*, 1987) imply for gKdV and gBBM the following.

gKdV. For all $c > 0$, $u_c(x)$ is linearly exponentially unstable (i.e. (2.1) holds) if and only if $p > p_0 = 4$.

gBBM. (*a*) If $p \leqslant p_0 = 4$, $u_c(x)$ is orbitally stable for all $c > 1$. (*b*) For $p > 4$, there is a constant $c_0(p) > 1$ such that for $c > c_0(p)$, $u_c(x)$ is orbitally stable while for $1 < c < c_0(p)$, $u_c(x)$ is linearly exponentially unstable. The value of $c_0(p)$ is

$$c_0(p) = \frac{p}{4+2p}[1+\surd(2+\tfrac{1}{2}p)] = \frac{(2(4+p))^{\frac{1}{2}}}{p+(2(4+p))^{\frac{1}{2}}}\frac{p-4}{4}+1. \qquad (4.1\,a)$$

We also conclude from §2 that for gBou we have:
gBou. For $p > 4$, $u_c(x)$ is linearly exponentially unstable if $1 < c^2 < c_0^2(p)$ where

$$c_0^2(p) = 3p/(4+2p). \qquad (4.1\,b)$$

In these examples, we see that a transition from stability to instability occurs as $K = (p, c)$ crosses a curve in the plane. Let us study Evans's function $D(\lambda, K)$ as a function of λ and the parameters $K = (p, c)$ near a point $(\lambda, K) = (0, K_0)$, where K_0 lies on the transition curve. From the explicit formulae in §2, and Remark 1.12, it is clear that D is analytic in some neighbourhood of $(0, K_0)$ in $\mathbb{C}^3$. For any $p > 0$, $c > 1$ ($c > 0$ for gKdV) we have computed

$$D(0,K) = 0, \quad D_\lambda(0,K) = 0. \qquad (4.2)$$

The *transition curve* is the curve in the (p, c)-plane on which:

$$D_{\lambda\lambda}(0,K_0) = \frac{\mathrm{d}}{\mathrm{d}c}\mathcal{N}[u_c] = 0.$$

Examining the Taylor expansion of D at $(0, K_0)$, therefore, we find that for (λ, K) close to $(0, K_0)$ we have

$$D(\lambda,K) = \lambda^2(\tfrac{1}{6}\partial_\lambda^3 D(0,K_0)\,\lambda+\tfrac{1}{2}\nabla_K\,\partial_\lambda^2 D(0,K_0)\cdot(K-K_0))\cdot(1+0(|\lambda|+\|K-K_0\|)). \quad (4.3)$$

Below we shall prove the following for the above examples.

Proposition 4.1. $\partial_\lambda^3 D(0,K_0) \neq 0$ *and* $\nabla_K\,\partial_\lambda^2 D(0,K_0) \neq 0$.

With this result, the mechanism for transition from stability to instability may be described as follows: As $K = (p, c)$ crosses the transition curve at K_0, a real root $\lambda_0(K)$ of $D(\lambda, K) = 0$ crosses from the negative real axis $\lambda_0 < 0$ to the positive axis $\lambda_0 > 0$, with $\lambda_0(K_0) = 0$. λ_0 is locally an analytic function of K, and $\nabla\lambda_0(K_0) \neq 0$.

Discussion. Let us discuss this mechanism in some more detail. The mechanism outlined here is very different from a typical transition to instability in finite-dimensional hamiltonian systems. In finite dimensions, unstable eigenvalues typically arise as purely imaginary eigenvalues collide and branch in pairs away from the imaginary axis (see, for example, Arnold & Avez 1968; Arnold 1978; MacKay 1987).

By contrast, in our examples, the transition to instability does not involve any purely imaginary eigenvalues. We have seen in §2 that when $\lambda_0(K) > 0$, an exponentially decaying eigenfunction exists; the solitary wave is unstable. When $\lambda_0(K) < 0$, however, it is not an eigenvalue in the usual sense (at least for gKdV and gBBM, where we know the solitary wave is stable in this case). What happens when $\lambda = \lambda_0(K) < 0$ is that because $D(\lambda_0(K), K) = 0$, the solution $Y^+(x, \lambda)$ of the eigenvalue equations (2.13), (2.27) or (2.37) exhibits maximal decay rate $Y^+ = O(\mathrm{e}^{\mu x})$ as $x \to \infty$, and submaximal growth $Y^+ = O(e^{\mu_* x})$ as $x \to -\infty$. But since $\lambda = \lambda_0(K) < 0$, in each example we have $\mu_*(\lambda) < 0$ (see §2 parts (iii) and (iv)); and compare (2.4), (2.5)). Hence Y^+ need not decay as $x \to -\infty$. Indeed it cannot, for gKdV and gBBM at least, since no unstable eigenvalue exists in these examples when $\lambda_0(K) < 0$.

