or

$$2x = 2\lambda x + \mu, \qquad 2y = 2\lambda y + \mu, \qquad 2z = \mu. \tag{5}$$

The scalar equations in Equations (5) yield

$$\begin{aligned} 2x &= 2\lambda x + 2z \Rightarrow (1 - \lambda)x = z, \\ 2y &= 2\lambda y + 2z \Rightarrow (1 - \lambda)y = z. \end{aligned} \tag{6}$$

Equations (6) are satisfied simultaneously if either $\lambda = 1$ and $z = 0$ or $\lambda \neq 1$ and $x = y = z/(1 - \lambda)$.

If $z = 0$, then solving Equations (3) and (4) simultaneously to find the corresponding points on the ellipse gives the two points (1, 0, 0) and (0, 1, 0). This makes sense when you look at Figure 14.59.

If $x = y$, then Equations (3) and (4) give

$$\begin{aligned} x^2 + x^2 - 1 &= 0 & \qquad x + x + z - 1 &= 0 \\ 2x^2 &= 1 & z &= 1 - 2x \\ x &= \pm\frac{\sqrt{2}}{2} & z &= 1 \mp \sqrt{2}. \end{aligned}$$

The corresponding points on the ellipse are

$$P_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 - \sqrt{2}\right) \quad \text{and} \quad P_2 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 + \sqrt{2}\right).$$

Here we need to be careful, however. Although P_1 and P_2 both give local maxima of f on the ellipse, P_2 is farther from the origin than P_1.

The points on the ellipse closest to the origin are (1, 0, 0) and (0, 1, 0). The point on the ellipse farthest from the origin is P_2. (See Figure 14.59.) ■

Exercises 14.8

Two Independent Variables with One Constraint

1. **Extrema on an ellipse** Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x, y) = xy$ has its extreme values.

2. **Extrema on a circle** Find the extreme values of $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 10 = 0$.

3. **Maximum on a line** Find the maximum value of $f(x, y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$.

4. **Extrema on a line** Find the local extreme values of $f(x, y) = x^2y$ on the line $x + y = 3$.

5. **Constrained minimum** Find the points on the curve $xy^2 = 54$ nearest the origin.

6. **Constrained minimum** Find the points on the curve $x^2y = 2$ nearest the origin.

7. Use the method of Lagrange multipliers to find
   a. **Minimum on a hyperbola** The minimum value of $x + y$, subject to the constraints $xy = 16$, $x > 0$, $y > 0$
   b. **Maximum on a line** The maximum value of xy, subject to the constraint $x + y = 16$.

   Comment on the geometry of each solution.

8. **Extrema on a curve** Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy-plane that are nearest to and farthest from the origin.

9. **Minimum surface area with fixed volume** Find the dimensions of the closed right circular cylindrical can of smallest surface area whose volume is 16π cm^3.

10. **Cylinder in a sphere** Find the radius and height of the open right circular cylinder of largest surface area that can be inscribed in a sphere of radius a. What *is* the largest surface area?

11. **Rectangle of greatest area in an ellipse** Use the method of Lagrange multipliers to find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $x^2/16 + y^2/9 = 1$ with sides parallel to the coordinate axes.

12. **Rectangle of longest perimeter in an ellipse** Find the dimensions of the rectangle of largest perimeter that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$ with sides parallel to the coordinate axes. What *is* the largest perimeter?

13. **Extrema on a circle** Find the maximum and minimum values of $x^2 + y^2$ subject to the constraint $x^2 - 2x + y^2 - 4y = 0$.

14. **Extrema on a circle** Find the maximum and minimum values of $3x - y + 6$ subject to the constraint $x^2 + y^2 = 4$.

15. **Ant on a metal plate** The temperature at a point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant on the plate walks around the circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the ant?

16. **Cheapest storage tank** Your firm has been asked to design a storage tank for liquid petroleum gas. The customer's specifications call for a cylindrical tank with hemispherical ends, and the tank is to hold 8000 m^3 of gas. The customer also wants to use the smallest amount of material possible in building the tank. What radius and height do you recommend for the cylindrical portion of the tank?

Three Independent Variables with One Constraint

17. Minimum distance to a point Find the point on the plane $x + 2y + 3z = 13$ closest to the point (1, 1, 1).

18. Maximum distance to a point Find the point on the sphere $x^2 + y^2 + z^2 = 4$ farthest from the point $(1, -1, 1)$.

19. Minimum distance to the origin Find the minimum distance from the surface $x^2 - y^2 - z^2 = 1$ to the origin.

20. Minimum distance to the origin Find the point on the surface $z = xy + 1$ nearest the origin.

21. Minimum distance to the origin Find the points on the surface $z^2 = xy + 4$ closest to the origin.

22. Minimum distance to the origin Find the point(s) on the surface $xyz = 1$ closest to the origin.

23. Extrema on a sphere Find the maximum and minimum values of

$$f(x, y, z) = x - 2y + 5z$$

on the sphere $x^2 + y^2 + z^2 = 30$.

