Question 1

(i) [**7 points**] Compute the determinant of the following matrix using cofactor expansion.

$$\begin{bmatrix} 2 & 0 & -4 & 0 \\ 1 & 3 & 1 & -3 \\ -1 & 0 & 2 & 4 \\ 0 & 0 & 2 & -3 \end{bmatrix}$$

Solution: Expand down the second column, since it has the most zeros. We get

$$\text{determinant} = +3\det \begin{bmatrix} 2 & -4 & 0 \\ -1 & 2 & 4 \\ 0 & 2 & -3 \end{bmatrix}.$$

Expanding now along the fist column (another quick way is to expand along the first row), we get

$$= 3\left[+(2)\det\begin{bmatrix} 2 & 4 \\ 2 & -3 \end{bmatrix} - (-1)\det\begin{bmatrix} -4 & 0 \\ 2 & -3 \end{bmatrix}\right] = 3[2(-6-8) + 1(12-0)] = 3(-16) = -48.$$

(ii) [**8 points**] Find a basis for the set of all vectors of the form

$$H = \left\{ \begin{bmatrix} a+3b-3c \\ -2a+b-8c \\ a-b+5c \\ a+2b-c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

Solution: By writing

$$\begin{bmatrix} a+3b-3c \\ -2a+b-8c \\ a-b+5c \\ a+2b-c \end{bmatrix} = a\begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 3 \\ 1 \\ -1 \\ 2 \end{bmatrix} + c\begin{bmatrix} -3 \\ -8 \\ 5 \\ -1 \end{bmatrix} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3.$$

we see that $H = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. We need to check if these vectors are independent. Putting them into the columns of a matrix and doing row reductions, we see that $\mathbf{v}_3 = 3\mathbf{v}_1 + -2\mathbf{v}_2$. Therefore, the basis is $\{\mathbf{v}_1, \mathbf{v}_2\}$.

Question 2 Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 2 & -4 & 1 \\ -3 & 1 & -4 & 14 & -3 \\ 4 & 0 & 8 & -16 & 4 \\ -2 & 1 & -4 & 10 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 & -4 & 1 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which are row equivalent. Please justify your answers to the following.

(i) [**5 points**] Find rank A and dim Nul A.
Solution: Since B has three pivot columns and two columns without pivots, rank $A = 3$ and dim Nul $A = 2$.

(ii) [**5 points**] Find a basis for Col A.
Solution: The basis for the column space consists of the pivot columns of A, so it is

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 14 \\ -16 \\ 10 \end{bmatrix}, \right\}.$$

(iii) [**5 points**] Find a basis for Row A.
Solution: The basis for the row space consists of the nonzero rows of B, so it is

$$\{(1, 0, 2, -4, 1), (0, 1, 2, 2, 0), (0, 0, 0, 2, -2)\}$$

(iv) [**5 points**] Find a basis for Nul A.
Solution: We need to put B into reduced row echelon form. We then obtain

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -3 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By finding the solution set to the homogeneous equation in parametric vector form, we obtain

$$\mathbf{x} = \begin{bmatrix} -2x_3 + 3x_5 \\ -2x_3 - 2x_5 \\ x_3 \\ x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

So, the basis for the null space is

$$\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Question 3 Are the following sets subspaces of an appropriate vector space? Explain why or why not.

(i) [**7 points**] The set

$$H = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{array}{c} a - 3b - d = c \\ 2a + c = d \\ -b + 2d = c \end{array} \right\}$$

Solution: Yes. The set H is the null space of the matrix

$$\begin{bmatrix} 1 & -3 & -1 & -1 \\ 2 & 0 & 1 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}.$$

Since we know from class that the null space of any matrix is a subspace, H must be a subspace (of $\mathbb{R}^4$).

(ii) [**8 points**] The set $W = \{\mathbf{p} \in \mathbb{P}_n : \mathbf{p}(-3) = 0\}$.

Solution: Yes. You can check this using the definition of a subspace. We check: (1) The zero vector $\mathbf{p}(t) = 0$ is in W because if we evaluate this polynomial at $t = -3$ we get 0; (2) W is closed under addition because, if $\mathbf{p}_1$ and $\mathbf{p}_2 \in W$, then $(\mathbf{p}_1 + \mathbf{p}_2)(-3) = \mathbf{p}_1(-3) + \mathbf{p}_2(-3) = 0 + 0 = 0$; (3) W is closed under scalar multiplication because, if $\mathbf{p} \in W$ and $c \in \mathbb{R}$, the $(c\mathbf{p})(-3) = c(\mathbf{p}(-3)) = c(0) = 0$. Therefore, W is a subspace of $\mathbb{P}_n$.

Question 4

(i) Suppose a 5×8 matrix A has four pivot columns.

(a) [**4 points**] Is Nul $A = \mathbb{R}^4$? Justify your answer.

