Evans function for the traveling wave (viscous shock) in Burgers equation

Consider the viscous Burgers equation

$$u_t = u_{xx} - uu_x. \tag{1}$$

An explicit traveling wave solution with speed $c = 0$ is given by $\bar{u}(x) = -\tanh(x/2)$. Linearizing (1) about $\bar{u}$ gives

$$v_t = v_{xx} + \tanh(x/2)v_x + (1/2)\mathrm{sech}(x/2)v. \tag{2}$$

The associated eigenvalue equation is

$$\lambda v = v_{xx} + \tanh(x/2)v_x + (1/2)\mathrm{sech}(x/2)v. \tag{3}$$

Taking asymptotic limits of the operator on the right hand sides leads to $\lambda v = v_{xx} \pm v_x$, and taking the Fourier transform then gives $\lambda v = (-k^2 \pm ik)v$. Thus, the boundary of the essential spectrum is given by

$$\partial\Sigma_{\text{ess}} = \{\lambda = -k^2 + ik : k \in \mathbb{R}\}.$$

Two independent solutions for (3) are given by

$$u_-(x;\lambda) = \left(-\frac{1}{2}\tanh(\frac{x}{2}) + \sqrt{\lambda + \frac{1}{4}}\right)\mathrm{sech}(\frac{x}{2})e^{\sqrt{\lambda+\frac{1}{4}}x}$$
$$u_+(x;\lambda) = \left(-\frac{1}{2}\tanh(\frac{x}{2}) - \sqrt{\lambda + \frac{1}{4}}\right)\mathrm{sech}(\frac{x}{2})e^{-\sqrt{\lambda+\frac{1}{4}}x},$$

which are bounded at $-\infty$ and $+\infty$, respectively, for all $\lambda \notin (-\infty, -1/4)$. In fact, one can see that both these solutions decay exponentially to zero at both $\pm\infty$ whenever λ lies to the left of the boundary of the essential spectrum. This is because $\mathrm{Re}(\sqrt{\lambda + \frac{1}{4}}) < 1/2$ there. Therefore, the region to the left of that parabola is filled with eigenvalues, and

$$\Sigma_{\text{ess}} = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq -(\mathrm{Im}(\lambda))^2\}.$$

To determine if this is all of the spectrum, we compute the Evans function:

$$E(\lambda) = \det\begin{pmatrix} u_-(0;\lambda) & u_+(0;\lambda) \\ u_-'(0;\lambda) & u_+'(0;\lambda)\end{pmatrix} = 2\lambda\sqrt{\lambda + \frac{1}{4}}.$$

Thus, $\Sigma = \Sigma_{\text{ess}}$. Note that the zero of the Evans function $\lambda = 0$ is embedded in the continuous spectrum. Using the Evans function alone, we do not know if this is a genuine eigenvalue or just a "resonance pole." However, we can explicitly see that the derivative of the wave, $\bar{u}'(x) = -(1/2)\mathrm{sech}^2(x/2)$, is an eigenfunction with eigenvalue 0.

Note that one can instead consider the stability of the wave in a weighted space by defining $w(x,t) = \cosh(\frac{x}{2})v(x,t)$, where v solves (2). We then have

$$w_t = w_{xx} + \left(\frac{1}{2}\mathrm{sech}^2(\frac{x}{2}) - \frac{1}{4}\right)w.$$

The self-adjiont operator on the right hand side has essential spectrum $\Sigma_{\text{ess}}^{\text{w}} = (-\infty, -1/4]$. The solutions to the associated eigenvalue equation that decay at $\pm\infty$ are given by

$$w_-(x) = \left(-\frac{1}{2}\tanh(\frac{x}{2}) + \sqrt{\lambda + \frac{1}{4}}\right)e^{\sqrt{\lambda+\frac{1}{4}}x}$$
$$w_+(x) = \left(-\frac{1}{2}\tanh(\frac{x}{2}) - \sqrt{\lambda + \frac{1}{4}}\right)e^{-\sqrt{\lambda+\frac{1}{4}}x}$$

and the Evans function is $E_w(\lambda) = 2\lambda\sqrt{\lambda + \frac{1}{4}}$. So it remains unchanged in this example.