24. Extrema on a sphere Find the points on the sphere $x^2 + y^2 + z^2 = 25$ where $f(x, y, z) = x + 2y + 3z$ has its maximum and minimum values.

25. Minimizing a sum of squares Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

26. Maximizing a product Find the largest product the positive numbers x, y, and z can have if $x + y + z^2 = 16$.

27. Rectangular box of largest volume in a sphere Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere.

28. Box with vertex on a plane Find the volume of the largest closed rectangular box in the first octant having three faces in the coordinate planes and a vertex on the plane $x/a + y/b + z/c = 1$, where $a > 0$, $b > 0$, and $c > 0$.

29. Hottest point on a space probe A space probe in the shape of the ellipsoid

$$4x^2 + y^2 + 4z^2 = 16$$

enters Earth's atmosphere and its surface begins to heat. After 1 hour, the temperature at the point (x, y, z) on the probe's surface is

$$T(x, y, z) = 8x^2 + 4yz - 16z + 600.$$

Find the hottest point on the probe's surface.

30. Extreme temperatures on a sphere Suppose that the Celsius temperature at the point (x, y, z) on the sphere $x^2 + y^2 + z^2 = 1$ is $T = 400xyz^2$. Locate the highest and lowest temperatures on the sphere.

31. Cobb-Douglas production function During the 1920s, Charles Cobb and Paul Douglas modeled total production output P (of a firm, industry, or entire economy) as a function of labor hours involved x and capital invested y (which includes the monetary worth of all buildings and equipment). The Cobb-Douglas production function is given by

$$P(x, y) = kx^{\alpha}y^{1-\alpha},$$

where k and α are constants representative of a particular firm or economy.

a. Show that a doubling of both labor and capital results in a doubling of production P.

b. Suppose a particular firm has the production function for $k = 120$ and $\alpha = 3/4$. Assume that each unit of labor costs \$250 and each unit of capital costs \$400, and that the total expenses for all costs cannot exceed \$100,000. Find the maximum production level for the firm.

32. (*Continuation of Exercise 31.*) If the cost of a unit of labor is c_1 and the cost of a unit of capital is c_2, and if the firm can spend only B dollars as its total budget, then production P is constrained by $c_1x + c_2y = B$. Show that the maximum production level subject to the constraint occurs at the point

$$x = \frac{\alpha B}{c_1} \quad \text{and} \quad y = \frac{(1 - \alpha)B}{c_2}.$$

33. Maximizing a utility function: an example from economics In economics, the usefulness or *utility* of amounts x and y of two capital goods G_1 and G_2 is sometimes measured by a function $U(x, y)$. For example, G_1 and G_2 might be two chemicals a pharmaceutical company needs to have on hand and $U(x, y)$ the gain from manufacturing a product whose synthesis requires different amounts of the chemicals depending on the process used. If G_1 costs a dollars per kilogram, G_2 costs b dollars per kilogram, and the total amount allocated for the purchase of G_1 and G_2 together is c dollars, then the company's managers want to maximize $U(x, y)$ given that $ax + by = c$. Thus, they need to solve a typical Lagrange multiplier problem.

Suppose that

$$U(x, y) = xy + 2x$$

and that the equation $ax + by = c$ simplifies to

$$2x + y = 30.$$

Find the maximum value of U and the corresponding values of x and y subject to this latter constraint.

34. Blood types Human blood types are classified by three gene forms A, B, and O. Blood types AA, BB, and OO are *homozygous*, and blood types AB, AO, and BO are *heterozygous.* If p, q, and r represent the proportions of the three gene forms to the population, respectively, then the *Hardy-Weinberg Law* asserts that the proportion Q of heterozygous persons in any specific population is modeled by

$$Q(p, q, r) = 2(pq + pr + qr),$$

subject to $p + q + r = 1$. Find the maximum value of Q.

35. Length of a beam In Section 4.6, Exercise 39, we posed a problem of finding the length L of the shortest beam that can reach over a wall of height h to a tall building located k units from the wall. Use Lagrange multipliers to show that

$$L = (h^{2/3} + k^{2/3})^{3/2}.$$

36. Locating a radio telescope You are in charge of erecting a radio telescope on a newly discovered planet. To minimize interference, you want to place it where the magnetic field of the planet is weakest. The planet is spherical, with a radius of 6 units. Based on a coordinate system whose origin is at the center of the planet, the strength of the magnetic field is given by $M(x, y, z) = 6x - y^2 + xz + 60$. Where should you locate the radio telescope?