Solution: No. We know that dim Nul $A = n -$rank $A = n -$#of pivot columns $= 8 - 4 = 4$. However, since A is 5×8, Nul $A \subset \mathbb{R}^8$. Therefore, Nul A is a four dimensional subspace of $\mathbb{R}^8$, which is not $\mathbb{R}^4$.

(b) [**4 points**] What is dim Row A? Justify your answer.

Solution: We know dim Row $A =$ dim Col $A =$ #of pivot columns $= 4$.

(ii) [**7 points**] Is it possible for a nonhomogeneous system of seven equation in six unknowns to have a unique solution for some right-hand side of constants? Is it possible for such a system to have a unique solution for every right-hand side? (Justify your answers.)

Solution: If we were to think of this system as a matrix equation, it would be of the form $A\mathbf{x} = \mathbf{b}$, where A was 7×6. To have unique solutions whenever the system was consistent, A would need a trivial null space, meaning 6 pivot columns. This is possible, so it is possible for the system to have a unique solution for some right-hand side. However, it is not possible for it to have a unique solution for every right-hand side. This could require a pivot in each row. Since there are 7 rows and only 6 columns, this can't happen.

Question 5

(i) [**7 points**] Prove that, if A is an $m \times n$ matrix, then Nul A is a subspace of $\mathbb{R}^n$.

Solution: Let $\mathbf{u}, \mathbf{v} \in \text{Nul } A$ and $c \in \mathbb{R}$. First, note that $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ because A is an $m \times n$ matrix. So, if Nul A is a subspace, it must be a subspace of $\mathbb{R}^n$. Checking the three parts of the definition of a subspace, we see that (1) The zero vector is in the null space, since $A\mathbf{0} = \mathbf{0}$; (2) The null space is closed under addition, because $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$; (3) The null space is closed under scalar multiplication, because $A(c\mathbf{u}) = c(A\mathbf{u}) = c(\mathbf{0}) = \mathbf{0}$.

(ii) [**7 points**] Prove that, if A is an $m \times n$ matrix with rank $A = n$, then the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.

Solution: Since rank A + dim Nul $A = n$, we see that dim Nul $A = 0$. Thus, there are no nontrivial solutions to the homogeneous problem. This means that, if $A\mathbf{x} = \mathbf{b}$ is consistent, it will have a unique solution, which is, by definition, the same as T being one-to-one.

(iii) [**6 points**] Prove that, if A is a 4×5 matrix with rank $A = 4$, then the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for Col A.

Solution: If the rank of the matrix is four, then the dimension of the column space is four. Since the column space is a subspace of $\mathbb{R}^4$, it must therefore be all of $\mathbb{R}^4$. Since the given vectors for the standard basis for $\mathbb{R}^4$, they must therefore also be a basis for the column space.

Question 6 Consider the transformation

$$T : \mathbb{P}_2 \to \mathbb{R}, \quad T(\mathbf{p}) = \int_0^1 \mathbf{p}(t)\mathrm{d}t.$$

(i) [**8 points**] Prove that T is a linear transformation.

Solution: Let $\mathbf{p}_1$, $\mathbf{p}_1 \in \mathbb{P}_2$ and let $c \in \mathbb{R}$. Using the properties of integration, we check (1) $T(\mathbf{p}_1 + \mathbf{p}_2) = \int_0^1 [\mathbf{p}_1(t) + \mathbf{p}_2(t)]\mathrm{d}t = \int_0^1 \mathbf{p}_1(t)\mathrm{d}t + \int_0^1 \mathbf{p}_2(t)\mathrm{d}t = T(\mathbf{p}_1) + T(\mathbf{p}_2)$; (2) $T(c\mathbf{p}_1) = \int_0^1 c\mathbf{p}_1(t)\mathrm{d}t = c\int_0^1 \mathbf{p}_1(t)\mathrm{d}t = cT(\mathbf{p}_1)$, so the transformation is linear.

(ii) [**7 points**] Find a basis for the kernel of T.

Solution: For an arbitrary element of $\mathbb{P}_2$, we compute

$$\int_0^1 [a_0 + a_1 t + a_2 t^2]\mathrm{d}t = a_0 + \frac{a_1}{2} + \frac{a_2}{3}.$$

Therefore, the kernel is the set of all polynomials in $\mathbb{P}_2$ such that $a_0 + \frac{a_1}{2} + \frac{a_2}{3} = 0$. Thinking in terms of the coordinates for the standard basis, this set corresponds to the null space of the matrix

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which has parametric solution set

$$\mathbf{x} = x_2 \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, the above two vectors give a basis for the null space. These vectors give the coordinates for the basis for the kernel of the transformation, and so the basis is

$$\left\{ -\frac{1}{2} + t, -\frac{1}{3} + t^2 \right\}.$$