Homework Assignment 4, Due Thursday, October 21

1) Consider an ODE of the form

$$\dot{x} = Sx + F(x, y), \quad \dot{y} = Cy + G(x, y), \quad (x, y) \in \mathbb{R}^k \times \mathbb{R}^l \tag{0.1}$$

where all the eigenvalues of S have negative real part, all the eigenvalues of C have zero real part, and F, G satisfy the hypotheses of the invariant manifold theorem we discussed in class. Thus, the "unstable" invariant manifold guaranteed by the theorem is really a center manifold, given by

$$W^c(0, 0) = \{(x, y) : x = \alpha(y)\}.$$

(a) Prove that the function $\alpha(y)$ must satisfy the differential equation

$$S\alpha + F(\alpha, y) = D\alpha[Cy + G(\alpha, y)] \quad \forall y \in \mathbb{R}^l.$$

(In this equation $D\alpha$ means the derivative of α with respect to y – so it is a matrix.)

(b) Given an initial condition (x_0, y_0), let $(x(t, x_0), y(t, y_0))$ denote the corresponding solution to (0.1). Prove that there exist constants $C, \gamma > 0$ such that

$$|x(t, x_0) - \alpha(y(t, y_0))| \le Ce^{-\gamma t}|x_0 - \alpha(y_0)| \quad \forall t \ge 0.$$

(c) Explain in words why the dynamics of (0.1) when restricted to the center manifold are given by the equation

$$\dot{y} = Cy + G(\alpha(y), y). \tag{0.2}$$

Prove that, if $y = 0$ is an unstable fixed point of (0.2), then the origin is an unstable fixed point of (0.1). (Recall: a fixed point y_0 is said to be stable if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $|y_0| < \delta$ then $|y(t, y_0)| < \epsilon$ for all $t \ge 0$. A fixed point is unstable if it is not stable.)

It is also possible to prove that, if $y = 0$ is a stable fixed point of (0.2), then the origin is a stable fixed point of (0.1). Sometimes this is encapsulated in the statement: "The nonlinear stability of the fixed point at the origin is determined by the dynamics on the center manifold."

2) (Chicone Ex 4.3) This problem essentially asks you to apply the result of the previous exercise in a particular context. This technique is referred to as a "center manifold reduction."

(a) For the following system, find an explicit formula for the center manifold at the origin:

$$\begin{aligned} \dot{x} &= -x + y^2 - 2x^2 \\ \dot{y} &= \epsilon y - xy. \end{aligned}$$

(Hint: you know the center manifold is given by a function of the form $x = \alpha(y)$. Try assuming the function α has a Taylor series expansion. Use invariance of the center manifold to determine the coefficients in the expansion.)

(b) Determine the dynamics on the center manifold. Is the origin stable? How does its stability depend on ϵ? Sketch the (x, y) phase plane near the origin for different values of ϵ.