Please note: for some of these problems, there is more than one valid line of reasoning. The answers I’ve written below are just one example of a way to solve each problem.

Question 1

(i) [7 points] Put the following matrix into reduced row echelon form.

\[
\begin{bmatrix}
2 & 0 & -4 & 0 \\
1 & 3 & 1 & -3 \\
0 & 4 & 8 & 4 \\
-1 & 0 & 2 & -9
\end{bmatrix}
\]

Solution: (Note that there are many correct ways to row-reduce a matrix, but the end result should be the same.)

\[
\begin{bmatrix}
2 & 0 & -4 & 0 \\
1 & 3 & 1 & -3 \\
0 & 4 & 8 & 4 \\
-1 & 0 & 2 & -9
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 3 & 1 & -3 \\
2 & 0 & -4 & 0 \\
0 & 4 & 8 & 4 \\
-1 & 0 & 2 & -9
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 3 & 1 & -3 \\
0 & 1 & 1 & -1 \\
0 & 1 & 2 & 1 \\
0 & 1 & 1 & -4
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 3 & 1 & -3 \\
0 & 1 & 1 & -1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & -3
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 3 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(ii) [8 points] Consider the following system of linear equations.

\[
x_1 - 4x_3 - x_5 = 1 \\
x_2 + 2x_3 - 4x_4 + 3x_5 = -2 \\
x_4 + x_5 = 1
\]

Write the solution set in parametric vector form.

Solution: The associated augmented matrix is

\[
\begin{bmatrix}
1 & 0 & -4 & 0 & -1 & 1 \\
0 & 1 & 2 & -4 & 3 & -2 \\
0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
\]
Row reducing this yields

$$\begin{bmatrix} 1 & 0 & -4 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 & 7 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$ 

Therefore, \( x_5 \) and \( x_3 \) are the free variables, and so the solution set in parametric form is

\[
x = \begin{bmatrix} 1 + 4x_3 + x_5 \\ 2 - 2x_3 - 7x_5 \\ x_3 \\ 1 - x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_5 + \begin{bmatrix} 1 \\ -7 \\ 0 \\ -1 \\ 1 \end{bmatrix}
\]

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**Question 2.** (Note: there is more than one correct way to justify the answers to this problem.)

(i) [5 points] Consider the following matrix

\[
B = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & -1 & 3 & 7 \end{bmatrix}.
\]

Do the columns of \( B \) span \( \mathbb{R}^2 \)?

**Solution:** Yes. In order for the columns to span \( \mathbb{R}^2 \), we need a pivot in each row, and the matrix \( B \) (already in echelon form) has a pivot in each row.

(ii) [5 points] Given an arbitrary \( m \times n \) matrix \( A \), is it possible for the columns of \( A \) to span \( \mathbb{R}^m \) if \( m > n \)? Why?

**Solution:** No. This matrix has more rows than columns, and so cannot possibly have a pivot in each row. (It can have at most \( n \) pivots.)

(iii) [5 points] Given an arbitrary \( m \times n \) matrix \( A \), is it possible for the columns of \( A \) to span \( \mathbb{R}^m \) if \( n > m \)? Why?

**Solution:** Yes. This matrix has more columns than rows, and so could potentially have a pivot in each row, as in the above matrix \( B \).

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**Question 3.** Consider the matrix

\[
A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & 2 \\ 0 & 1 & 1 \\ 4 & -1 & 0 \end{bmatrix}.
\]

(i) [7 points] Find the solution set for the equation \( Ax = 0 \).
Solution: Construct the augmented matrix and perform row reductions:

\[
\begin{bmatrix}
1 & 0 & 2 & 0 \\
-3 & 1 & 2 & 0 \\
0 & 1 & 1 & 0 \\
4 & -1 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 8 & 0 \\
0 & 1 & 1 & 0 \\
0 & -1 & -8 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 8 & 0 \\
0 & 0 & -7 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Since there is a pivot in each column of the echelon form of \(A\), the only solution is the trivial solution, \(x = 0\) (ie \(x_1 = x_2 = x_3 = 0\)).

(ii) [5 points] Describe, in terms of the columns of the matrix \(A\), the vectors \(b\) such that \(Ax = b\) is consistent.

Solution: \(Ax = b\) is consistent only for vectors \(b\) that are linear combinations of the columns of \(A\). This can be seen by writing

\[b = Ax = [a_1 \ldots a_n]x = x_1a_1 + \cdots + x_na_n.\]

(iii) [5 points] Given a vector \(b\) such that a solution of \(Ax = b\) exists, is the solution unique?

