Homework Assignment 3, Due Thursday, Oct 29

1) (From Evans: "Equipartition of Energy") Let $u \in C^2(\mathbb{R} \times [0,\infty))$ solve the initial value problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \quad \text{in } \mathbb{R} \times (0,\infty) \\ u = g,\ u_t &= h \quad \text{on } \mathbb{R} \times \{0\}. \end{aligned}$$

Suppose g and h have compact support. The kinetic and potential energies are, respectively,

$$k(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\mathrm{d}x, \qquad p(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\mathrm{d}x.$$

Prove that

(a) $k(t) + p(t)$ is constant in t.

(b) $k(t) = p(t)$ for all large enough times t.

2) (From Evans) Let u solve

$$\begin{aligned} u_{tt} - \Delta u &= 0 \quad \text{in } \mathbb{R}^3 \times (0,\infty) \\ u = g,\ u_t &= h \quad \text{on } \mathbb{R}^3 \times \{0\}, \end{aligned}$$

where g and h are smooth and have compact support. Show there exists a constant C such that

$$|u(x,t)| \le \frac{C}{t}, \qquad \forall \qquad x \in \mathbb{R}^3, \quad t > 0.$$

3) Let Ω be a bounded region in $\mathbb{R}^2$ with smooth boundary. The motion of a thin, vibrating plate with shape Ω and clamped edges is approximated by the equation

$$\begin{aligned} u_{tt} &= -\Delta^2 u \quad \text{in } \Omega \times (0,\infty) \\ u(x,t) = 0, \quad Du(x,t)\cdot\nu &= 0 \quad \text{on } \partial\Omega \times (0,\infty), \end{aligned}$$

where ν is the outward pointing normal vector on $\partial\Omega$. Show that if we specify initial conditions $u(x,0) = g(x)$ and $u_t(x,0) = h(x)$, then this problem has at most one solution. Hint: Try to find a conserved "energy" for this problem.

4) (From Evans) Solve the following equations using the method of characteristics:

(a) $x_1 u_{x_1} + x_2 u_{x_2} = 2u$, with boundary data $u(x_1, 1) = g(x_1)$.

(b) $u u_{x_1} + u_{x_2} = 1$, with boundary data $u(x_1, x_1) = x_1/2$.

(c) $x_1 u_{x_1} + 2x_2 u_{x_2} = 3u$, with boundary data $u(x_1, x_2, 0) = g(x_1, x_2)$.

Hint: for the last one, something goes wrong. Why?

5) (From Evans) Assume $F(0) = 0$, u is a continuous integral solution of the conservation law

$$\begin{aligned} u_t + F(u)_x &= 0 \quad \text{in } \mathbb{R} \times (0,\infty) \\ u &= g \quad \text{on } \mathbb{R} \times \{0\}, \end{aligned}$$

and u has compact support in $\mathbb{R} \times [0,\infty]$. Prove that, for all $t > 0$,

$$\int_{-\infty}^{\infty} u(x,t)\mathrm{d}x = \int_{-\infty}^{\infty} g(x)\mathrm{d}x.$$

6) (From Evans) Explicitly compute the unique entropy solution of

$$\begin{aligned} u_t + \left(\frac{u^2}{2}\right)_x &= 0 \qquad \text{in } \mathbb{R} \times (0,\infty) \\ u &= g \qquad \text{on } \mathbb{R} \times \{0\}, \end{aligned}$$

where

$$g(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } -1 < x < 0 \\ 2 & \text{if } 0 < x < 1 \\ 0 & \text{if } 1 < x. \end{cases}$$

Draw a picture illustrating your answer, being sure to illustrate what happens for all times $t > 0$.

7) The equation

$$u_t + uu_x = \nu u_{xx}$$

is called Burgers equation with viscosity, where $\nu > 0$ is the viscosity parameter. Show that if one defines $w(x,t)$ via

$$-2\nu \log w(x,t) = \int_{-\infty}^{x} u(y,t)\mathrm{d}y,$$

then w solves the one-dimensional heat equation $w_t = \nu w_{xx}$. (You can assume u is smooth with compact support.) This change of variables is known as the Cole-Hopf transformation. Use it to solve Burgers equation with viscosity for $u(x,0) = e^{-x^2}$. Plot the solution for several values of $\nu > 0$ that approach the limit $\nu = 0$. Describe how shocks are forming. What effect does the viscosity, ie the presence of the Laplacian u_{xx}, have? Does this make sense, based on what you know about the properties of solutions of the heat equation and Poisson's equation?