Homework Assignment 2, Due Tuesday, September 28

1) (Ex 2.19, Chicone) Find a fundamental matrix solution of

$$\dot{x} = \begin{pmatrix} 1 & -\frac{1}{t} \\ 1+t & -1 \end{pmatrix} x, \quad t > 0.$$

(Hint: one solution is $x(t) = (1, t)$.)

2) (Ex 2.30, Chicone) Consider the space $\mathcal{L}(\mathbb{C}^n)$ with the topology generated by the operator norm,

$$\|A\| = \sup_{|v|=1} |Av|.$$

Prove that the set of matrices with n distinct eigenvalues is open and dense. (A property that is defined on the countable intersection of open dense sets is called *generic*.)

3) (Ex 2.36, Chicone) (Laplace transform) Recall that the definition of the Laplace transform of a (perhaps matrix valued) function f is

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-s\tau} f(\tau) \mathrm{d}\tau.$$

(a) Prove that if A is an $n \times n$ matrix then

$$e^{tA} - I = \int_0^t A e^{\tau A} \mathrm{d}\tau.$$

(b) Prove that if all eigenvalues of A have negative real parts, then

$$-A^{-1} = \int_0^\infty e^{\tau A} \mathrm{d}\tau.$$

(c) Prove that if $s \in \mathbb{R}$ is sufficiently large, then

$$(sI - A)^{-1} = \int_0^\infty e^{-s\tau} e^{\tau A} \mathrm{d}\tau;$$

that is, the Laplace transform of e^{tA} is $(sI - A)^{-1}$.

(d) Solve the initial value problem $\dot{x} = Ax$, $x(0) = x_0$, using the method of Laplace transform: start by taking the Laplace transform of both sides of the equation, solve the resulting equation, then take the inverse Laplace transform.

4) (Ex 2.42, Chicone)

(a) Suppose A is an $n \times n$ matrix such that $A^2 = I$. Find an explicit formula for e^{tA}.

(b) Do the same if $A^2 = -I$.

(c) Solve

$$\dot{x} = \begin{pmatrix} 2 & -5 & 8 & -12 \\ 1 & -2 & 4 & -8 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 1 & -2 \end{pmatrix} x, \qquad x(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

5) (Ex 2.50, Chicone) Find a matrix function $t \to A(t)$ such that

$$t \to \exp\left(\int_0^t A(s)\mathrm{d}s\right)$$

is not a matrix solution of the system $\dot{x} = A(t)x$. Show that the given exponential formula is a solution in the scalar case. When is it a solution for the matrix case?

6) (Ex 2.59, Chicone) Consider the following ODE on the Banach space $E = C([0,1])$, the space of continuous functions on the interval $[0,1]$, equipped with the norm

$$\|f\| = \sup_{x\in[0,1]} |f(x)|.$$

Define the linear operator $U \in \mathcal{L}(E)$ as $(Uf)(x) = f(ax)$ for some fixed $a \in [0,1]$. Let $g \in E$ be the function $g(x) = bx$. Find the solution to the initial value problem

$$\dot{f} = Uf, \quad f_0 = g.$$