

Whenever it is correctly defined, the composite of continuous functions is also continuous. The only requirement is that each function be continuous where it is applied. The proof, omitted here, is similar to that for functions of a single variable (Theorem 9 in Section 2.5).

Continuity of Composites

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0) .

For example, the composite functions

$$e^{x-y}, \quad \cos \frac{xy}{x^2 + 1}, \quad \ln(1 + x^2y^2)$$

are continuous at every point (x, y) .

Functions of More Than Two Variables

The definitions of limit and continuity for functions of two variables and the conclusions about limits and continuity for sums, products, quotients, powers, and composites all extend to functions of three or more variables. Functions like

$$\ln(x + y + z) \quad \text{and} \quad \frac{y \sin z}{x - 1}$$

are continuous throughout their domains, and limits like

$$\lim_{P \rightarrow (1,0,-1)} \frac{e^{x+z}}{z^2 + \cos \sqrt{xy}} = \frac{e^{1-1}}{(-1)^2 + \cos 0} = \frac{1}{2},$$

where P denotes the point (x, y, z) , may be found by direct substitution.

Extreme Values of Continuous Functions on Closed, Bounded Sets

The Extreme Value Theorem (Theorem 1, Section 4.1) states that a function of a single variable that is continuous throughout a closed, bounded interval $[a, b]$ takes on an absolute maximum value and an absolute minimum value at least once in $[a, b]$. The same holds true of a function $z = f(x, y)$ that is continuous on a closed, bounded set R in the plane (like a line segment, a disk, or a filled-in triangle). The function takes on an absolute maximum value at some point in R and an absolute minimum value at some point in R . The function may take on a maximum or minimum value more than once over R .

Similar results hold for functions of three or more variables. A continuous function $w = f(x, y, z)$, for example, must take on absolute maximum and minimum values on any closed, bounded set (solid ball or cube, spherical shell, rectangular solid) on which it is defined. We will learn how to find these extreme values in Section 14.7.

Exercises 14.2

Limits with Two Variables

Find the limits in Exercises 1–12.

1. $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$

2. $\lim_{(x,y) \rightarrow (0,4)} \frac{x}{\sqrt{y}}$

3. $\lim_{(x,y) \rightarrow (3,4)} \sqrt{x^2 + y^2 - 1}$

4. $\lim_{(x,y) \rightarrow (2,-3)} \left(\frac{1}{x} + \frac{1}{y} \right)^2$

5. $\lim_{(x,y) \rightarrow (0,\pi/4)} \sec x \tan y$

6. $\lim_{(x,y) \rightarrow (0,0)} \cos \frac{x^2 + y^3}{x + y + 1}$

7. $\lim_{(x,y) \rightarrow (0, \ln 2)} e^{x-y}$ 8. $\lim_{(x,y) \rightarrow (1,1)} \ln |1 + x^2 y^2|$
9. $\lim_{(x,y) \rightarrow (0,0)} \frac{e^y \sin x}{x}$ 10. $\lim_{(x,y) \rightarrow (1/27, \pi^3)} \cos \sqrt[3]{xy}$
11. $\lim_{(x,y) \rightarrow (1, \pi/6)} \frac{x \sin y}{x^2 + 1}$ 12. $\lim_{(x,y) \rightarrow (\pi/2, 0)} \frac{\cos y + 1}{y - \sin x}$

Limits of Quotients

Find the limits in Exercises 13–24 by rewriting the fractions first.

13. $\lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq y}} \frac{x^2 - 2xy + y^2}{x - y}$ 14. $\lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq y}} \frac{x^2 - y^2}{x - y}$
15. $\lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq 1}} \frac{xy - y - 2x + 2}{x - 1}$
16. $\lim_{\substack{(x,y) \rightarrow (2,-4) \\ x \neq -4, x \neq x^2}} \frac{y + 4}{x^2 y - xy + 4x^2 - 4x}$
17. $\lim_{\substack{(x,y) \rightarrow (0,0) \\ x \neq y}} \frac{x - y + 2\sqrt{x} - 2\sqrt{y}}{\sqrt{x} - \sqrt{y}}$
18. $\lim_{\substack{(x,y) \rightarrow (2,2) \\ x+y \neq 4}} \frac{x + y - 4}{\sqrt{x} + y - 2}$ 19. $\lim_{\substack{(x,y) \rightarrow (2,0) \\ 2x-y \neq 4}} \frac{\sqrt{2x-y} - y - 2}{2x - y - 4}$
20. $\lim_{\substack{(x,y) \rightarrow (4,3) \\ x \neq y+1}} \frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1}$
21. $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}$ 22. $\lim_{(x,y) \rightarrow (0,0)} \frac{1 - \cos(xy)}{xy}$
23. $\lim_{(x,y) \rightarrow (1,-1)} \frac{x^3 + y^3}{x + y}$ 24. $\lim_{(x,y) \rightarrow (2,2)} \frac{x - y}{x^4 - y^4}$

Limits with Three Variables

Find the limits in Exercises 25–30.

