Homework Assignment 2, Due Tuesday, September 28

1) (Ex 2.19, Chicone) Find a fundamental matrix solution of
\[
\dot{x} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 1 + t & -1 \end{pmatrix} x, \quad t > 0.
\]
(Hint: one solution is \(x(t) = (1, t)\).)

2) (Ex 2.30, Chicone) Consider the space \(\mathcal{L}(\mathbb{C}^n)\) with the topology generated by the operator norm,
\[
\|A\| = \sup_{|v|=1} |Av|.
\]
Prove that the set of matrices with \(n\) distinct eigenvalues is open and dense. (A property that is defined on the countable intersection of open dense sets is called \textit{generic}.)

3) (Ex 2.36, Chicone) (Laplace transform) Recall that the definition of the Laplace transform of a (perhaps matrix valued) function \(f\) is
\[
\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t) dt.
\]
(a) Prove that if \(A\) is an \(n \times n\) matrix then
\[
e^{tA} - I = \int_0^t A e^{\tau A} d\tau.
\]
(b) Prove that if all eigenvalues of \(A\) have negative real parts, then
\[
-A^{-1} = \int_0^\infty e^{\tau A} d\tau.
\]
(c) Prove that if \(s \in \mathbb{R}\) is sufficiently large, then
\[
(sI - A)^{-1} = \int_0^\infty e^{-s\tau} e^{\tau A} d\tau;
\]
that is, the Laplace transform of \(e^{tA}\) is \((sI - A)^{-1}\).
(d) Solve the initial value problem \(\dot{x} = Ax, \ x(0) = x_0\), using the method of Laplace transform: start by taking the Laplace transform of both sides of the equation, solve the resulting equation, then take the inverse Laplace transform.

4) (Ex 2.42, Chicone)
(a) Suppose \(A\) is an \(n \times n\) matrix such that \(A^2 = I\). Find an explicit formula for \(e^{tA}\).
(b) Do the same if \(A^2 = -I\).
(c) Solve
\[
\dot{x} = \begin{pmatrix} 2 & -5 & 8 & -12 \\ 1 & -2 & 4 & -8 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 1 & -2 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\]
5) (Ex 2.50, Chicone) Find a matrix function \( t \to A(t) \) such that

\[
t \to \exp \left( \int_0^t A(s) \, ds \right)
\]

is not a matrix solution of the system \( \dot{x} = A(t)x \). Show that the given exponential formula is a solution in the scalar case. When is it a solution for the matrix case?

6) (Ex 2.59, Chicone) Consider the following ODE on the Banach space \( E = C([0,1]) \), the space of continuous functions on the interval \([0,1]\), equipped with the norm

\[
\|f\| = \sup_{x \in [0,1]} |f(x)|.
\]

Define the linear operator \( U \in \mathcal{L}(E) \) as \((U f)(x) = f(ax)\) for some fixed \( a \in [0,1] \). Let \( g \in E \) be the function \( g(x) = bx \). Find the solution to the initial value problem

\[
\dot{f} = U f, \quad f_0 = g.
\]