Evans function for the traveling wave (viscous shock) in Burgers equation

Consider the viscous Burgers equation

\[ u_t = u_{xx} - uu_x. \]  (1)

An explicit traveling wave solution with speed \( c = 0 \) is given by \( \bar{u}(x) = -\tanh(x/2) \). Linearizing (1) about \( \bar{u} \) gives

\[ v_t = v_{xx} + \tanh(x/2)v_x + (1/2)\text{sech}(x/2)v. \] (2)

The associated eigenvalue equation is

\[ \lambda v = v_{xx} + \tanh(x/2)v_x + (1/2)\text{sech}(x/2)v. \] (3)

Taking asymptotic limits of the operator on the right hand sides leads to \( \lambda v = v_{xx} \pm v_x \), and taking the Fourier transform then gives \( \lambda v = -(k^2 \pm ik)v \). Thus, the boundary of the essential spectrum is given by

\[ \partial \Sigma_{\text{ess}} = \{ \lambda = -k^2 + ik : k \in \mathbb{R} \}. \]

Two independent solutions for (3) are given by

\[ u_-(x; \lambda) = \left( -\frac{1}{2} \tanh(x/2) + \sqrt{\lambda + \frac{1}{4}} \right) \text{sech}(x/2) e^{\sqrt{\lambda + \frac{1}{4}}x} \]

\[ u_+(x; \lambda) = \left( -\frac{1}{2} \tanh(x/2) - \sqrt{\lambda + \frac{1}{4}} \right) \text{sech}(x/2) e^{-\sqrt{\lambda + \frac{1}{4}}x}, \]

which are bounded at \(-\infty\) and \(+\infty\), respectively, for all \( \lambda \notin (-\infty, -1/4) \). In fact, one can see that both these solutions decay exponentially to zero at both \( \pm\infty \) whenever \( \lambda \) lies to the left of the boundary of the essential spectrum. This is because \( \text{Re}(\sqrt{\lambda + \frac{1}{4}}) < 1/2 \) there. Therefore, the region to the left of that parabola is filled with eigenvalues, and

\[ \Sigma_{\text{ess}} = \{ \lambda \in \mathbb{C} : \text{Re}(\lambda) \leq -\text{Im}(\lambda)^2 \}. \]

To determine if this is all of the spectrum, we compute the Evans function:

\[ E(\lambda) = \det \begin{pmatrix} u_-(0; \lambda) & u_+(0; \lambda) \\ u'_-(0; \lambda) & u'_+(0; \lambda) \end{pmatrix} = 2\lambda \sqrt{\lambda + \frac{1}{4}}. \]

Thus, \( \Sigma = \Sigma_{\text{ess}} \). Note that the zero of the Evans function \( \lambda = 0 \) is embedded in the continuous spectrum. Using the Evans function alone, we do not know if this is a genuine eigenvalue or just a “resonance pole.” However, we can explicitly see that the derivative of the wave, \( \bar{u}'(x) = -(1/2)\text{sech}^2(x/2) \), is an eigenfunction with eigenvalue 0.

Note that one can instead consider the stability of the wave in a weighted space by defining \( w(x, t) = \cosh(\frac{x}{2})v(x, t) \), where \( v \) solves (2). We then have

\[ w_t = w_{xx} + \left( \frac{1}{2} \text{sech}^2(\frac{x}{2}) - \frac{1}{4} \right) w. \]

The self-adjoint operator on the right hand side has essential spectrum \( \Sigma_{\text{ess}}^w = (-\infty, -1/4] \). The solutions to the associated eigenvalue equation that decay at \( \pm\infty \) are given by

\[ w_-(x) = \left( -\frac{1}{2} \tanh(\frac{x}{2}) + \sqrt{\lambda + \frac{1}{4}} \right) e^{\sqrt{\lambda + \frac{1}{4}}x} \]

\[ w_+(x) = \left( -\frac{1}{2} \tanh(\frac{x}{2}) - \sqrt{\lambda + \frac{1}{4}} \right) e^{-\sqrt{\lambda + \frac{1}{4}}x} \]

and the Evans function is \( E_w(\lambda) = 2\lambda \sqrt{\lambda + \frac{1}{4}} \). So it remains unchanged in this example.