Homework Assignment 4, Due Tuesday, Nov 17

1) (From Evans) Let U and V be open sets with $V \subset\subset U$. Show there exists a smooth function ζ such that $\zeta \equiv 1$ on V and $\zeta = 0$ near ∂U. (Hint: Take $V \subset\subset W \subset\subset U$ and mollify the characteristic function on W.)

2) (From Evans) Assume U is bounded and $U \subset\subset \bigcup_{i=1}^N V_i$. Show there exist C^∞ functions ζ_i, $i = 1, \ldots, N$, such that
$$0 \leq \zeta_1 \leq 1, \qquad \operatorname{supp}(\zeta_i) \subset V_i, \qquad i = 1, \ldots, N$$
and
$$\sum_{i=1}^N \zeta_i = 1 \qquad \text{on} \qquad U.$$
The functions $\{\zeta_i\}_{i=1}^N$ form a partition of unity.

3) (From Evans) Prove directly that if $u \in W^{1,p}(0,1)$ for some $1 < p < \infty$, then
$$|u(x) - u(y)| \leq |x - y|^{1-1/p} \left(\int_0^1 |u'(z)|^p \mathrm{d}z \right)^{1/p}$$
for almost all $x, y \in [0,1]$. (Note that this shows that any $u \in W^{1,p}(0,1)$ is also Hölder continuous on $(0,1)$.)

4) Use the Fourier transform to solve
$$u_t = u_{xx} + cu_x, \qquad x \in \mathbb{R}, \qquad u(x,0) = f(x),$$
from a nonzero constant c. Your answer should be expressed in terms of a convolution integral involving the initial condition $f(x)$.