Final – MA 242 A1 – Fall 2009

Question 1

(i) [**5 points**] Compute the inverse of the following matrix using any method you like.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & 2 \\ 0 & 1 & 7 \end{bmatrix}.$$

Solution: Augment the matrix with the identity and do row reductions:

$$\left[\begin{array}{cccccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -3 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 7 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 8 & 3 & 1 & 0 \\ 0 & 1 & 7 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 8 & 3 & 1 & 0 \\ 0 & 0 & -1 & -3 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccccc} 1 & 0 & 0 & -5 & -2 & 2 \\ 0 & 1 & 0 & -21 & -7 & 8 \\ 0 & 0 & 1 & 3 & 1 & -1 \end{array}\right] \to A^{-1} = \begin{bmatrix} -5 & -2 & 2 \\ -21 & -7 & 8 \\ 3 & 1 & -1 \end{bmatrix}$$

You can check your answer by multiplying A and A^{-1} and checking that you get the identity matrix.

(ii) [**5 points**] Suppose A and B are $n \times n$ matrices such that $\det A = 3$ and $\det B = -2$. Compute the following quantities:

(a) **Solution:** $\det AB^2 = \det A(\det B)^2 = 3(-2)^2 = 12$.

(b) **Solution:** $\det A^{-1} = 1/\det A = 1/3$.

(c) $\det kB$, for some scalar $k \in \mathbb{R}$.
Solution: The rule says that each time a row is multiplied by a scalar k, we must multiply the determinant by k. Since there are n rows, we have to multiply by k^n. Thus, $\det kB = k^n \det B = -2k^n$.

Question 2

(i) [**5 points**] Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that first rotates points $\pi/2$ radians (counterclockwise) about the origin and then reflects points through the horizontal x_1-axis. Find the standard matrix of T.
Solution: The standard matrix is $A = [T(\mathbf{e}_1)T(\mathbf{e}_2)]$, so we need to compute $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$. By drawing a picture, we see that $T(\mathbf{e}_1) = -\mathbf{e}_2$ and $T(\mathbf{e}_2) = -\mathbf{e}_1$. Hence, the matrix is

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

(ii) [**5 points**] Consider the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ defined by the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

Let S be the region inside the right triangle that has vertices $(0,0)$, $(3,2)$, and $(3,0)$. Determine the area of the region $T(S)$.

Solution: We know that $\text{area}(T(S)) = |\det A|\text{area}(S)$. We compute $\det A = 1(1) - 2(1) = -1$ and $\text{area}(S) = \text{base} \cdot \text{height}/2 = (3 \cdot 2)/2 = 3$. Therefore, $\text{area}(T(S)) = 3|-1| = 3$.

Question 3

(i) [**5 points**] Find the eigenvalues of the matrix

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix}.$$

Solution: Expanding along the first column, we compute

$$\det\,(A - \lambda I) = \begin{vmatrix} (1-\lambda) & 2 & -1 \\ 0 & (2-\lambda) & -2 \\ 0 & -1 & (1-\lambda) \end{vmatrix} = (1-\lambda)[(2-\lambda)(1-\lambda) - 2] = (1-\lambda)\lambda(\lambda - 3).$$

Therefore, the eigenvalues are $0, 1, 3$.

(ii) [**5 points**] The following matrix has eigenvalues $\lambda = 1, 2, 2, 3$. Determine if it is diagonalizable.

$$\begin{bmatrix} 3 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Solution: We need to determine if there are two independent eigenvectors associated to the eigenvalue 2, so we need to solve the augmented system $[(A - 2I)\ \mathbf{0}]$:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

There is only one free variable (x_4), and so there will only be one independent eigenvector. Hence, the matrix is not diagonalizable.

Question 4

(i) [**5 points**] Consider the vectors

$$\mathbf{u} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}.$$

Find an orthogonal basis for $V = \operatorname{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

Solution: We apply the Gram-Schmidt process to obtain the orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$:

$$\mathbf{u}_1 = \mathbf{u} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_2 = \mathbf{v} - \left(\frac{\mathbf{v} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \left(\frac{-36}{12}\right)\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\mathbf{u}_3 = \mathbf{w} - \left(\frac{\mathbf{w} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 - \left(\frac{\mathbf{w} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\right)\mathbf{u}_2 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \left(\frac{6}{12}\right)\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{30}{12}\right)\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

(ii) [**5 points**] Let $W = \operatorname{span}\{\mathbf{a}, \mathbf{b}\}$, where

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix},$$

and let $T(\mathbf{x}) = \operatorname{proj}_W \mathbf{x}$. Find a matrix A such that Ker T = Nul A.

