

parallelogram. The number $|\mathbf{w}|\cos\theta$ is the parallelepiped's height. Because of this geometry, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is also called the **box product** of \mathbf{u} , \mathbf{v} , and \mathbf{w} .

By treating the planes of \mathbf{v} and \mathbf{w} and of \mathbf{w} and \mathbf{u} as the base planes of the parallelepiped determined by \mathbf{u} , \mathbf{v} , and \mathbf{w} , we see that

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}.$$

Since the dot product is commutative, we also have

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

The triple scalar product can be evaluated as a determinant:

$$\begin{aligned} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \left[\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \right] \cdot \mathbf{w} \\ &= w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \end{aligned}$$

Calculating the Triple Scalar Product as a Determinant

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

EXAMPLE 6 Find the volume of the box (parallelepiped) determined by $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{k}$, and $\mathbf{w} = 7\mathbf{j} - 4\mathbf{k}$.

Solution Using the rule for calculating a 3×3 determinant, we find

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = (1) \begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix} - (2) \begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix} + (-1) \begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix} = -23.$$

The volume is $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = 23$ units cubed. ■

Exercises 12.4

Cross Product Calculations

In Exercises 1–8, find the length and direction (when defined) of $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$.

- $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{k}$
- $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j}$
- $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
- $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{0}$
- $\mathbf{u} = 2\mathbf{i}$, $\mathbf{v} = -3\mathbf{j}$
- $\mathbf{u} = \mathbf{i} \times \mathbf{j}$, $\mathbf{v} = \mathbf{j} \times \mathbf{k}$

7. $\mathbf{u} = -8\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

8. $\mathbf{u} = \frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$

In Exercises 9–14, sketch the coordinate axes and then include the vectors \mathbf{u} , \mathbf{v} , and $\mathbf{u} \times \mathbf{v}$ as vectors starting at the origin.

9. $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{j}$

10. $\mathbf{u} = \mathbf{i} - \mathbf{k}$, $\mathbf{v} = \mathbf{j}$

11. $\mathbf{u} = \mathbf{i} - \mathbf{k}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$

12. $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$

13. $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$

14. $\mathbf{u} = \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = \mathbf{i}$

Triangles in Space

In Exercises 15–18,

- a. Find the area of the triangle determined by the points P , Q , and R .
 - b. Find a unit vector perpendicular to plane PQR .
15. $P(1, -1, 2)$, $Q(2, 0, -1)$, $R(0, 2, 1)$
 16. $P(1, 1, 1)$, $Q(2, 1, 3)$, $R(3, -1, 1)$
 17. $P(2, -2, 1)$, $Q(3, -1, 2)$, $R(3, -1, 1)$
 18. $P(-2, 2, 0)$, $Q(0, 1, -1)$, $R(-1, 2, -2)$

Triple Scalar Products

In Exercises 19–22, verify that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$ and find the volume of the parallelepiped (box) determined by \mathbf{u} , \mathbf{v} , and \mathbf{w} .

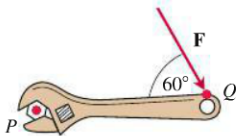
\mathbf{u}	\mathbf{v}	\mathbf{w}
19. $2\mathbf{i}$	$2\mathbf{j}$	$2\mathbf{k}$
20. $\mathbf{i} - \mathbf{j} + \mathbf{k}$	$2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$	$-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
21. $2\mathbf{i} + \mathbf{j}$	$2\mathbf{i} - \mathbf{j} + \mathbf{k}$	$\mathbf{i} + 2\mathbf{k}$
22. $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$	$-\mathbf{i} - \mathbf{k}$	$2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

Theory and Examples

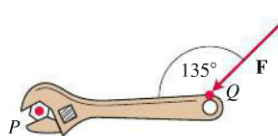
23. **Parallel and perpendicular vectors** Let $\mathbf{u} = 5\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{j} - 5\mathbf{k}$, $\mathbf{w} = -15\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$. Which vectors, if any, are (a) perpendicular? (b) Parallel? Give reasons for your answers.
24. **Parallel and perpendicular vectors** Let $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{w} = \mathbf{i} + \mathbf{k}$, $\mathbf{r} = -(\pi/2)\mathbf{i} - \pi\mathbf{j} + (\pi/2)\mathbf{k}$. Which vectors, if any, are (a) perpendicular? (b) Parallel? Give reasons for your answers.