We remark at this point that small values of $\lambda_0(K) < 0$ can be regarded as true eigenvalues on a *weighted* L^2 space consisting of measurable functions u such that $e^{ax}u$ is square-integrable; provided $\mu < -a < \mu_*$, the function $e^{ax}Y^+$ will decay exponentially as $|x| \to \infty$. Introducing the weight can also serve to shift the essential spectrum from the imaginary axis strictly into the left-plane. The technique of analytic dilation serves the same purpose in quantum scattering theory (Reed & Simon 1978). Weights are used to shift essential spectrum in the study of travelling waves of some parabolic systems (see Sattinger 1976). In work in progress, nonlinear stability with an exponential decay rate is established for gKdV solitary waves in such weighted spaces (Pego & Weinstein 1992).

Another interpretation of $\lambda_0(K)$ when negative is that it corresponds to what is known in quantum scattering theory as a resonance pole (Reed & Simon 1978). The same phenomenon is associated with what is called Landau damping in the Vlasov–Poisson system (Crawford & Hislop 1989*a*, *b*). To fix ideas, consider the resolvent equation for gKdV in $L^2(\mathbb{R})$, written as

$$(JL-\lambda)\,u = g,$$

where $J = \partial_x$ and $L = -\partial_x^2 + c - f'(u_c)$ as in §3*a*. Suppose we are near the transition with $\lambda_0(K) < 0$, so JL has no eigenvalues off the imaginary axis by Theorem 3.4. We denote the resolvent of JL by $\mathscr{R}_1(\lambda) = (JL-\lambda)^{-1}$. For $\operatorname{Re}\lambda \neq 0$, $\mathscr{R}_1(\lambda)$ is a bounded operator on $L^2(\mathbb{R})$. For $\operatorname{Re}\lambda > 0$, it is given in terms of the first component in formula (1.39) by

$$u = \mathscr{R}_1(\lambda)\,g = e_1^{\mathrm{t}}\,\mathscr{R}(\lambda)\,(0, \ldots, g)^{\mathrm{t}}, \qquad (*)$$

where $e_1^{\mathrm{t}} = (1, 0, \ldots, 0)$. (For $\operatorname{Re}\lambda < 0$, an analogous representation can be obtained.) In operator norm, the resolvent $\mathscr{R}_1(\lambda)$ becomes singular as $\operatorname{Re}\lambda \to 0$; the imaginary axis is essential spectrum. The formula above using (1.39) is not correct when $\operatorname{Re}\lambda < 0$; in this case $\mu^* < 0$ and the right-hand side of (*) does not define an operator on L^2. But for a dense set of right-hand sides g in L^2, namely those which are continuous with compact support, the formula (1.39) yields an analytic continuation of $\mathscr{R}_1(\lambda)\,g(x)$ (for fixed x) from the region $\operatorname{Re}\lambda > 0$, across the essential spectrum (avoiding a double pole at $\lambda = 0$), into the region $-\epsilon < \operatorname{Re}\lambda < 0$ for some $\epsilon > 0$. This analytic continuation exhibits a pole (called a *resonance pole*) at $\lambda = \lambda_0(K)$, where $D(\lambda) = 0$. During the transition from stability to instability, this resonance pole moves across the imaginary axis, and emerges as an eigenvalue in the right half-plane.

This description accounts for the emergence of the unstable eigenvalue λ during the transition in terms of $\lambda_0(K)$, but something should be said about the emergence of the symmetry-related eigenvalue $-\lambda$. Recall that for JL, $-\lambda$ is an eigenvalue if λ is. When $\lambda_0(K) > 0$, the eigenvalue $-\lambda = -\lambda_0(K)$, while not a zero of $D(\lambda)$, is a pole of the resolvent $\mathscr{R}_1(\lambda)$. We claim that, as $\lambda_0(K)$ becomes negative, the pole at $-\lambda_0(K)$ becomes a resonance pole, in the *right* half-plane.

Formula (1.39) does not apply directly to this situation, but can be used with the odd symmetry of JL in the following way. Let T denote the reflection operator $Tu(x) = u(-x)$. Then $TJL = -JLT$, from which one can easily verify that

$$\mathscr{R}_1(\lambda) = -T\mathscr{R}_1(-\lambda)\,Tg = -Te_1^{\mathrm{t}}\,\mathscr{R}(-\lambda)\,(0, \ldots, 0, Tg)^{\mathrm{t}}.$$

Using this relation for $\mathrm{Re}\,\lambda < 0$ we see clearly that $\lambda = -\lambda_0(K)$ is a pole or a resonance pole of the resolvent, respectively, when $\lambda_0(K) > 0$ or $\lambda_0(K) < 0$.

