$$\iint_R (9 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (9 - r^2)\, r\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^1 (9r - r^3)\, dr\, d\theta$$
$$= \int_0^{2\pi} \left[\frac{9}{2} r^2 - \frac{1}{4} r^4 \right]_{r=0}^{r=1} d\theta$$
$$= \frac{17}{4} \int_0^{2\pi} d\theta = \frac{17\pi}{2}.$$ ■

EXAMPLE 6 Using polar integration, find the area of the region R in the xy-plane enclosed by the circle $x^2 + y^2 + 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.

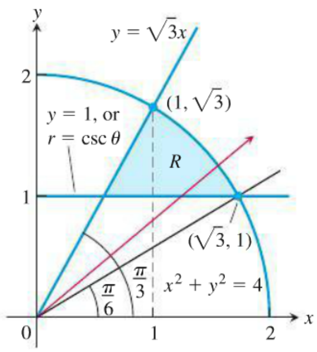


FIGURE 15.29 The region R in Example 6.

Solution A sketch of the region R is shown in Figure 15.29. First we note that the line $y = \sqrt{3}x$ has slope $\sqrt{3} = \tan\theta$, so $\theta = \pi/3$. Next we observe that the line $y = 1$ intersects the circle $x^2 + y^2 = 4$ when $x^2 + 1 = 4$, or $x = \sqrt{3}$. Moreover, the radial line from the origin through the point $(\sqrt{3}, 1)$ has slope $1/\sqrt{3} = \tan\theta$, giving its angle of inclination as $\theta = \pi/6$. This information is shown in Figure 15.29.

Now, for the region R, as θ varies from $\pi/6$ to $\pi/3$, the polar coordinate r varies from the horizontal line $y = 1$ to the circle $x^2 + y^2 = 4$. Substituting $r \sin\theta$ for y in the equation for the horizontal line, we have $r\sin\theta = 1$, or $r = \csc\theta$, which is the polar equation of the line. The polar equation for the circle is $r = 2$. So in polar coordinates, for $\pi/6 \le \theta \le \pi/3$, r varies from $r = \csc\theta$ to $r = 2$. It follows that the iterated integral for the area then gives

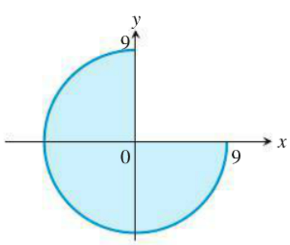
$$\iint_R dA = \int_{\pi/6}^{\pi/3} \int_{\csc\theta}^{2} r\, dr\, d\theta$$
$$= \int_{\pi/6}^{\pi/3} \left[\frac{1}{2} r^2 \right]_{r=\csc\theta}^{r=2} d\theta$$
$$= \int_{\pi/6}^{\pi/3} \frac{1}{2} [4 - \csc^2\theta]\, d\theta$$
$$= \frac{1}{2} \left[4\theta + \cot\theta \right]_{\pi/6}^{\pi/3}$$
$$= \frac{1}{2}\left(\frac{4\pi}{3} + \frac{1}{\sqrt{3}} \right) - \frac{1}{2}\left(\frac{4\pi}{6} + \sqrt{3} \right) = \frac{\pi - \sqrt{3}}{3}.$$ ■

Exercises 15.4

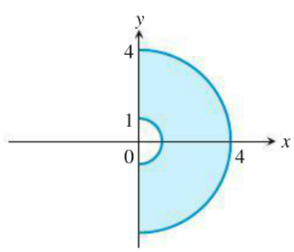
Regions in Polar Coordinates

In Exercises 1–8, describe the given region in polar coordinates.

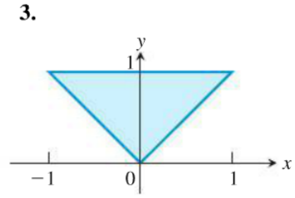
1.



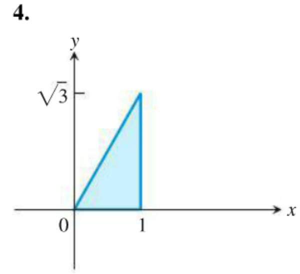
2.



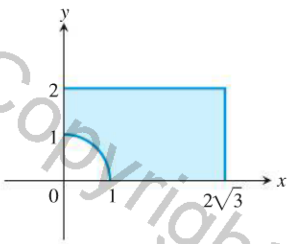
3.



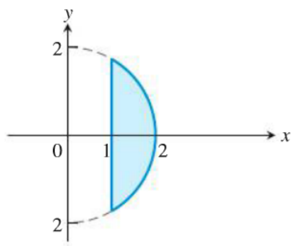
4.



5.



6.



7. The region enclosed by the circle $x^2 + y^2 = 2x$

8. The region enclosed by the semicircle $x^2 + y^2 = 2y,\ y \ge 0$

Evaluating Polar Integrals

In Exercises 9–22, change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral.