In Exercises 25 and 26, find the magnitude of the torque exerted by \mathbf{F} on the bolt at P if $|\overrightarrow{PQ}| = 8$ in. and $|\mathbf{F}| = 30$ lb. Answer in foot-pounds.

25.



26.



27. Which of the following are *always true*, and which are *not always true*? Give reasons for your answers.
 - a. $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$
 - b. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|$
 - c. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
 - d. $\mathbf{u} \times (-\mathbf{u}) = \mathbf{0}$
 - e. $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$
 - f. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
 - g. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$
 - h. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$
28. Which of the following are *always true*, and which are *not always true*? Give reasons for your answers.
 - a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
 - b. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
 - c. $(-\mathbf{u}) \times \mathbf{v} = -(\mathbf{u} \times \mathbf{v})$

- d. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$ (any number c)
- e. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$ (any number c)
- f. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
- g. $(\mathbf{u} \times \mathbf{u}) \cdot \mathbf{u} = 0$
- h. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$

29. Given nonzero vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} , use dot product and cross product notation, as appropriate, to describe the following.
 - a. The vector projection of \mathbf{u} onto \mathbf{v}
 - b. A vector orthogonal to \mathbf{u} and \mathbf{v}
 - c. A vector orthogonal to $\mathbf{u} \times \mathbf{v}$ and \mathbf{w}
 - d. The volume of the parallelepiped determined by \mathbf{u} , \mathbf{v} , and \mathbf{w}
 - e. A vector orthogonal to $\mathbf{u} \times \mathbf{v}$ and $\mathbf{u} \times \mathbf{w}$
 - f. A vector of length $|\mathbf{u}|$ in the direction of \mathbf{v}
30. Compute $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j}$ and $\mathbf{i} \times (\mathbf{j} \times \mathbf{j})$. What can you conclude about the associativity of the cross product?
31. Let \mathbf{u} , \mathbf{v} , and \mathbf{w} be vectors. Which of the following make sense, and which do not? Give reasons for your answers.
 - a. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
 - b. $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$
 - c. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$
 - d. $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$
32. **Cross products of three vectors** Show that except in degenerate cases, $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ lies in the plane of \mathbf{u} and \mathbf{v} , whereas $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ lies in the plane of \mathbf{v} and \mathbf{w} . What *are* the degenerate cases?
33. **Cancellation in cross products** If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \neq \mathbf{0}$, then does $\mathbf{v} = \mathbf{w}$? Give reasons for your answer.
34. **Double cancellation** If $\mathbf{u} \neq \mathbf{0}$ and if $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then does $\mathbf{v} = \mathbf{w}$? Give reasons for your answer.

Area of a Parallelogram

Find the areas of the parallelograms whose vertices are given in Exercises 35–40.

35. $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$, $D(0, -1)$
36. $A(0, 0)$, $B(7, 3)$, $C(9, 8)$, $D(2, 5)$
37. $A(-1, 2)$, $B(2, 0)$, $C(7, 1)$, $D(4, 3)$
38. $A(-6, 0)$, $B(1, -4)$, $C(3, 1)$, $D(-4, 5)$
39. $A(0, 0, 0)$, $B(3, 2, 4)$, $C(5, 1, 4)$, $D(2, -1, 0)$
40. $A(1, 0, -1)$, $B(1, 7, 2)$, $C(2, 4, -1)$, $D(0, 3, 2)$

Area of a Triangle

Find the areas of the triangles whose vertices are given in Exercises 41–47.

41. $A(0, 0)$, $B(-2, 3)$, $C(3, 1)$
42. $A(-1, -1)$, $B(3, 3)$, $C(2, 1)$
43. $A(-5, 3)$, $B(1, -2)$, $C(6, -2)$
44. $A(-6, 0)$, $B(10, -5)$, $C(-2, 4)$
45. $A(1, 0, 0)$, $B(0, 2, 0)$, $C(0, 0, -1)$
46. $A(0, 0, 0)$, $B(-1, 1, -1)$, $C(3, 0, 3)$
47. $A(1, -1, 1)$, $B(0, 1, 1)$, $C(1, 0, -1)$