Test 1 Solutions – MA 242 A1 – Fall 2009

Please note: for some of these problems, there is more than one valid line of reasoning. The answers I've written below are just one example of a way to solve each problem.

Question 1

(i) [**7 points**] Put the following matrix into reduced row echelon form.

$$\begin{bmatrix} 2 & 0 & -4 & 0 \\ 1 & 3 & 1 & -3 \\ 0 & 4 & 8 & 4 \\ -1 & 0 & 2 & -9 \end{bmatrix}$$

Solution: (Note that there are many correct ways to row-reduce a matrix, but the end result should be the same.)

$$\begin{bmatrix} 2 & 0 & -4 & 0 \\ 1 & 3 & 1 & -3 \\ 0 & 4 & 8 & 4 \\ -1 & 0 & 2 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & -3 \\ 2 & 0 & -4 & 0 \\ 0 & 4 & 8 & 4 \\ -1 & 0 & 2 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & -3 \\ 0 & -6 & -6 & 6 \\ 0 & 4 & 8 & 4 \\ 0 & 3 & 3 & -12 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 3 & 1 & -3 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & -3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & -3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) [**8 points**] Consider the following system of linear equations.

$$\begin{aligned} x_1 - 4x_3 - x_5 &= 1 \\ x_2 + 2x_3 - 4x_4 + 3x_5 &= -2 \\ x_4 + x_5 &= 1 \end{aligned}$$

Write the solution set in parametric vector form.

Solution: The associated augmented matrix is

$$\begin{bmatrix} 1 & 0 & -4 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 & 3 & -2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Row reducing this yields

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 & 7 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Therefore, x_5 and x_3 are the free variables, and so the solution set in parametric form is

$$\mathbf{x} = \begin{bmatrix} 1+4x_3+x_5 \\ 2-2x_3-7x_5 \\ x_3 \\ 1-x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -7 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Question 2. (Note: there is more than one correct way to justify the answers to this problem.)

(i) [**5 points**] Consider the following matrix

$$B = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & -1 & 3 & 7 \end{bmatrix}.$$

Do the columns of B span $\mathbb{R}^2$?

Solution: Yes. In order for the columns to span $\mathbb{R}^2$, we need a pivot in each row, and the matrix B (already in echelon form) has a pivot in each row.

(ii) [**5 points**] Given an arbitrary $m \times n$ matrix A, is it possible for the columns of A to span $\mathbb{R}^m$ if $m > n$? Why?

Solution: No. This matrix has more rows than columns, and so cannot possibly have a pivot in each row. (It can have at most n pivots.)

(iii) [**5 points**] Given an arbitrary $m \times n$ matrix A, is it possible for the columns of A to span $\mathbb{R}^m$ if $n > m$? Why?

Solution: Yes. This matrix has more columns than rows, and so could potentially have a pivot in each row, as in the above matrix B.

Question 3. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & 2 \\ 0 & 1 & 1 \\ 4 & -1 & 0 \end{bmatrix}.$$

(i) [**7 points**] Find the solution set for the equation $A\mathbf{x} = \mathbf{0}$.

Solution: Construct the augmented matrix and perform row reductions:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ -3 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 4 & -1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -8 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a pivot in each column of the echelon form of A, the only solution is the trivial solution, $\mathbf{x} = \mathbf{0}$ (ie $x_1 = x_2 = x_3 = 0$).

(ii) [**5 points**] Describe, in terms of the columns of the matrix A, the vectors $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ is consistent.

Solution: $A\mathbf{x} = \mathbf{b}$ is consistent only for vectors b that are linear combinations of the columns of A. This can be seen by writing

$$\mathbf{b} = A\mathbf{x} = [\mathbf{a}_1 \ldots \mathbf{a}_n]\mathbf{x} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n.$$

(iii) [**5 points**] Given a vector $\mathbf{b}$ such that a solution of $A\mathbf{x} = \mathbf{b}$ exists, is the solution unique?

Solution: (Note: more than one correct way to justify this answer.) Yes, the solution is unique because $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, ie the echelon form of A implies there will be no free variables when finding a solution.

Question 4 [**8 points**] Consider a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, and suppose that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}.$$

Find the standard matrix for the transformation.

