Exercises 14.7

Finding Local Extrema

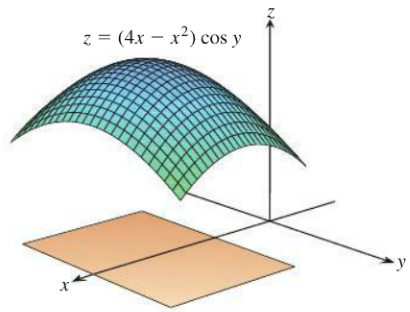
Find all the local maxima, local minima, and saddle points of the functions in Exercises 1–30.

1. $f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4$
2. $f(x, y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$
3. $f(x, y) = x^2 + xy + 3x + 2y + 5$
4. $f(x, y) = 5xy - 7x^2 + 3x - 6y + 2$
5. $f(x, y) = 2xy - x^2 - 2y^2 + 3x + 4$
6. $f(x, y) = x^2 - 4xy + y^2 + 6y + 2$
7. $f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$
8. $f(x, y) = x^2 - 2xy + 2y^2 - 2x + 2y + 1$
9. $f(x, y) = x^2 - y^2 - 2x + 4y + 6$
10. $f(x, y) = x^2 + 2xy$
11. $f(x, y) = \sqrt{56x^2 - 8y^2 - 16x - 31} + 1 - 8x$
12. $f(x, y) = 1 - \sqrt[3]{x^2 + y^2}$
13. $f(x, y) = x^3 - y^3 - 2xy + 6$
14. $f(x, y) = x^3 + 3xy + y^3$
15. $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$
16. $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$
17. $f(x, y) = x^3 + 3xy^2 - 15x + y^3 - 15y$
18. $f(x, y) = 2x^3 + 2y^3 - 9x^2 + 3y^2 - 12y$
19. $f(x, y) = 4xy - x^4 - y^4$
20. $f(x, y) = x^4 + y^4 + 4xy$
21. $f(x, y) = \dfrac{1}{x^2 + y^2 - 1}$
22. $f(x, y) = \dfrac{1}{x} + xy + \dfrac{1}{y}$
23. $f(x, y) = y \sin x$
24. $f(x, y) = e^{2x} \cos y$
25. $f(x, y) = e^{x^2+y^2-4x}$
26. $f(x, y) = e^y - ye^x$
27. $f(x, y) = e^{-y}(x^2 + y^2)$
28. $f(x, y) = e^x(x^2 - y^2)$
29. $f(x, y) = 2 \ln x + \ln y - 4x - y$
30. $f(x, y) = \ln(x + y) + x^2 - y$

Finding Absolute Extrema

In Exercises 31–38, find the absolute maxima and minima of the functions on the given domains.

31. $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0, y = 2, y = 2x$ in the first quadrant
32. $D(x, y) = x^2 - xy + y^2 + 1$ on the closed triangular plate in the first quadrant bounded by the lines $x = 0, y = 4, y = x$
33. $f(x, y) = x^2 + y^2$ on the closed triangular plate bounded by the lines $x = 0, y = 0, y + 2x = 2$ in the first quadrant
34. $T(x, y) = x^2 + xy + y^2 - 6x$ on the rectangular plate $0 \le x \le 5, -3 \le y \le 3$
35. $T(x, y) = x^2 + xy + y^2 - 6x + 2$ on the rectangular plate $0 \le x \le 5, -3 \le y \le 0$
36. $f(x, y) = 48xy - 32x^3 - 24y^2$ on the rectangular plate $0 \le x \le 1, 0 \le y \le 1$
37. $f(x, y) = (4x - x^2) \cos y$ on the rectangular plate $1 \le x \le 3$, $-\pi/4 \le y \le \pi/4$ (see accompanying figure)



38. $f(x, y) = 4x - 8xy + 2y + 1$ on the triangular plate bounded by the lines $x = 0, y = 0, x + y = 1$ in the first quadrant
39. Find two numbers a and b with $a \le b$ such that
$$\int_a^b (6 - x - x^2)\, dx$$
has its largest value.
40. Find two numbers a and b with $a \le b$ such that
$$\int_a^b (24 - 2x - x^2)^{1/3}\, dx$$
has its largest value.
41. **Temperatures** A flat circular plate has the shape of the region $x^2 + y^2 \le 1$. The plate, including the boundary where $x^2 + y^2 = 1$, is heated so that the temperature at the point (x, y) is
$$T(x, y) = x^2 + 2y^2 - x.$$
Find the temperatures at the hottest and coldest points on the plate.
42. Find the critical point of
$$f(x, y) = xy + 2x - \ln x^2y$$
in the open first quadrant $(x > 0, y > 0)$ and show that f takes on a minimum there.

Theory and Examples

43. Find the maxima, minima, and saddle points of $f(x, y)$, if any, given that
    a. $f_x = 2x - 4y$ and $f_y = 2y - 4x$
    b. $f_x = 2x - 2$ and $f_y = 2y - 4$
    c. $f_x = 9x^2 - 9$ and $f_y = 2y + 4$

    Describe your reasoning in each case.
44. The discriminant $f_{xx}f_{yy} - f_{xy}{}^2$ is zero at the origin for each of the following functions, so the Second Derivative Test fails there. Determine whether the function has a maximum, a minimum, or neither at the origin by imagining what the surface $z = f(x, y)$ looks like. Describe your reasoning in each case.
    a. $f(x, y) = x^2y^2$
    b. $f(x, y) = 1 - x^2y^2$
    c. $f(x, y) = xy^2$
    d. $f(x, y) = x^3y^2$
    e. $f(x, y) = x^3y^3$
    f. $f(x, y) = x^4y^4$

45. Show that (0, 0) is a critical point of $f(x, y) = x^2 + kxy + y^2$ no matter what value the constant k has. (*Hint:* Consider two cases: $k = 0$ and $k \neq 0$.)