9. $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\,dx$

10. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} (x^2 + y^2)\,dx\,dy$

11. $\displaystyle\int_{0}^{2}\int_{0}^{\sqrt{4-y^2}} (x^2 + y^2)\,dx\,dy$

12. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$

13. $\displaystyle\int_{0}^{6}\int_{0}^{y} x\,dx\,dy$

14. $\displaystyle\int_{0}^{2}\int_{0}^{x} y\,dy\,dx$

15. $\displaystyle\int_{1}^{\sqrt{3}}\int_{1}^{x} dy\,dx$

16. $\displaystyle\int_{\sqrt{2}}^{2}\int_{\sqrt{4-y^2}}^{y} dx\,dy$

17. $\displaystyle\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1 + \sqrt{x^2 + y^2}}\,dy\,dx$

18. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1 + x^2 + y^2)^2}\,dy\,dx$

19. $\displaystyle\int_{0}^{\ln 2}\int_{0}^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\,dx\,dy$

20. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1)\,dx\,dy$

21. $\displaystyle\int_{0}^{1}\int_{x}^{\sqrt{2-x^2}} (x + 2y)\,dy\,dx$

22. $\displaystyle\int_{1}^{2}\int_{0}^{\sqrt{2x-x^2}} \frac{1}{(x^2 + y^2)^2}\,dy\,dx$

In Exercises 23–26, sketch the region of integration and convert each polar integral or sum of integrals to a Cartesian integral or sum of integrals. Do not evaluate the integrals.

23. $\displaystyle\int_{0}^{\pi/2}\int_{0}^{1} r^3 \sin\theta\cos\theta\,dr\,d\theta$

24. $\displaystyle\int_{\pi/6}^{\pi/2}\int_{1}^{\csc\theta} r^2 \cos\theta\,dr\,d\theta$

25. $\displaystyle\int_{0}^{\pi/4}\int_{0}^{2\sec\theta} r^5 \sin^2\theta\,dr\,d\theta$

26. $\displaystyle\int_{0}^{\tan^{-1}\frac{4}{3}}\int_{0}^{3\sec\theta} r^7\,dr\,d\theta + \int_{\tan^{-1}\frac{4}{3}}^{\pi/2}\int_{0}^{4\csc\theta} r^7\,dr\,d\theta$

Area in Polar Coordinates

27. Find the area of the region cut from the first quadrant by the curve $r = 2(2 - \sin 2\theta)^{1/2}$.

28. Cardioid overlapping a circle Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

29. One leaf of a rose Find the area enclosed by one leaf of the rose $r = 12\cos 3\theta$.

30. Snail shell Find the area of the region enclosed by the positive x-axis and spiral $r = 4\theta/3,\ 0 \le \theta \le 2\pi$. The region looks like a snail shell.

31. Cardioid in the first quadrant Find the area of the region cut from the first quadrant by the cardioid $r = 1 + \sin\theta$.

32. Overlapping cardioids Find the area of the region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.

Average Values

In polar coordinates, the **average value** of a function over a region R (Section 15.3) is given by

$$\frac{1}{\text{Area}(R)}\iint_R f(r, \theta)\,r\,dr\,d\theta.$$

33. Average height of a hemisphere Find the average height of the hemispherical surface $z = \sqrt{a^2 - x^2 - y^2}$ above the disk $x^2 + y^2 \le a^2$ in the xy-plane.

34. Average height of a cone Find the average height of the (single) cone $z = \sqrt{x^2 + y^2}$ above the disk $x^2 + y^2 \le a^2$ in the xy-plane.

35. Average distance from interior of disk to center Find the average distance from a point $P(x, y)$ in the disk $x^2 + y^2 \le a^2$ to the origin.

36. Average distance squared from a point in a disk to a point in its boundary Find the average value of the *square* of the distance from the point $P(x, y)$ in the disk $x^2 + y^2 \le 1$ to the boundary point $A(1, 0)$.

Theory and Examples

37. Converting to a polar integral Integrate $f(x, y) = [\ln(x^2 + y^2)]/\sqrt{x^2 + y^2}$ over the region $1 \le x^2 + y^2 \le e$.

38. Converting to a polar integral Integrate $f(x, y) = [\ln(x^2 + y^2)]/(x^2 + y^2)$ over the region $1 \le x^2 + y^2 \le e^2$.

39. Volume of noncircular right cylinder The region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ is the base of a solid right cylinder. The top of the cylinder lies in the plane $z = x$. Find the cylinder's volume.

40. Volume of noncircular right cylinder The region enclosed by the lemniscate $r^2 = 2\cos 2\theta$ is the base of a solid right cylinder whose top is bounded by the sphere $z = \sqrt{2 - r^2}$. Find the cylinder's volume.

