Homework Assignment 2, Due Oct 15

1) (From Evans) Prove that there exists a constant C, depending only on the dimension n, such that

$$\max_{B(0,1)} |u| \le C \left(\max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f| \right)$$

whenever u is a smooth solution of

$$\begin{aligned} -\Delta u &= f \quad \text{in } B^0(0,1) \\ u &= g \quad \text{on } \partial B(0,1). \end{aligned}$$

Hint: Consider splitting up the solution into two pieces: one that solves a homogeneous equation with nonzero boundary condition, and one that solves an inhomogeneous equation with zero boundary condition. You may also want to use the Green's function for Laplace's equation in a ball (which can be found in Evans).

2) Consider the initial value problem for the $n = 1$ heat equation. For general initial values $g \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$, one can show that

$$\|u(\cdot, t)\|_{L^\infty} \le \frac{C}{\sqrt{t}}.$$

Using the representation of solutions via the fundamental solution, prove that if $\int_{\mathbb{R}} g(y)\mathrm{d}y = 0$ and if $G(y) = \int_{-\infty}^{y} g(z)\mathrm{d}z \in L^1(\mathbb{R})$, then this can be improved to

$$\|u(\cdot, t)\|_{L^\infty} \le \frac{C}{t}.$$

3) Extend the class of self-similar solutions we found in the lecture by looking for additional solutions of the one dimensional heat equation of the form

$$u(x,t) = \frac{1}{t^\alpha} w\left(\frac{x}{\sqrt{t}}\right).$$

Hint: try to find solutions of this form with $w(\xi) = H(\xi)e^{-\xi^2/4}$, where $H(\xi)$ is a polynomial. The solution we found in lecture corresponded to the case $H(\xi) = 1$.

4) (From Evans) Write down an explicit formula for a solution of

$$\begin{aligned} u_t - \Delta u + cu &= f \quad \text{in } \mathbb{R}^n \times (0, \infty) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{0\}, \end{aligned}$$

where $c \in \mathbb{R}$ is a given constant.

5) (From Evans) Given $g : [0, \infty) \to \mathbb{R}$ with $g(0) = 0$, derive the formula

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s)\mathrm{d}s$$

for a solution of the initial/boundary value problem

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{in } \mathbb{R}_+ \times (0, \infty) \\ u &= 0 \quad \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u &= g \quad \text{on } \{x = 0\} \times [0, \infty). \end{aligned}$$

Hint: Let $v(x,t) = u(x,t) - g(t)$ and extend v to $\{x < 0\}$ by odd reflection.

6) Suppose that $f \in L^1(\mathbb{R})$ and $g \in C_c^\infty(\mathbb{R})$. Show that the convolution of f and g,

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\mathrm{d}y = \int_{-\infty}^{\infty} f(y)g(x-y)\mathrm{d}y$$

is in $C^\infty(\mathbb{R})$. Hint: Use the dominated convergence theorem to show that $f * g \in C^1(\mathbb{R})$ and then use induction.