46. For what values of the constant k does the Second Derivative Test guarantee that $f(x, y) = x^2 + kxy + y^2$ will have a saddle point at (0, 0)? A local minimum at (0, 0)? For what values of k is the Second Derivative Test inconclusive? Give reasons for your answers.

47. If $f_x(a, b) = f_y(a, b) = 0$, must f have a local maximum or minimum value at (a, b)? Give reasons for your answer.

48. Can you conclude anything about $f(a, b)$ if f and its first and second partial derivatives are continuous throughout a disk centered at the critical point (a, b) and $f_{xx}(a, b)$ and $f_{yy}(a, b)$ differ in sign? Give reasons for your answer.

49. Among all the points on the graph of $z = 10 - x^2 - y^2$ that lie above the plane $x + 2y + 3z = 0$, find the point farthest from the plane.

50. Find the point on the graph of $z = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0$.

51. Find the point on the plane $3x + 2y + z = 6$ that is nearest the origin.

52. Find the minimum distance from the point $(2, -1, 1)$ to the plane $x + y - z = 2$.

53. Find three numbers whose sum is 9 and whose sum of squares is a minimum.

54. Find three positive numbers whose sum is 3 and whose product is a maximum.

55. Find the maximum value of $s = xy + yz + xz$ where $x + y + z = 6$.

56. Find the minimum distance from the cone $z = \sqrt{x^2 + y^2}$ to the point $(-6, 4, 0)$.

57. Find the dimensions of the rectangular box of maximum volume that can be inscribed inside the sphere $x^2 + y^2 + z^2 = 4$.

58. Among all closed rectangular boxes of volume 27 cm^3, what is the smallest surface area?

59. You are to construct an open rectangular box from 12 ft^2 of material. What dimensions will result in a box of maximum volume?

60. Consider the function $f(x, y) = x^2 + y^2 + 2xy - x - y + 1$ over the square $0 \le x \le 1$ and $0 \le y \le 1$.

a. Show that f has an absolute minimum along the line segment $2x + 2y = 1$ in this square. What *is* the absolute minimum value?

b. Find the absolute maximum value of f over the square.

Extreme Values on Parametrized Curves To find the extreme values of a function $f(x, y)$ on a curve $x = x(t)$, $y = y(t)$, we treat f as a function of the single variable t and use the Chain Rule to find where df/dt is zero. As in any other single-variable case, the extreme values of f are then found among the values at the

a. critical points (points where df/dt is zero or fails to exist), and

b. endpoints of the parameter domain.

Find the absolute maximum and minimum values of the following functions on the given curves.

61. Functions:

a. $f(x, y) = x + y$ **b.** $g(x, y) = xy$ **c.** $h(x, y) = 2x^2 + y^2$

Curves:

i) The semicircle $x^2 + y^2 = 4, \quad y \ge 0$

ii) The quarter circle $x^2 + y^2 = 4, \quad x \ge 0, \quad y \ge 0$

Use the parametric equations $x = 2\cos t$, $y = 2\sin t$.

62. Functions:

a. $f(x, y) = 2x + 3y$

b. $g(x, y) = xy$

c. $h(x, y) = x^2 + 3y^2$

Curves:

i) The semiellipse $(x^2/9) + (y^2/4) = 1, \quad y \ge 0$

ii) The quarter ellipse $(x^2/9) + (y^2/4) = 1, \quad x \ge 0, \quad y \ge 0$

Use the parametric equations $x = 3\cos t$, $y = 2\sin t$.

63. Function: $f(x, y) = xy$

Curves:

i) The line $x = 2t, \quad y = t + 1$

ii) The line segment $x = 2t, \quad y = t + 1, \quad -1 \le t \le 0$

iii) The line segment $x = 2t, \quad y = t + 1, \quad 0 \le t \le 1$

64. Functions:

a. $f(x, y) = x^2 + y^2$

b. $g(x, y) = 1/(x^2 + y^2)$

Curves:

i) The line $x = t, \quad y = 2 - 2t$

ii) The line segment $x = t, \quad y = 2 - 2t, \quad 0 \le t \le 1$

65. Least squares and regression lines When we try to fit a line $y = mx + b$ to a set of numerical data points (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$, we usually choose the line that minimizes the sum of the squares of the vertical distances from the points to the line. In theory, this means finding the values of m and b that minimize the value of the function

$$w = (mx_1 + b - y_1)^2 + \cdots + (mx_n + b - y_n)^2. \qquad (1)$$

(See the accompanying figure.) Show that the values of m and b that do this are

$$m = \frac{\left(\sum x_k\right)\left(\sum y_k\right) - n\sum x_k y_k}{\left(\sum x_k\right)^2 - n\sum x_k^{\,2}}, \qquad (2)$$

$$b = \frac{1}{n}\left(\sum y_k - m\sum x_k\right), \qquad (3)$$

with all sums running from $k = 1$ to $k = n$. Many scientific calculators have these formulas built in, enabling you to find m and b with only a few keystrokes after you have entered the data.

The line $y = mx + b$ determined by these values of m and b is called the **least squares line**, **regression line**, or **trend line** for the data under study. Finding a least squares line lets you

1. summarize data with a simple expression,
2. predict values of y for other, experimentally untried values of x,
3. handle data analytically.