For g continuous and of compact support, it is appropriate to view the function $\lambda \mapsto \mathscr{R}_1(\lambda)\,g(x)$, for fixed x, as extending to a Riemann surface, cut by the imaginary axis but continuing across it from both sides. See figure 2. Crawford & Hislop (1989*a*, *b*) have a somewhat different visualization.) In the region $-\epsilon < \mathrm{Re}\,\lambda < 0$, let $\mathscr{R}_2(\lambda)\,g(x)$ denote the analytic continuation of $\mathscr{R}_1(\lambda)\,g(x)$ from the region $\mathrm{Re}\,\lambda > 0$ to a second sheet of the Riemann surface. Then the analytic continuation of $\mathscr{R}_1(\lambda)\,g(x)$ from the region $\mathrm{Re}\,\lambda < 0$ to a second sheet over the strip $0 < \mathrm{Re}\,\lambda < \epsilon$, is given by $\mathscr{R}_2(\lambda)\,g(x) = -T\mathscr{R}_2(-\lambda)\,Tg(x)$. We remark that from (1.39) one can see that $\mathscr{R}_2(\lambda)\,g(x)$ is typically not a bounded function of x; for $\mathrm{Re}\,\lambda < 0$, $\mu_*(\lambda) < 0$ and (1.1) has solutions of the form $\zeta^-(x,\lambda)\,c$ which grow as $x \to \infty$. The global structure of this Riemann surface is unknown.

To prove Proposition 4.1 for each example, it is useful to show that certain simplifications occur when computing higher derivatives of $D(\lambda)$ at a high-order zero. For our purposes it suffices to prove the following formulae, which are quite generally valid under the assumptions of Theorem 1.11 and Proposition 1.21. Below, the subscript λ denotes differentiation with respect to λ.

Proposition 4.2. *Make the assumptions of Theorem* 1.11 *and Proposition* 1.21. *Assume that* $0 = D(0) = D_\lambda(0) = D_{\lambda\lambda}(0)$. *Then*

$$-D_{\lambda\lambda\lambda}(0) = \int_{-\infty}^{\infty} 6\eta_\lambda^- A_\lambda \zeta_\lambda^+ + 3(\eta_\lambda^- A_{\lambda\lambda}\zeta^+ + \eta^- A_{\lambda\lambda}\zeta_\lambda^+) + \eta^- A_{\lambda\lambda\lambda}\zeta^+. \tag{4.4}$$

Moreover, for higher-order scalar equations as discussed in §1*e*, *we have*

$$D_{\lambda\lambda\lambda}(0) = \int_{-\infty}^{\infty} 6Z_\lambda^- \mathscr{A}_\lambda Y_\lambda^+ + 3(Z_\lambda^- \mathscr{A}_{\lambda\lambda} Y^+ + Z^- \mathscr{A}_{\lambda\lambda} Y_\lambda^+) + Z^- \mathscr{A}_{\lambda\lambda\lambda} Y^+. \tag{4.5}$$

Proof. Start with formula (1.21), differentiate twice and evaluate at $\lambda = 0$. We obtain

$$-D_{\lambda\lambda\lambda}(0) = \int_{-\infty}^{\infty} (\eta^- A_\lambda \zeta^+)_{\lambda\lambda}. \tag{4.6}$$

The convergence of this improper integral is guaranteed by Theorem 1.11, but more is true. Using Proposition 1.21 and (1.8), (1.9) we find that if $\epsilon > 0$ is sufficiently small,

$$\zeta^+, \zeta_\lambda^+, \zeta_{\lambda\lambda}^+ = \begin{cases} O(\mathrm{e}^{\mu x}\,\mathrm{e}^{-\epsilon|x|}) & \text{as} \quad x \to -\infty, \\ O(x^2\,\mathrm{e}^{\mu x}) & \text{as} \quad x \to +\infty, \end{cases} \tag{4.7}$$

$$\eta^-, \eta_\lambda^-, \eta_{\lambda\lambda}^- = \begin{cases} O(x^2\,\mathrm{e}^{-\mu x}) & \text{as} \quad x \to -\infty, \\ O(\mathrm{e}^{-\mu x}\,\mathrm{e}^{-\epsilon|x|}) & \text{as} \quad x \to +\infty. \end{cases} \tag{4.8}$$

Therefore, when the differentiation in (4.6) is carried out, each term is $O(x^2\,\mathrm{e}^{-\epsilon|x|})$, so decays exponentially. Using the equations

$$\frac{\mathrm{d}}{\mathrm{d}x}\zeta^+ = A\zeta^+, \quad \frac{\mathrm{d}}{\mathrm{d}x}\zeta_\lambda^+ = A\zeta_\lambda^+ + A\zeta_\lambda^+, \quad \frac{\mathrm{d}}{\mathrm{d}x}\zeta_{\lambda\lambda}^+ = A\zeta_{\lambda\lambda}^+ + 2A_\lambda\zeta_\lambda^+ + A\zeta_{\lambda\lambda}^+,$$

$$-\frac{\mathrm{d}}{\mathrm{d}x}\eta^- = \eta^- A, \quad -\frac{\mathrm{d}}{\mathrm{d}x}\eta_\lambda^- = \eta_\lambda^- A + \eta^- A_\lambda, \quad -\frac{\mathrm{d}}{\mathrm{d}x}\eta_{\lambda\lambda}^- = \eta_{\lambda\lambda}^- A + 2\eta_\lambda^- A_\lambda + \eta^- A_{\lambda\lambda},$$