Solution: Notice that

$$T(\mathbf{x}) = \left(\frac{\mathbf{x} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a} + \left(\frac{\mathbf{x} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\right)\mathbf{b}.$$

This is a linear combination of $\mathbf{a}$ and $\mathbf{b}$. Since they are orthogonal, the only way $T(\mathbf{x}) = 0$ is if both the weights are zero. This will happen exactly when $\mathbf{x} \cdot \mathbf{a} = 0$ and $\mathbf{x} \cdot \mathbf{b} = 0$. Writing this out, we have

$$\begin{aligned} 1x_1 + 2x_2 + 0x_3 - 1x_4 &= 0 \\ 2x_1 + 1x_2 + 3x_3 + 4x_4 &= 0 \end{aligned} \rightarrow \underbrace{\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 1 & 3 & 4 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, Ker T = Nul A.

Question 5

(i) [**5 points**] Let A be an $n \times n$ matrix. Provide five statements that are equivalent to the statement "A is invertible." (Note: you do not need to justify your answer. However, you will lose 0.5 points for each incorrect statement you list. So if you list 10 statements and only 5 are correct, you will only earn 2.5 points.)

Solution: There are at least 20 correct statements, and they can be found in the "Invertible Matrix Theorems": Theorem 8, page 129 (chapter 2.3), the Theorem on page 267 (chapter 4.6), and the Theorem on page 312 (chapter 5.2).

(ii) [**5 points**] Let B be an 8×6 matrix. What is the smallest possible dimension of Nul B? What is the largest possible dimension of Col B? Justify your answer.

Solution: The maximum number of pivots this matrix can have is 6. In this case, there will be a pivot in each column and so no free variables. Thus, the smallest possible dimension of the null space is 0. Similarly, there can be at most 6 rows with pivots, and so the largest possible dimension of the column space is 6.

Question 6

(i) [**5 points**] Consider the matrix A =

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 & 5 \\ -1 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

Find a basis for Col A.

Solution: To find a basis for the column space, we use row reduction to find the pivots, and then take the pivot columns from the original matrix (not the row-reduced one):

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 1 \\ 2 & 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 & 5 \\ -1 & 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & 0 & 1 \\ 0 & -1 & 4 & 1 & -2 \\ 0 & 0 & 2 & 3 & 5 \\ 0 & 1 & -1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & 0 & 1 \\ 0 & -1 & 4 & 1 & -2 \\ 0 & 0 & 2 & 3 & 5 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & -1 & 0 & 1 \\ 0 & -1 & 4 & 1 & -2 \\ 0 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & -5/2 & -13/2 \end{bmatrix}.$$

Thus, the first four columns are pivot columns, so the basis for the column space is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

(ii) [**5 points**] Consider the vectors

$$\mathbf{b}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, find the $\mathcal{B}$-coordinate of the vector $\mathbf{x}$.

Solution: We need to write $\mathbf{x}$ as a linear combination of the basis elements

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = \mathbf{x},$$

Which is equivalent to solving the linear system:

$$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 2 & -2 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & 0 & 4 & 4 \\ 0 & 2 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Question 7

(i) [**5 points**] Consider the polynomials

$$\mathbf{p}_1(t) = 1 + t, \quad \mathbf{p}_2(t) = 1 - 2t, \quad \mathbf{p}_3(t) = 1, \quad \mathbf{p}_4(t) = t + t^2, \quad \mathbf{p}_5(t) = 1 + 3t + t^2.$$

Define $H = \text{span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5\} \subset \mathbb{P}_2$. Find a basis for H.

Solution: We need to determine which of the polynomials are independent. To do so, we write them in terms of their coordinate vectors, put those into the columns of a matrix, and do row reductions:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & -2 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & -3 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Since the first, second, and fourth columns have pivots, the basis for H is $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_4\}$.

