Homework Assignment 1, Due Thursday, October 1

1) Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is C^∞. Prove that

$$f(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + \mathcal{O}(|x|^{k+1}),$$

as $|x| \to 0$. This is Taylor's Theorem in multi-index notation. Note that, if $\alpha = (\alpha_1, \ldots, \alpha_n)$, then $\alpha! = \alpha_1! \cdots \alpha_n!$, and the $\mathcal{O}(|x|^{k+1})$ notation means that there exists a constant $C = C(r)$ such that

$$\left| f(x) - \sum_{|\alpha| \leq k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha \right| \leq C|x|^{k+1}$$

for all $x \in B(x, r)$. To prove this you may use facts about C^∞ functions on $\mathbb{R}$, including Talyor's Theorem for functions on $\mathbb{R}$.

2) (From Evans) Write down an explicit formula for a function u solving the initial value problem

$$\begin{aligned} u_t + b \cdot Du + cu &= 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t = 0\} \end{aligned}$$

where $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$ are given constants, and g is a given smooth function.

3) Given some open, bounded, connected domain U with smooth boundary, the Neumann problem for Laplace's equation is to solve

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } U \\ Du \cdot n &= g \quad \text{on } \partial U. \end{aligned}$$

where n is the outward normal vector on ∂U.

(a) Show that this problem can have a solution only if $\int_{\partial U} g \mathrm{d}S = 0$.

(b) If this problem has a solution, is it unique?