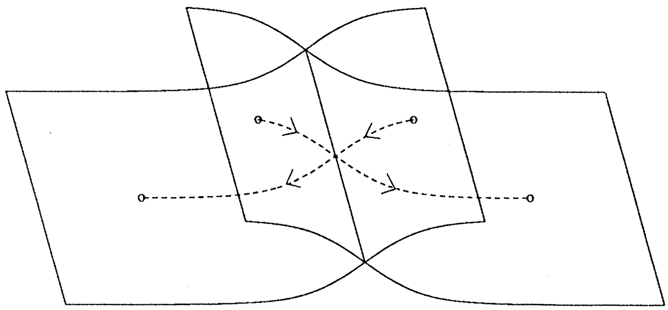

Figure 2. Riemann surface for resolvent. During transition to instability, resonance poles on upper sheet move across the imaginary axis, and become eigenvalues on lower sheet.

along with (4.7), (4.8) we may write, for $\lambda = 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} \eta_{\lambda\lambda}^{-} A_{\lambda} \zeta^{+} &= \int_{-\infty}^{\infty} \eta_{\lambda\lambda}^{-}\left(\frac{\mathrm{d}}{\mathrm{d}x}\zeta_{\lambda}^{+} - A\zeta_{\lambda}^{+}\right) \\ &= \lim_{R\to\infty} (\eta_{\lambda\lambda}^{-}\zeta_{\lambda}^{+})_{-R}^{R} + \int_{-R}^{R}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\eta_{\lambda\lambda}^{-} - \eta_{\lambda\lambda}^{-}A\right)\zeta_{\lambda}^{+} \\ &= \int_{-\infty}^{\infty} 2\eta_{\lambda}^{-} A_{\lambda}\zeta_{\lambda}^{+} + \eta^{-}A_{\lambda\lambda}\zeta_{\lambda}^{+}\end{aligned}$$

and similarly
$$\int_{-\infty}^{\infty} \eta^{-} A_{\lambda}\zeta_{\lambda\lambda}^{+} = \int_{-\infty}^{\infty} 2\eta_{\lambda}^{-} A_{\lambda}\zeta_{\lambda}^{+} + \eta_{\lambda}^{-} A_{\lambda\lambda}\zeta^{+}.$$

Using these equations in (4.6), the result (4.4) follows easily. Equation (4.5) is an immediate corollary. □

Let us prove Proposition 4.1 for each example:

gKdV. The transition curve is given by $K_0 = (4, c)$ for $c > 0$. We have $\mathscr{A}_{\lambda} =$ identity; using (4.5) and (2.21) we compute that

$$\tfrac{1}{6}D_{\lambda\lambda\lambda}(0, K_0) = \int_{-\infty}^{\infty} Z_{\lambda}^{-} Y_{\lambda}^{+} = -\frac{1}{2\mu c\beta^2}\int_{-\infty}^{\infty}\left(\partial_c u_c \int_{-\infty}^{x} \partial_c u_c\right) = -\frac{1}{4\mu c\beta^2}\left[\int_{-\infty}^{\infty} \partial_c u_c\right]^2$$

(all other terms vanish since $\int \partial_c u_c u_c = 0$ for $p = 4$). Since

$$\int_{-\infty}^{\infty} u_c = \frac{\alpha}{\gamma}\int_{-\infty}^{\infty} \operatorname{sech}^{2/p} y \,\mathrm{d}y,$$

where α and γ are given in §2*b* (vii), we find that

$$\partial_c \int_{-\infty}^{\infty} u_c = \text{const.} \frac{\mathrm{d}}{\mathrm{d}c} c^{-\frac{1}{4}} \neq 0.$$

Hence $D_{\lambda\lambda\lambda}(0, K_0) \neq 0$, as claimed.

To compute $\partial_p \partial_{\lambda}^2 D(0, K_0)$ we use the result of §2*b* (vi), that $-\mu c\beta^2 \partial_{\lambda}^2 D(0, K_0) = \mathrm{d}\mathscr{N}[u_c]/\mathrm{d}c$ and the fact that $\partial_{\lambda}^2 D(0, K_0) = 0$. Hence

$$-\mu c\beta^2 \partial_p \partial_{\lambda}^2 D(0, K_0) = \partial_p \partial_c \mathscr{N}[u_c] = \frac{\partial}{\partial p}\left[\left(\frac{2}{pc} - \frac{1}{2c}\right)\mathscr{N}[u_c]\right]\bigg|_{(p,\,c)=(4,\,c)} = -\frac{1}{8c}\mathscr{N}[u_c] \neq 0$$

using the calculation of §2*b* (vii). This proves Proposition 4.1 for gKdV.