(ii) Consider the transformation $T : \mathbb{P}_2 \to \mathbb{P}_3$ defined by $T(\mathbf{p}(t)) = t\mathbf{p}(t) - \mathbf{p}(t)$. (Part (b) on next page.)

(a) [**5 points**] Determine the matrix for the transformation relative to the bases $\{1, t, t^2\}$ for $\mathbb{P}_2$ and $\{1, 2+t, -1+t^2, t^3\}$ for $\mathbb{P}_3$.

Solution: The formula for the matrix is

$$M = [[T(\mathbf{b}_1)]_{\mathcal{C}}[T(\mathbf{b}_2)]_{\mathcal{C}}[T(\mathbf{b}_3)]_{\mathcal{C}}],$$

where the basis $\mathcal{B}$ is given by $\mathbf{b}_1 = 1$, $\mathbf{b}_2 = t$, and $\mathbf{b}_3 = t^2$, and the basis $\mathcal{C}$ is given by $\mathbf{c}_1 = 1$, $\mathbf{c}_2 = 2+t$, $\mathbf{c}_3 = -1+t^2$, and $\mathbf{c}_4 = t^3$. First, we find

$$T(1) = t-1, \quad T(t) = t^2 - t, \quad T(t^2) = t^3 - t^2.$$

Now we need to find the $\mathcal{C}$-coordinate of these polynomials. Consider first $T(1) = t-1$. We need to write $t - 1 = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3 + x_4\mathbf{c}_4$. This is equivalent to solving the linear system

$$\begin{bmatrix} 1 & 2 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

For $T(t) = t^2 - t$, we solve

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

and for $T(t^2) = t^3 - t^2$ we solve

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Therefore,

$$M = [[T(\mathbf{b}_1)]_{\mathcal{C}}[T(\mathbf{b}_2)]_{\mathcal{C}}[T(\mathbf{b}_3)]_{\mathcal{C}}] = \begin{bmatrix} -3 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) [**5 points**] Find a basis for the range of T.

Solution: Notice that

$$T(a_0 + a_1t + a_2t^2) = t(a_0 + a_1t + a_2t^2) - (a_0 + a_1t + a_2t^2) = a_0(t-1) + a_1(t^2 - t) + a_2(t^3 - t^2).$$

Therefore, the basis is $\{(t-1), (t^2 - t), (t^3 - t^2)\}$. (We know this is a basis because we can check that these polynomials are all independent and the dimension of the range can't be bigger than 3, since the matrix of the transformation can have at most three pivots.)

(iii) [**5 points**] Is the following is an inner product on $\mathbb{P}_2$? If so, prove it. If not, explain why not.

$$\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = \mathbf{p}_1'(1)\mathbf{p}_2(0).$$

Solution: This is not an inner product. To explain why, we just need to show that it violates any of the four properties that an inner product must have. For example, the fourth one says

$$\langle \mathbf{p}_1, \mathbf{p}_1 \rangle = 0 \quad \text{if and only if} \quad \mathbf{p}_1 = \mathbf{0}.$$

This fails, as can be seen by choosing $\mathbf{p}_1(t) = t^2$. We have

$$\langle \mathbf{p}_1, \mathbf{p}_1 \rangle = (2)(0) = 0,$$

but $\mathbf{p}_1(t) \neq \mathbf{0}$. Alternatively, you could show that it fails the first property,

$$\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = \langle \mathbf{p}_2, \mathbf{p}_1 \rangle.$$

If we let $\mathbf{p}_1(t) = t$ and $\mathbf{p}_2(t) = t + 1$, then

$$\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = 1(1) = 1 \neq 0 = (1)(0) = \langle \mathbf{p}_2, \mathbf{p}_1 \rangle.$$

Note, however, that it does satisfy the second and third properties of an inner product (about distributing addition and scalar multiplication).

Question 8

(i) [**5 points**] Prove that, if A and B are $n \times n$ matrices and the columns of B are linearly dependent, then the columns of AB are linearly dependent.