Solution: The matrix is given by

$$A = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ T(\mathbf{e}_3)],$$

where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We are told $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$, so we only need to find $T(\mathbf{e}_3)$. To do so, notice

$$T(\mathbf{e}_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = T\left(-2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right) = -2\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 2 \end{bmatrix}$$

Therefore,

$$A = \begin{bmatrix} -3 & 1 & -2 \\ 1 & -2 & 5 \\ 2 & 1 & 2 \end{bmatrix}$$

Question 5 Suppose A is a 4×3 matrix that satisfies $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. Define a linear transformation by $T(\mathbf{x}) = A\mathbf{x}$.

(i) [**5 points**] What is the domain and codomain of T?

Solution: The domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^4$.

(ii) [**5 points**] Is $T(\mathbf{x}) = A\mathbf{x}$ one-to-one?

Solution: (Note: more than one correct way to justify this answer.) No. Since $T(\mathbf{x}) = \mathbf{0}$ has a nontrivial solution, for any $\mathbf{b}$ in the range of T, there will necessarily be infinitely many points $\mathbf{y}$ that get mapped to it.

(iii) [**5 points**] Is $T(\mathbf{x}) = A\mathbf{x}$ onto?

Solution: (Note: more than one correct way to justify this answer.) No. In order for T to be onto, its standard matrix would need a pivot in each row. The above matrix has four rows but only three columns, and so can have at most three pivots, which isn't enough.

Question 6

(i) [**5 points**] Compute the determinant of

$$A = \begin{bmatrix} 3 & k \\ -1 & 1 \end{bmatrix}.$$

Solution: Using the formula for the determinant of a 2×2 matrix, we find $\det A = 3(1) - (k)(-1) = 3 + k$.

(ii) [**5 points**] For what values of k does the system $A\mathbf{x} = \mathbf{b}$ have a solution for all $\mathbf{b} \in \mathbb{R}^2$? Why?

Solution: (Note: more than one correct way to justify this answer.) The system will be consistent for all $\mathbf{b}$ whenever A is invertible, which is whenever $\det A \neq 0$. Therefore, whenever $k \neq -3$.

Question 7

(i) [**7 points**] Suppose that the columns of an $m \times n$ matrix A are linearly dependent and the vector $\mathbf{b}$ in $\mathbb{R}^m$ is such that the system $A\mathbf{x} = \mathbf{b}$ is consistent. Prove that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

Solution: (Note: there is more than one correct way to prove this.) Since the columns of A, $\mathbf{a}_1, \ldots \mathbf{a}_n$, are dependent, there exist weights $x_1, \ldots, x_n$, not all zero, such that

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}.$$

Therefore, if these weights are put into the entries of the vector $\mathbf{x}$, then $A\mathbf{x} = 0$. Since the system is consistent, there exists a $\mathbf{y}$ such that $A\mathbf{y} = \mathbf{b}$. We can compute

$$A(\mathbf{y} + c\mathbf{x}) = A\mathbf{y} + cA\mathbf{x} = \mathbf{b} + \mathbf{0} = \mathbf{b},$$

for any scalar c. Thus, we have found infinitely many solutions.

(ii) [**6 points**] Let A and B be $n \times n$ matrices. Prove that if the matrix AB is invertible, then so is the matrix B.

Solution: (Note: there is more than one correct way to prove this.) Since AB is invertible, there exists a matrix C such that $C(AB) = I$. Therefore, $(CA)B = I$. Define $D = CA$. By a theorem we discussed in class and that is in the book, if there exists a matrix D such that $DB = I$, then B is invertible and $D = B^{-1}$. Therefore, we have shown B must be invertible.

(iii) [**7 points**] Suppose that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ has the property that $T(\mathbf{u}) = T(\mathbf{v})$ for two vectors $\mathbf{u} \neq \mathbf{v}$. Is it possible for T to be onto? Why or why not?

Solution: (Note: there is more than one correct way to justify this answer.) By definition, this transformation is not one-to-one, and therefore it can't have a pivot in each column. So, it has less than n pivots. Since its matrix is $n \times n$, it can't possibly have a pivot in each row, which is what is required for it to be onto. Therefore, it is not possible for T to be onto.