gBBM. From §2c(vii) we may compute that the transition curve is $K_0 = (p, c_0(p))$, where $c_0(p)$ is given by (4.1a). We have $c\mathscr{A}_\lambda = (I - \partial_x^2)$. Using (4.5) and (2.32) we compute that

$$\tfrac{1}{6}cD_{\lambda\lambda\lambda}(0,K_0) = \int_{-\infty}^{\infty} Z_\lambda^-(I-\partial_x^2)\,Y_\lambda^+ = -\frac{1}{2\mu(c-1)\,\beta^2}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x}\partial_c u_c\right)(I-\partial_x^2)\,\partial_c u_c$$

$$= -\frac{1}{4\mu(c-1)\,\beta^2}\left[\int_{-\infty}^{\infty}\partial_c u_c\right]^2.$$

From §2c(vii) we compute that $\int_{-\infty}^{\infty} u_c = \alpha/\gamma$ times a constant independent of c, so that

$$\left(\int_{-\infty}^{\infty} u_c\right)^{-1}\partial_c\int_{-\infty}^{\infty} u_c = \frac{\partial}{\partial c}\ln\int u_c = \left(\frac{1}{p}-\frac{1}{2}\right)\frac{1}{c-1}+\frac{1}{2c} = \frac{1}{2c(c-1)}\left(\frac{2c}{p}-1\right) < 0$$

for $c > 1$ and (p, c) on the transition curve. Hence $D_{\lambda\lambda\lambda}(0,K_0) \neq 0$ as claimed.

We compute $\partial_c\partial_\lambda^2 D(0,K_0)$ by using the results of $2c$(vi) and (vii), finding that

$$-\mu c(c-1)\,\beta^2\,\partial_c\partial_\lambda^2 D(0,K_0) = \frac{\mathrm{d}^2}{\mathrm{d}c^2}\mathscr{N}[u_c]$$

and (on the transition curve)

$$\frac{p(c-1)\,c^2\gamma}{\alpha^2 I(4/p)}\frac{\mathrm{d}^2}{\mathrm{d}c^2}\mathscr{N}[u_c] = 2(1+k)\,c-2k > 0.$$

Hence $\partial_c\partial_\lambda^2 D(0,K_0) \neq 0$, establishing Proposition 4.1 for gBBM.

gBou. From §2d(vii) we find that the transition curve is $K_0 = (p, c_0(p))$, where $c_0(p)$ is given by (4.1b). We have $-c^2\mathscr{A}_\lambda = 2(\lambda - c\partial_x)(I-\partial_x^2)$, so using (4.5) and (2.43) we compute that

$$-\tfrac{1}{6}c^2 D_{\lambda\lambda\lambda}(0,K_0) = -2c\int_{-\infty}^{\infty} Z_\lambda^-\,\partial_x(I-\partial_x^2)\,Y_\lambda^+ + \int_{-\infty}^{\infty} Z_\lambda^-(I-\partial_x^2)\,Y^+ + Z^-(I-\partial_x^2)\,Y_\lambda^+.$$

After some integrations by parts,

$$-\tfrac{1}{6}2c^2\beta^2\mu^3 D_{\lambda\lambda\lambda}(0,K_0) = 2c\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x}\partial_c u_c\right)(I-\partial_x^2)\,\partial_c u_c$$

$$+\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x}\partial_c u_c\right)(I-\partial_x^2)\,u_c + \int_{-\infty}^{\infty}\left(\int_{-\infty}^{x} u_c\right)(I-\partial_x^2)\,\partial_c u_c$$

$$= c\left(\int_{-\infty}^{\infty}\partial_c u_c\right)^2 + \int_{-\infty}^{\infty}\partial_c u_c\int_{-\infty}^{\infty} u_c.$$

As previously, we have $\int_{-\infty}^{\infty} u_c = (\alpha/\gamma)\,I(2/p)$, so

$$\left(\int_{-\infty}^{\infty} u_c\right)^{-1}\partial_c\int_{-\infty}^{\infty} u_c = \frac{\mathrm{d}}{\mathrm{d}c}(\ln\alpha - \ln\gamma) = \frac{1}{c(c^2-1)}\left(\frac{2c^2}{p}-1\right) = \frac{\frac{1}{2}-c^2}{c(c^2-1)}$$

on the transition curve, so

$$-\tfrac{1}{6}2c^2\beta^2\mu^3 D_{\lambda\lambda\lambda}(0,K_0) = \int_{-\infty}^{\infty} \partial_c u_c \int_{-\infty}^{\infty} u_c\left(\frac{-\frac{1}{2}}{c^2-1}\right) \neq 0,$$

hence $D_{\lambda\lambda\lambda}(0,K_0) \neq 0$ as claimed.