41. Converting to polar integrals

a. The usual way to evaluate the improper integral $I = \int_0^{\infty} e^{-x^2}\,dx$ is first to calculate its square:

$$I^2 = \left(\int_0^{\infty} e^{-x^2}\,dx\right)\left(\int_0^{\infty} e^{-y^2}\,dy\right) = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy.$$

Evaluate the last integral using polar coordinates and solve the resulting equation for I.

b. Evaluate

$$\lim_{x\to\infty} \operatorname{erf}(x) = \lim_{x\to\infty}\int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}}\,dt.$$

42. Converting to a polar integral Evaluate the integral

$$\int_0^{\infty}\int_0^{\infty} \frac{1}{(1+x^2+y^2)^2}\,dx\,dy.$$

43. Existence Integrate the function $f(x, y) = 1/(1 - x^2 - y^2)$ over the disk $x^2 + y^2 \le 3/4$. Does the integral of $f(x, y)$ over the disk $x^2 + y^2 \le 1$ exist? Give reasons for your answer.

44. Area formula in polar coordinates Use the double integral in polar coordinates to derive the formula

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2\,d\theta$$

for the area of the fan-shaped region between the origin and polar curve $r = f(\theta), \alpha \le \theta \le \beta$.

45. Average distance to a given point inside a disk Let P_0 be a point inside a circle of radius a and let h denote the distance from P_0 to the center of the circle. Let d denote the distance from an arbitrary point P to P_0. Find the average value of d^2 over the region enclosed by the circle. (*Hint:* Simplify your work by placing the center of the circle at the origin and P_0 on the x-axis.)

46. Area Suppose that the area of a region in the polar coordinate plane is

$$A = \int_{\pi/4}^{3\pi/4}\int_{\csc\theta}^{2\sin\theta} r\,dr\,d\theta.$$

Sketch the region and find its area.

47. Evaluate the integral $\iint_R \sqrt{x^2 + y^2}\,dA$, where R is the region inside the upper semicircle of radius 2 centered at the origin, but outside the circle $x^2 + (y - 1)^2 = 1$.

48. Evaluate the integral $\iint_R (x^2 + y^2)^{-2}\,dA$, where R is the region inside the circle $x^2 + y^2 = 2$ for $x \le -1$.

COMPUTER EXPLORATIONS

In Exercises 49–52, use a CAS to change the Cartesian integrals into an equivalent polar integral and evaluate the polar integral. Perform the following steps in each exercise.

a. Plot the Cartesian region of integration in the xy-plane.

b. Change each boundary curve of the Cartesian region in part (a) to its polar representation by solving its Cartesian equation for r and θ.

c. Using the results in part (b), plot the polar region of integration in the $r\theta$-plane.

d. Change the integrand from Cartesian to polar coordinates. Determine the limits of integration from your plot in part (c) and evaluate the polar integral using the CAS integration utility.

49. $\displaystyle\int_0^1\int_x^1 \frac{y}{x^2+y^2}\,dy\,dx$

50. $\displaystyle\int_0^1\int_0^{x/2} \frac{x}{x^2+y^2}\,dy\,dx$

51. $\displaystyle\int_0^1\int_{-y/3}^{y/3} \frac{y}{\sqrt{x^2+y^2}}\,dx\,dy$

52. $\displaystyle\int_0^1\int_y^{2-y} \sqrt{x+y}\,dx\,dy$

15.5 Triple Integrals in Rectangular Coordinates

Just as double integrals allow us to deal with more general situations than could be handled by single integrals, triple integrals enable us to solve still more general problems. We use triple integrals to calculate the volumes of three-dimensional shapes and the average value of a function over a three-dimensional region. Triple integrals also arise in the study of vector fields and fluid flow in three dimensions, as we will see in Chapter 16.

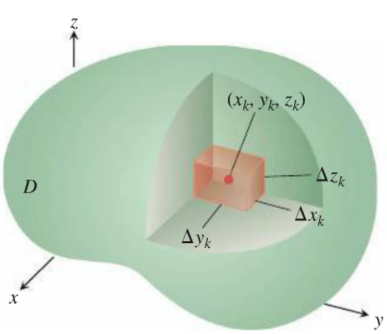


FIGURE 15.30 Partitioning a solid with rectangular cells of volume ΔV_k.

Triple Integrals

If $F(x, y, z)$ is a function defined on a closed bounded region D in space, such as the region occupied by a solid ball or a lump of clay, then the integral of F over D may be defined in the following way. We partition a rectangular boxlike region containing D into rectangular cells by planes parallel to the coordinate axes (Figure 15.30). We number the cells that lie completely inside D from 1 to n in some order, the kth cell having dimensions Δx_k by Δy_k by Δz_k and volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$. We choose a point (x_k, y_k, z_k) in each cell and form the sum

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\,\Delta V_k. \tag{1}$$