Homework Assignment 6, Due Wednesday, Dec 16, 930am
(Please put your solutions in my mailbox or slide them under my door by 930am on 12/16.)

Unless stated otherwise, in the following problems assume that $U \subset \mathbb{R}^n$ is open and bounded with smooth boundary and that the various coefficients appearing in the PDEs are smooth and such that the uniform ellipticity condition is satisfied.

1) (From Evans) Let
$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + cu.$$
Prove there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\cdot,\cdot]$ satisfies the hypotheses of the Lax-Milgram theorem, provide that $c(x) \geq -\mu$ for al $x \in U$.

2) (From Evans) A function $u \in H_0^2(U)$ is a weak solution of the following boundary value problem for the biharmonic equation
$$\begin{aligned} \Delta^2 u &= f \text{ in } U \\ u = \frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial U \end{aligned}$$
provided
$$\int_U \Delta u \Delta v \mathrm{d}x = \int_U f u \mathrm{d}x$$
for all $v \in H_0^2(U)$. Given $f \in L^2(U)$, prove there exists such a weak solution.

3) (From Evans) Assume U is connected. A function $u \in H^1(U)$ is a weak solution of Neumann's problem
$$\begin{aligned} -\Delta u &= f \text{ in } U \\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial U \end{aligned}$$
provided
$$\int_U Du \cdot Dv \mathrm{d}x = \int_U f u \mathrm{d}x$$
for all $v \in H^1(U)$. Suppose $f \in L^2(U)$. Prove that a weak solution exists if and only if
$$\int_U f \mathrm{d}x = 0.$$

4) Let $U = \{x \in \mathbb{R}^3 : |x| < \pi\}$. Show that a necessary condition for $-\Delta u - u = f$ to have a weak solution in $H_0^1(U)$ is that
$$\int_U f(x) \frac{\sin|x|}{|x|} \mathrm{d}x = 0.$$