From the results of §2*d*(vi) and (vii) we have that $-\mu^3\beta^2c^2\,\partial_c\partial_\lambda^2 D(0,K_0) = \mathrm{d}^2\mathcal{N}[u_c]/\mathrm{d}c^2$ and, on the transition curve

$$\frac{c^2(c^2-1)\,\gamma}{\alpha I(4/p)}\frac{\mathrm{d}^2}{\mathrm{d}c^2}\mathcal{N}[u_c] = 2c\left(2c^2\left(\frac{1+k}{k}\right)-3\right) = 6c \neq 0.$$

Hence $\partial_c\partial_\lambda^2 D(0,K_0) \neq 0$, proving Proposition 4.1 for gBou.

5. Remarks on the generalized KdV–Burgers equation

(*a*) *Construction of travelling waves*

Consider the generalized KdV–Burgers equation:

$$\partial_t u + \partial_x f(u) + \partial_x^3 u = \alpha\,\partial_x^2 u. \tag{5.1}$$

Here $\alpha > 0$ is a parameter, and in addition to the previous assumptions on f, we assume f is strictly convex. The properties of the travelling waves which we study are summarized as follows.

Theorem 5.1. *Given any $c > 0$, $\alpha > 0$, there is a travelling wave solution $u(x,t) = u_c(x-ct)$ to (5.1), unique up to translation, which satisfies the* ODE

$$-\partial_x^2 u_c + \alpha\partial_x u_c + cu_c - f(u_c) = 0, \quad x \in \mathbb{R} \tag{5.2}$$

and has the limiting behaviour

$$u_c \to \begin{cases} 0 & \text{as} \quad x \to +\infty, \\ u_L & \text{as} \quad x \to -\infty, \end{cases}$$

where u_L is the unique positive solution of $f(u_L) = cu_L$. The wave profile u_c satisfies

$$(u_c(x), \partial_x u_c(x))\,\mathrm{e}^{-\mu x} \to \beta(1,\mu) \quad \text{as} \quad x \to +\infty \tag{5.3}$$

for some $\beta > 0$ where $\mu = \tfrac{1}{2}(\alpha - \surd(\alpha^2+4c)) < 0$. For $\alpha \geqslant 2\surd(f'(u_L)-c)$, u_c is monotone decreasing, while for $\alpha < 2\surd(f'(u_L)-c)$, $u_c(x)-u_L$ decays to zero exponentially in an oscillatory fashion as $x \to -\infty$.

Proof. The proof is little different from that given for $f(u) = \tfrac{1}{2}u^2$ by Bona & Schonbek (1985). Consider the phase plane for (5.2): Let $u = u_c$, $v = \partial_x u_c$ and consider the system

$$u' = v, \quad v' = cu - f(u) + \alpha v. \tag{5.4}$$

Our hypotheses on f guarantee that for $u \geqslant 0$, the only critical points of (5.4) are at $(0,0)$ and $(u_L, 0)$. By convexity of f, $0 < c < f'(u_L)$, so the point $(0,0)$ is always a saddle while the point $(0, u_L)$ is an unstable spiral point or node, depending on whether $\tfrac{1}{4}\alpha^2 < f'(u_L) - c$ or not.

Put

$$E(u,v) = \tfrac{1}{2}v^2 - \tfrac{1}{2}cu^2 + \int_0^u f(s)\,\mathrm{d}s,$$

then $E(u,v)' = \alpha v^2 \geqslant 0$. The level curve $E(u,v) = 0$ contains $(0,0)$ and encloses $(u_L, 0)$ in a single loop in the half-plane $u \geqslant 0$. When $\alpha = 0$ this loop corresponds to a homoclinic orbit for (5.4) that yields the solitary wave for gKdV. For $\alpha > 0$, the part of the stable manifold of $(0,0)$ in the half-plane $u \geqslant 0$ is a trajectory lying in the bounded region where $E(u,v) < 0$. As we decrease x, this trajectory is trapped in this region and E is non-increasing. It is easy to show that the trajectory must approach the critical point $(u_L, 0)$ as $x \to -\infty$. This is the unique trajectory connecting $(u_L, 0)$ to $(0,0)$: the only other trajectory approaching $(0,0)$ as $x \to +\infty$ lies entirely in the half-plane $u < 0$, since $E(0,v) > 0$ for $v \neq 0$.