Solution: (Note: more than one correct way to justify this answer.) Yes, the solution is unique because \(Ax = 0\) has only the trivial solution, ie the echelon form of \(A\) implies there will be no free variables when finding a solution.

Question 4 [8 points] Consider a linear transformation \(T : \mathbb{R}^3 \rightarrow \mathbb{R}^3\), and suppose that

\[
T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \quad T \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad T \left( \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}.
\]

Find the standard matrix for the transformation.

Solution: The matrix is given by

\[A = [T(e_1) \ T(e_2) \ T(e_3)],\]

where

\[e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.\]

We are told \(T(e_1)\) and \(T(e_2)\), so we only need to find \(T(e_3)\). To do so, notice

\[T(e_3) = T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = T \left( -2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = -2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 2 \end{bmatrix}.\]
Therefore,

\[
A = \begin{bmatrix}
-3 & 1 & -2 \\
1 & -2 & 5 \\
2 & 1 & 2
\end{bmatrix}
\]

**Question 5** Suppose \( A \) is a \( 4 \times 3 \) matrix that satisfies \( Ax = 0 \) for \( x = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \). Define a linear transformation by \( T(x) = Ax \).

(i) **[5 points]** What is the domain and codomain of \( T? \)

**Solution:** The domain is \( \mathbb{R}^3 \) and the codomain is \( \mathbb{R}^4 \).

(ii) **[5 points]** Is \( T(x) = Ax \) one-to-one?

**Solution:** (Note: more than one correct way to justify this answer.) No. Since \( T(x) = 0 \) has a nontrivial solution, for any \( b \) in the range of \( T \), there will necessarily be infinitely many points \( y \) that get mapped to it.

(iii) **[5 points]** Is \( T(x) = Ax \) onto?

**Solution:** (Note: more than one correct way to justify this answer.) No. In order for \( T \) to be onto, its standard matrix would need a pivot in each row. The above matrix has four rows but only three columns, and so can have at most three pivots, which isn’t enough.

**Question 6**

(i) **[5 points]** Compute the determinant of

\[
A = \begin{bmatrix}
3 & k \\
-1 & 1
\end{bmatrix}
\]

**Solution:** Using the formula for the determinant of a \( 2 \times 2 \) matrix, we find \( \det A = 3(1) - (k)(-1) = 3 + k \).

(ii) **[5 points]** For what values of \( k \) does the system \( Ax = b \) have a solution for all \( b \in \mathbb{R}^2 \)? Why?

**Solution:** (Note: more than one correct way to justify this answer.) The system will be consistent for all \( b \) whenever \( A \) is invertible, which is whenever \( \det A \neq 0 \). Therefore, whenever \( k \neq -3 \).
Question 7

(i) **[7 points]** Suppose that the columns of an $m \times n$ matrix $A$ are linearly dependent and the vector $b$ in $\mathbb{R}^m$ is such that the system $Ax = b$ is consistent. Prove that $Ax = b$ has infinitely many solutions.

**Solution:** (Note: there is more than one correct way to prove this.) Since the columns of $A$, $a_1, \ldots, a_n$, are dependent, there exist weights $x_1, \ldots, x_n$, not all zero, such that

$$x_1a_1 + \cdots + x_na_n = 0.$$ 

Therefore, if these weights are put into the entries of the vector $x$, then $Ax = 0$. Since the system is consistent, there exists a $y$ such that $Ay = b$. We can compute

$$A(y + cx) = Ay + cAx = b + 0 = b,$$

for any scalar $c$. Thus, we have found infinitely many solutions.

(ii) **[6 points]** Let $A$ and $B$ be $n \times n$ matrices. Prove that if the matrix $AB$ is invertible, then so is the matrix $B$.

**Solution:** (Note: there is more than one correct way to prove this.) Since $AB$ is invertible, there exists a matrix $C$ such that $C(AB) = I$. Therefore, $(CA)B = I$. Define $D = CA$. By a theorem we discussed in class and that is in the book, if there exists a matrix $D$ such that $DB = I$, then $B$ is invertible and $D = B^{-1}$. Therefore, we have shown $B$ must be invertible.

(iii) **[7 points]** Suppose that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ has the property that $T(u) = T(v)$ for two vectors $u \neq v$. Is it possible for $T$ to be onto? Why or why not?

**Solution:** (Note: there is more than one correct way to justify this answer.) By definition, this transformation is not one-to-one, and therefore it can’t have a pivot in each column. So, it has less than $n$ pivots. Since its matrix is $n \times n$, it can’t possibly have a pivot in each row, which is what is required for it to be onto. Therefore, it is not possible for $T$ to be onto.