25. $\lim_{P \rightarrow (1,3,4)} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)$ 26. $\lim_{P \rightarrow (1,-1,-1)} \frac{2xy + yz}{x^2 + z^2}$
27. $\lim_{P \rightarrow (\pi, \pi, 0)} (\sin^2 x + \cos^2 y + \sec^2 z)$
28. $\lim_{P \rightarrow (-1/4, \pi/2, 2)} \tan^{-1} xyz$ 29. $\lim_{P \rightarrow (\pi, 0, 3)} ze^{-2y} \cos 2x$
30. $\lim_{P \rightarrow (2, -3, 6)} \ln \sqrt{x^2 + y^2 + z^2}$

Continuity for Two VariablesAt what points (x, y) in the plane are the functions in Exercises 31–34 continuous?

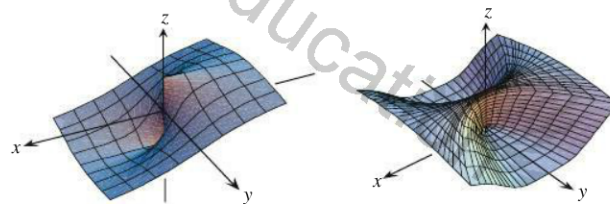
31. a. $f(x, y) = \sin(x + y)$ b. $f(x, y) = \ln(x^2 + y^2)$
32. a. $f(x, y) = \frac{x + y}{x - y}$ b. $f(x, y) = \frac{y}{x^2 + 1}$
33. a. $g(x, y) = \sin \frac{1}{xy}$ b. $g(x, y) = \frac{x + y}{2 + \cos x}$
34. a. $g(x, y) = \frac{x^2 + y^2}{x^2 - 3x + 2}$ b. $g(x, y) = \frac{1}{x^2 - y}$

Continuity for Three VariablesAt what points (x, y, z) in space are the functions in Exercises 35–40 continuous?

35. a. $f(x, y, z) = x^2 + y^2 - 2z^2$
b. $f(x, y, z) = \sqrt{x^2 + y^2 - 1}$
36. a. $f(x, y, z) = \ln xyz$ b. $f(x, y, z) = e^{x+y} \cos z$
37. a. $h(x, y, z) = xy \sin \frac{1}{z}$ b. $h(x, y, z) = \frac{1}{x^2 + z^2 - 1}$
38. a. $h(x, y, z) = \frac{1}{|y| + |z|}$ b. $h(x, y, z) = \frac{1}{|xy| + |z|}$
39. a. $h(x, y, z) = \ln(z - x^2 - y^2 - 1)$
b. $h(x, y, z) = \frac{1}{z - \sqrt{x^2 + y^2}}$
40. a. $h(x, y, z) = \sqrt{4 - x^2 - y^2 - z^2}$
b. $h(x, y, z) = \frac{1}{4 - \sqrt{x^2 + y^2 + z^2 - 9}}$

No Limit Exists at the OriginBy considering different paths of approach, show that the functions in Exercises 41–48 have no limit as $(x, y) \rightarrow (0, 0)$.

41. $f(x, y) = \frac{x}{\sqrt{x^2 + y^2}}$ 42. $f(x, y) = \frac{x^4}{x^4 + y^2}$



43. $f(x, y) = \frac{x^4 - y^2}{x^4 + y^2}$ 44. $f(x, y) = \frac{xy}{|xy|}$
45. $g(x, y) = \frac{x - y}{x + y}$ 46. $g(x, y) = \frac{x^2 - y}{x - y}$
47. $h(x, y) = \frac{x^2 + y}{y}$ 48. $h(x, y) = \frac{x^2 y}{x^4 + y^2}$

Theory and Examples

In Exercises 49 and 50, show that the limits do not exist.

49. $\lim_{(x,y) \rightarrow (1,1)} \frac{xy^2 - 1}{y - 1}$ 50. $\lim_{(x,y) \rightarrow (1,-1)} \frac{xy + 1}{x^2 - y^2}$

51. Let $f(x, y) = \begin{cases} 1, & y \geq x^4 \\ 1, & y \leq 0 \\ 0, & \text{otherwise.} \end{cases}$

Find each of the following limits, or explain that the limit does not exist.

- a. $\lim_{(x,y) \rightarrow (0,1)} f(x, y)$
b. $\lim_{(x,y) \rightarrow (2,3)} f(x, y)$
c. $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$

52. Let $f(x, y) = \begin{cases} x^2, & x \geq 0 \\ x^3, & x < 0 \end{cases}$.

Find the following limits.

a. $\lim_{(x,y) \rightarrow (3,-2)} f(x, y)$

b. $\lim_{(x,y) \rightarrow (-2, 1)} f(x, y)$

c. $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$

53. Show that the function in Example 6 has limit 0 along every straight line approaching (0, 0).

54. If $f(x_0, y_0) = 3$, what can you say about

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$$

if f is continuous at (x_0, y_0) ? If f is not continuous at (x_0, y_0) ? Give reasons for your answers.