It remains to show that when $\alpha \geqslant 2\sqrt{(f'(u_L)-c)}$, the connecting trajectory lies in the quarter plane $u > 0$, $v < 0$, so that u_c is monotone decreasing. Assume $\frac{1}{4}\alpha^2 > f'(u_L)-c$, the case of equality follows by continuity. Consider a line segment with slope $m > 0$ passing through $(u_L, 0)$, parametrized by (u_L+s, ms) for $-u_L < s < 0$. At any point of this line segment, (5.4), the equation $f(u_L) = cu_L$, and Taylor's theorem imply that for some $\xi \in (u_L+s, u_L)$,

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \alpha + \frac{cs + f(u_L) - f(u_L+s)}{ms} = \alpha + \frac{c - f'(\xi)}{m} > \alpha + \frac{c - f'(u_L)}{m}.$$

The inequality follows since f is strictly convex. Because $\frac{1}{4}\alpha^2 > f'(u_L)-c$, we may choose m so that $m^2 - \alpha m - (c - f'(u_L)) < 0$. Hence $\mathrm{d}v/\mathrm{d}u > m$ on the line segment above, which implies that no trajectory above it can cross this line segment as x decreases in (5.4). The connecting trajectory in particular is trapped between this line segment and the axis $v = 0$. Hence $v < 0$ on this trajectory, as claimed. □

(b) *The spectrum, and constraints on transitions to instability*

Let us now consider the linear stability of these waves. In a coordinate frame moving with the travelling wave, the eigenvalue equation for small perturbations of u_c is

$$\lambda Y = \partial_x(-\partial_x^2 + \alpha\partial_x + c - f'(u_c))\, Y = JLY, \tag{5.5}$$

where $J = \partial_x$, and $L = -\partial_x^2 + \alpha\partial_x + c - f'(u_c)$ is not self-adjoint. We may consider the spectrum of JL in any of the spaces $X = L^p(\mathbb{R})$, $1 \leqslant p < \infty$ or $C_u(\mathbb{R})$ (the space of bounded uniformly continuous functions); the results are the same. For simplicity, take $X = L^2(\mathbb{R})$. Applying the theory laid out by Henry (1985), the essential spectrum of JL may be determined as follows.

Reduce the equation $(-JL+\lambda)\, Y = 0$ to a first-order system in the manner of §1*e*. Let

$$S_e^{\pm} = \{\lambda \mid A^{\pm\infty}(\lambda) \quad \text{has an imaginary eigenvalue}\}.$$

We have

$$S_e^{+} = \{\lambda \mid \lambda = -\alpha\tau^2 + \mathrm{i}\tau(\tau^2 + c) \quad \text{for some real } \tau\},$$

$$S_e^{-} = \{\lambda \mid \lambda = -\alpha\tau^2 + \mathrm{i}\tau(\tau^2 + c - f'(u_L)) \quad \text{for some real } \tau\}.$$

These sets are curves in the left half-plane $\operatorname{Re}\lambda \leqslant 0$, which pass through $\lambda = 0$.

Proposition 5.2. *The essential spectrum of JL contains $S_e^+ \cup S_e^-$, but contains no point of the component Ω_+ of $\mathbb{C}\backslash(S_e^+ \cup S_e^-)$ that includes the right half-plane.*

Proof. It is straightforward to show that $S_e^+ \cup S_e^-$ lies in the approximate point spectrum of JL. As in the proof of Theorem A.2 of ch. 5 in Henry (1981) one may show that either Ω_+ consists entirely of eigenvalues, or it does not intersect the essential spectrum. But for $\lambda > 0$ and large, if Y satisfies (5.5), we may assume Y is real and compute

$$\begin{aligned}\lambda\int_{-\infty}^{\infty} Y^2 &= \int_{-\infty}^{\infty} Y(-Y''' + \alpha Y'' + cY' - (f'(u_c)\,Y)') \\ &= -\alpha\lambda\int_{-\infty}^{\infty}(Y')^2 - \frac{1}{2}\int_{-\infty}^{\infty} Y^2 f''(u_c)\,u_c' \\ &\leqslant C\int_{-\infty}^{\infty} Y^2 < \lambda\int_{-\infty}^{\infty} Y^2,\end{aligned}$$

which implies $Y = 0$. So Ω_+ does not consist entirely of eigenvalues, and the result follows.

Lemma 5.3. *For λ in Ω_+, $A^{\pm\infty}(\lambda)$ has exactly one eigenvalue $\mu = \mu^{\pm}(\lambda)$ with negative real part. We have*

$$\mu^+(0) = \tfrac{1}{2}(\alpha - \surd(\alpha^2 + 4c)), \mu^-(0) = 0. \tag{5.6}$$

Proof. It is clear that as λ varies in Ω_+, the number of eigenvalues of $A^{\pm\infty}(\lambda)$ with negative real part is constant. Any eigenvalue ν of $A^{+\infty}(\lambda)$ must satisfy

$$\mathscr{P}_+(\nu) = \nu^3 - \alpha\nu^2 - c\nu + \lambda = 0.$$

At $\lambda = 0$ the roots of $\mathscr{P}_+$ are $\nu = 0, \tfrac{1}{2}(\alpha \pm \surd(\alpha^2 + 4c))$. These are simple, so are locally analytic functions of λ. Considering the root $\nu = 0$ and differentiating, we find $c\,\mathrm{d}\nu/\mathrm{d}\lambda = 1$ at $\lambda = 0$, so for $\lambda > 0$ small we have $\nu > 0$, hence the Lemma holds for $A^{+\infty}$. Similarly, any eigenvalue of $A^{-\infty}(\lambda)$ must satisfy

$$\mathscr{P}_-(\nu) = \nu^3 - \alpha\nu^2 + (-c + f'(u_L))\,\nu + \lambda = 0.$$

