Take Home Final, Due Monday, December 13, by Noon (No exceptions!!!)

Instructions: You may consult with your notes and any other books or references you like. You can also come ask me for hints if you get stuck. However, you may not consult with any other person.

1) (An alternate proof of the existence of solutions) Consider

$$\dot{x} = f(x,t), \quad x \in \mathbb{R}^n,$$

where f is Lipschitz in the region $R = \{(x,t) : |t - t_0| \le \alpha, |x - x_0| \le \beta\}$. Suppose $|f(x,t)| \le M$ on R and let $\delta = \min\{\alpha, \beta/M\}$. Prove that there exists a solution to the ODE with initial condition (x_0, t_0) that is defined on the interval $|t - t_0| \le \delta$ using the following steps.

(a) Define the sequence of approximate solutions

$$x^0(t) = x_0, \quad x^n(t) = x_0 + \int_{t_0}^{t} f(x^{n-1}(s), s)\mathrm{d}s, \quad n = 1, 2, 3, ...$$

Prove that, for all t such that $|t - t_0| \le \delta$, $|x^n(t) - x_0| \le \beta$ for all n. Thus, the approximations stay within R for such t.

(b) Show the sequence has a limit in the space of continuous functions, $C^0([t_0 - \delta, t_0 + \delta])$, with the sup norm. (You may use the fact that that function space is complete.)

(c) Show the limiting function is a solution to the ODE.

Note that this procedure also provides a way to approximate solutions.

(d) Apply the above approximation procedure to $\dot{x} = Ax$, where $A \in \mathbb{R}^{n \times n}$. What do you find?

2) The linearized Hill's equations for the relative motion of two satellites with respect to a circular reference orbit about the earth are given by

$$\begin{aligned} \ddot{x} - 2n\dot{y} - 3n^2 x &= 0, \\ \ddot{y} + 2n\dot{x} &= 0, \\ \ddot{z} + n^2 z &= 0, \end{aligned}$$

where n is a physical constant. There is a five-dimensional manifold in the phase space corresponding to periodic orbits. Determine this manifold explicitly and explain what happens to solutions that are not on it. (It is OK to work in $\mathbb{C}^6$ rather than $\mathbb{R}^6$.)

3) Consider the ODE

$$\dot{x} = x - y + e^{-t}, \quad \dot{y} = x + y + e^{-t}$$

(a) Use variation of parameters to write down the solution with initial condition (x_0, y_0) at $t = 0$.

(b) Fine the set of initial conditions such that the solution satisfies $(x(t; x_0), y(t; y_0)) \to (0, 0)$ as $t \to \infty$.

4) (a) Find a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that solutions to the scalar equation $\dot{x} = -x + f(t)$ are not bounded for all $t \ge 0$.

(b) Find a smooth function $f : \mathbb{R} \to \mathbb{R}^n$ such that there exists a solution to the n-dimensional equation $\dot{x} = Ax + f(t)$, where all eigenvalues of A have negative real part, that is not bounded for all $t \geq 0$.

(c) Prove that if the n-dimensional system $\dot{x} = Ax + f(t)$, for smooth $f : \mathbb{R} \to \mathbb{R}^n$ and A having eigenvalues with negative real part, has a solution that is bounded for all $t \geq 0$, then all solutions are bounded for all $t \geq 0$.

5) Consider the ODE

$$\dot{x} = Ax + B(t)x + g(x,t), \quad x \in \mathbb{R}^n.$$

Suppose that all eigenvalues of A have negative real part, the matrix $B(t)$ is continuous for all t with $\int_0^\infty \|B(t)\| \mathrm{d}t < \infty$, $g \in C^1(\mathbb{R} \times \mathbb{R}^+)$, and there exist constants $a, k > 0$ such that $|g(x,t)| \leq k|x|^2$ for all $t \geq 0$ and for all x such that $|x| \leq a$.

Prove that the origin is asymptotically stable using the following steps:

(a) First show that, given any $\epsilon > 0$ (sufficiently small), there exists a $\tau, \delta > 0$ such that, for all $x_0 \in B_\delta(0)$, the corresponding solution satisfies $|x(t;x_0)| \leq \epsilon$ for all $t \in [0, \tau]$.

(b) Next assume that, given a particular $x_0 \in B_\delta(0)$, T is the first time such that $|x(T;x_0)| = \epsilon < a$. Use the integral form of solutions and Gronwall's inequality to derive an estimate for $|x(t;x_0)|$ for all $t \in [0,T]$.

(c) Use your estimate to show that, in fact, $|x(T;x_0)| \leq \epsilon/2$ if ϵ and δ are chosen sufficiently small. Conclude that no such T exists, so $|x(t;x_0)| < \epsilon$ for all $t \geq 0$ if ϵ is small enough.

(d) Explain why the work you have done so far implies that the origin is not only stable, but also asymptotically stable.

Remark: If B were zero, this would just be asking you to show that a linearly asymptotically stable fixed point is also nonlinearly asymptotically stable, if the nonlinearity is sufficiently nice. However, we already know this is true from, for example, Hartman-Grobman or the invariant manifold theorem, at least in the case where g is independent of t. Thus, this problem is asking you to prove that nonautonomous (non)linear perturbations that are sufficiently nice also cannot destroy asymptotic stability of a fixed point.

6) Consider the ODE

$$\begin{aligned} \dot{x} &= \alpha - x^2 + xy \\ \dot{y} &= -2y + x^2 + y^2. \end{aligned}$$

(a) Use the theorem we proved using Lyapunov-Schmidt reduction to show that there is a saddle node bifurcation at $(x, y, \alpha) = (0, 0, 0)$.

(b) Using a Taylor series expansion, determine an approximate formula for the center manifold (up to and including all quadratic terms).

(c) Draw the phase plane near the origin for $\alpha > 0$, $\alpha = 0$, and $\alpha < 0$.

7) Consider the planar system

$$\begin{aligned} \dot{x} &= -y - xy + 2y^2 \\ \dot{y} &= x - x^2 y. \end{aligned}$$

Use the theory we have developed regarding Hopf bifurcations to determine whether or not the origin is asymptotically stable.