The Sandwich Theorem for functions of two variables states that if $g(x, y) \leq f(x, y) \leq h(x, y)$ for all $(x, y) \neq (x_0, y_0)$ in a disk centered at (x_0, y_0) and if g and h have the same finite limit L as $(x, y) \rightarrow (x_0, y_0)$, then

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = L.$$

Use this result to support your answers to the questions in Exercises 55–58.

55. Does knowing that

$$1 - \frac{x^2y^2}{3} < \frac{\tan^{-1}xy}{xy} < 1$$

tell you anything about

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\tan^{-1}xy}{xy}?$$

Give reasons for your answer.

56. Does knowing that

$$2|xy| - \frac{x^2y^2}{6} < 4 - 4 \cos \sqrt{|xy|} < 2|xy|$$

tell you anything about

$$\lim_{(x,y) \rightarrow (0,0)} \frac{4 - 4 \cos \sqrt{|xy|}}{|xy|}?$$

Give reasons for your answer.

57. Does knowing that $|\sin(1/x)| \leq 1$ tell you anything about

$$\lim_{(x,y) \rightarrow (0,0)} y \sin \frac{1}{x}?$$

Give reasons for your answer.

58. Does knowing that $|\cos(1/y)| \leq 1$ tell you anything about

$$\lim_{(x,y) \rightarrow (0,0)} x \cos \frac{1}{y}?$$

Give reasons for your answer.

59. (Continuation of Example 5.)

a. Reread Example 5. Then substitute $m = \tan \theta$ into the formula

$$f(x, y) \Big|_{y=mx} = \frac{2m}{1+m^2}$$

and simplify the result to show how the value of f varies with the line's angle of inclination.

b. Use the formula you obtained in part (a) to show that the limit of f as $(x, y) \rightarrow (0, 0)$ along the line $y = mx$ varies from -1 to 1 depending on the angle of approach.

60. **Continuous extension** Define $f(0, 0)$ in a way that extends

$$f(x, y) = xy \frac{x^2 - y^2}{x^2 + y^2}$$

to be continuous at the origin.

Changing Variables to Polar Coordinates

If you cannot make any headway with $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ in rectangular coordinates, try changing to polar coordinates. Substitute $x = r \cos \theta$, $y = r \sin \theta$, and investigate the limit of the resulting expression as $r \rightarrow 0$. In other words, try to decide whether there exists a number L satisfying the following criterion:

Given $\epsilon > 0$, there exists a $\delta > 0$ such that for all r and θ ,

$$|r| < \delta \implies |f(r, \theta) - L| < \epsilon. \quad (1)$$

If such an L exists, then

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{r \rightarrow 0} f(r \cos \theta, r \sin \theta) = L.$$

For instance,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^3 \cos^3 \theta}{r^2} = \lim_{r \rightarrow 0} r \cos^3 \theta = 0.$$

To verify the last of these equalities, we need to show that Equation (1) is satisfied with $f(r, \theta) = r \cos^3 \theta$ and $L = 0$. That is, we need to show that given any $\epsilon > 0$, there exists a $\delta > 0$ such that for all r and θ ,

$$|r| < \delta \implies |r \cos^3 \theta - 0| < \epsilon.$$

Since

$$|r \cos^3 \theta| = |r| |\cos^3 \theta| \leq |r| \cdot 1 = |r|,$$

the implication holds for all r and θ if we take $\delta = \epsilon$.

In contrast,

$$\frac{x^2}{x^2 + y^2} = \frac{r^2 \cos^2 \theta}{r^2} = \cos^2 \theta$$

takes on all values from 0 to 1 regardless of how small $|r|$ is, so that $\lim_{(x,y) \rightarrow (0,0)} x^2/(x^2 + y^2)$ does not exist.

In each of these instances, the existence or nonexistence of the limit as $r \rightarrow 0$ is fairly clear. Shifting to polar coordinates does not always help, however, and may even tempt us to false conclusions. For example, the limit may exist along every straight line (or ray) $\theta = \text{constant}$ and yet fail to exist in the broader sense. Example 5 illustrates this point. In polar coordinates, $f(x, y) = (2x^2y)/(x^4 + y^2)$ becomes

$$f(r \cos \theta, r \sin \theta) = \frac{r \cos \theta \sin 2\theta}{r^2 \cos^4 \theta + \sin^2 \theta}$$

for $r \neq 0$. If we hold θ constant and let $r \rightarrow 0$, the limit is 0. On the path $y = x^2$, however, we have $r \sin \theta = r^2 \cos^2 \theta$ and

$$\begin{aligned} f(r \cos \theta, r \sin \theta) &= \frac{r \cos \theta \sin 2\theta}{r^2 \cos^4 \theta + (r \cos^2 \theta)^2} \\ &= \frac{2r \cos^2 \theta \sin \theta}{2r^2 \cos^4 \theta} = \frac{r \sin \theta}{r^2 \cos^2 \theta} = 1. \end{aligned}$$

In Exercises 61–66, find the limit of f as $(x, y) \rightarrow (0, 0)$ or show that the limit does not exist.