But now, $f'(u_L) > c$, so at $\lambda = 0$ the only root with $\mathrm{Re}\,\nu \leqslant 0$ is at $\nu = 0$. This root satisfies $(f'(u_L) - c)\,\mathrm{d}\nu/\mathrm{d}\lambda = -1$ at $\lambda = 0$, so $\nu < 0$ for $\lambda > 0$. This proves the Lemma. □

It is clear from the Lemma that the theory of §1 may be applied to define $D(\lambda)$ on a domain Ω that contains Ω_+ and some neighbourhood of $\lambda = 0$, and that for $\lambda \in \Omega_+$, λ is an eigenvalue of JL if and only if $D(\lambda) = 0$. Our main result in this section is the following:

Theorem 5.4. 1. *If u_c is monotone decreasing, then $D(\lambda) \neq 0$ for* $\mathrm{Re}\,\lambda > 0$.
2. *In all cases, for all $\alpha > 0$, we have $D(0) = 0$, but $D'(0) \neq 0$.*

We remark that when u_c is monotone decreasing, it may be shown as in Pego (1985) that the travelling wave is nonlinearly orbitally stable. The theorem above creates the following dilemma. Suppose that for $\alpha = 0$, $c > 0$ fixed, the solitary wave of gKdV is exponentially unstable. For $\alpha > 0$ large, the corresponding travelling wave of (5.1) is monotone and stable. As α decreases to zero, the travelling wave profiles develops an oscillatory 'tail'; the right-most hump approaches the solitary wave in form. We conjecture that there should be a transition to instability as $\alpha \to 0$. But the mechanism is mysterious: Theorem 5.4 seems to forbid the emergence of a positive eigenvalue out of $\lambda = 0$ as α decreases.

Proof. To prove part 1, suppose $\mathrm{Re}\,\lambda > 0$ and $D(\lambda) = 0$. Then (5.5) has as a solution Y which decays to zero as $|x| \to \infty$ together with its derivatives at an exponential rate. Integrating (5.5) yields

$$\lambda\int_{-\infty}^{\infty} Y = 0,$$

so, if we define

$$W(x) = \int_{-\infty}^{x} Y(s)\,\mathrm{d}s,$$

then W and its derivatives decay exponentially as $|x|\to\infty$ and W satisfied the 'integrated equation'

$$\lambda W = (-\partial_x^2+\alpha\partial_x+c-f'(u_c))\,\partial_x W. \quad (5.7)$$

Then we compute

$$0 < \mathrm{Re}\,\lambda \int_{-\infty}^{\infty} W\overline{W} = \mathrm{Re}\int_{-\infty}^{\infty} (-W'''+\alpha W''+cW'-f'(u_c)\,W')\,\overline{W}$$

$$= -\alpha\int_{-\infty}^{\infty} |W'|^2+\frac{1}{2}\int_{-\infty}^{\infty} |W|^2 f''(u_c)\,u_c' \leqslant 0$$

since $f'' \geqslant 0$ and $u_c' \leqslant 0$. This contradiction establishes the result.

For part 2, we compute $D'(0)$ from equation (1.35) in a manner similar to §2 parts (v) and (vi). For $\lambda = 0$, $Y^+ = \zeta_1^+$ and $Z^- = \eta_m^-$ are the unique solutions of

$$0 = \partial_x(-\partial_x^2+\alpha\partial_x+c-f'(u_c))\,Y^+ = JLY^+,$$

$$0 = (-\partial_x^2-\alpha\partial_x+c-f'(u_c))\,\partial_x Z^- = L^{\mathrm{t}}JZ^-,$$

such that

$$Y^+(x)\,\mathrm{e}^{-\mu^+x}\to 1 \quad \text{as} \quad x\to+\infty,$$

$$Z^-(x)\to 1/\mathscr{P}_-'(0) \quad \text{as} \quad x\to-\infty,$$

where we use that $\mu^-(0) = 0$. From differentiating (5.2), we know that

$$L\,\partial_x u_c = 0, \quad L^{\mathrm{t}}\,\partial_x 1 = 0.$$

Using (5.3) we find

$$Y^+(x) = (\mu^+\beta)^{-1}\,\partial_x u_c, \quad Z^-(x) = (f'(u_L)-c)^{-1}.$$

Since Y^+ decays exponentially as $x\to-\infty$, $D(0) = 0$, and using (1.35) we compute that

$$(f'(u_L)-c)\,(\mu^+\beta)\,D'(0) = \int_{-\infty}^{\infty} 1\cdot\partial_x u_c = -u_L \neq 0.$$

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