Solution: We have a theorem that tells us that the columns of a matrix are linearly dependent if and only if the matrix is invertible, and that a matrix is invertible if and only if its determinant is nonzero. Thus, the columns are dependent if and only if the determinant is zero. Hence, $\det\, B = 0$. Also,

$$\det\, AB = \det\, A \det\, \mathrm{B} = \det\, A(0) = 0.$$

Therefore, the columns of AB must be dependent as well.

(ii) [**5 points**] Is it possible for a matrix to be diagonalizable but not invertible? Explain why or why not.

Solution: Yes, this is possible. For example, it the matrix is $n \times n$ and has n distinct eigenvalues, then we know the matrix is diagonalizable. However, if one of those eigenvalues is 0, the its determinant will be zero. This can be seen, for example, using

$$\det\, A = \det\, PDP^{-1} = \det\, P(\det\, D)\det\, P^{-1} = \det\, D = \text{product of diagonal entries} = 0,$$

since the diagonal entries of D are just the eigenvalues, one of which is zero. Hence, the matrix would not be invertible. An example of this would be an upper triangular matrix like

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{bmatrix},$$

which has eigenvalues $1, 0, 3$ and determinant $(1)(0)(3) = 0$.

(iii) [**5 points**] Let W be a subspace of $\mathbb{R}^n$. Prove that $W^\perp$ is a subspace of $\mathbb{R}^n$.
Solution: We check the three conditions of a subspace, using the definition

$$W^\perp = \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{v} \in W\}.$$

(a) $\mathbf{0} \cdot \mathbf{v} = 0$ for all $\mathbf{v} \in W$, so $\mathbf{0} \in W^\perp$.

(b) If $\mathbf{u}_1, \mathbf{u}_2 \in W^\perp$, then $(\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{v} = \mathbf{u}_1 \cdot \mathbf{v} + \mathbf{u}_2 \cdot \mathbf{v} = 0 + 0 = 0$, so $\mathbf{u}_1 + \mathbf{u}_2 \in W^\perp$.

(c) If $c \in \mathbb{R}$ and $\mathbf{u} \in W^\perp$, then $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = c(0) = 0$, so $c\mathbf{u} \in W^\perp$.

Thus, $W^\perp$ is a subspace.

(iv) [**5 points**] Let $T : V \to W$, be a linear transformation. Let $\mathcal{B} = \{\mathbf{b}_1, \dots \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots \mathbf{c}_m\}$ be bases for V and W, respectively. Let M be the matrix of the transformation relative to these bases. Prove that if rank $M = m$ then the transformation T is onto.
Solution: We need to show that, given any $\mathbf{w} \in W$, there exists a $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$. Let $[\mathbf{w}]_{\mathbf{C}} = \mathbf{y}$. The matrix M is $m \times n$ and has rank m, which means that it has a pivot in every row, hence the transformation $\mathbf{x} \to M\mathbf{x}$ is onto. Therefore, there is some $\mathbf{x} \in \mathbb{R}^n$ such that $M\mathbf{x} = \mathbf{y}$. Define $\mathbf{v} \in V$ to be the vector with coordinates $[\mathbf{v}]_{\mathcal{B}} = \mathbf{x}$. Then, we have $T(\mathbf{v}) = \mathbf{w}$ as needed. You can see this more explicitly by writing

$$M = \begin{bmatrix} M_{11} & \dots & M_{1n} \\ \vdots & \vdots & \vdots \\ M_{m1} & \dots & M_{mn} \end{bmatrix},$$

which implies

$$\begin{aligned} T(\mathbf{v}) &= T(x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n) = x_1T(\mathbf{b}_1) + \cdots + x_nT(\mathbf{b}_n) \\ &= x_1(M_{11}\mathbf{c}_1 + \cdots + M_{m1}\mathbf{c}_m) + \cdots + x_n(M_{1n}\mathbf{c}_1 + \cdots + M_{mn}\mathbf{c}_m) \\ &= \underbrace{(x_1M_{11} + \cdots + x_nM_{1n})}_{\text{1st row of } M\mathbf{x}}\mathbf{c}_1 + \cdots + \underbrace{(x_1M_{m1} + \cdots + x_nM_{mn})}_{\text{mth row of } M\mathbf{x}}\mathbf{c}_m \\ &= y_1\mathbf{c}_1 + \cdots + y_m\mathbf{c}_m \\ &= \mathbf{w}. \end{aligned}$$