61. $f(x, y) = \frac{x^3 - xy^2}{x^2 + y^2}$ 62. $f(x, y) = \cos\left(\frac{x^3 - y^3}{x^2 + y^2}\right)$

63. $f(x, y) = \frac{y^2}{x^2 + y^2}$ 64. $f(x, y) = \frac{2x}{x^2 + x + y^2}$

65. $f(x, y) = \tan^{-1}\left(\frac{|x| + |y|}{x^2 + y^2}\right)$

66. $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$

In Exercises 67 and 68, define $f(0, 0)$ in a way that extends f to be continuous at the origin.

67. $f(x, y) = \ln\left(\frac{3x^2 - x^2y^2 + 3y^2}{x^2 + y^2}\right)$

68. $f(x, y) = \frac{3x^2y}{x^2 + y^2}$

Using the Limit Definition

Each of Exercises 69–74 gives a function $f(x, y)$ and a positive number ϵ . In each exercise, show that there exists a $\delta > 0$ such that for all (x, y) ,

$$\sqrt{x^2 + y^2} < \delta \implies |f(x, y) - f(0, 0)| < \epsilon.$$

69. $f(x, y) = x^2 + y^2$, $\epsilon = 0.01$

70. $f(x, y) = y/(x^2 + 1)$, $\epsilon = 0.05$

71. $f(x, y) = (x + y)/(x^2 + 1)$, $\epsilon = 0.01$

72. $f(x, y) = (x + y)/(2 + \cos x)$, $\epsilon = 0.02$

73. $f(x, y) = \frac{xy^2}{x^2 + y^2}$ and $f(0, 0) = 0$, $\epsilon = 0.04$

74. $f(x, y) = \frac{x^3 + y^4}{x^2 + y^2}$ and $f(0, 0) = 0$, $\epsilon = 0.02$

Each of Exercises 75–78 gives a function $f(x, y, z)$ and a positive number ϵ . In each exercise, show that there exists a $\delta > 0$ such that for all (x, y, z) ,

$$\sqrt{x^2 + y^2 + z^2} < \delta \implies |f(x, y, z) - f(0, 0, 0)| < \epsilon.$$

75. $f(x, y, z) = x^2 + y^2 + z^2$, $\epsilon = 0.015$

76. $f(x, y, z) = xyz$, $\epsilon = 0.008$

77. $f(x, y, z) = \frac{x + y + z}{x^2 + y^2 + z^2 + 1}$, $\epsilon = 0.015$

78. $f(x, y, z) = \tan^2 x + \tan^2 y + \tan^2 z$, $\epsilon = 0.03$

79. Show that $f(x, y, z) = x + y - z$ is continuous at every point (x_0, y_0, z_0) .

80. Show that $f(x, y, z) = x^2 + y^2 + z^2$ is continuous at the origin.

14.3 Partial Derivatives

The calculus of several variables is similar to single-variable calculus applied to several variables one at a time. When we hold all but one of the independent variables of a function constant and differentiate with respect to that one variable, we get a “partial” derivative. This section shows how partial derivatives are defined and interpreted geometrically, and how to calculate them by applying the rules for differentiating functions of a single variable. The idea of *differentiability* for functions of several variables requires more than the existence of the partial derivatives because a point can be approached from so many different directions. However, we will see that differentiable functions of several variables behave in the same way as differentiable single-variable functions, so they are continuous and can be well approximated by linear functions.

Partial Derivatives of a Function of Two Variables

If (x_0, y_0) is a point in the domain of a function $f(x, y)$, the vertical plane $y = y_0$ will cut the surface $z = f(x, y)$ in the curve $z = f(x, y_0)$ (Figure 14.16). This curve is the graph of the function $z = f(x, y_0)$ in the plane $y = y_0$. The horizontal coordinate in this plane is x ; the vertical coordinate is z . The y -value is held constant at y_0 , so y is not a variable.

We define the partial derivative of f with respect to x at the point (x_0, y_0) as the ordinary derivative of $f(x, y_0)$ with respect to x at the point $x = x_0$. To distinguish partial derivatives from ordinary derivatives we use the symbol ∂ rather than the d previously used. In the definition, h represents a real number, positive or negative.