

FIGURE 15.34 The region of integration in Example 4.

EXAMPLE 4 Find the average value of $F(x, y, z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$ in the first octant.

Solution We sketch the cube with enough detail to show the limits of integration (Figure 15.34). We then use Equation (2) to calculate the average value of F over the cube.

The volume of the region D is $(2)(2)(2) = 8$. The value of the integral of F over the cube is

$$\begin{aligned} \int_0^2 \int_0^2 \int_0^2 xyz \, dx \, dy \, dz &= \int_0^2 \int_0^2 \left[\frac{x^2}{2} yz \right]_{x=0}^{x=2} dy \, dz = \int_0^2 \int_0^2 2yz \, dy \, dz \\ &= \int_0^2 \left[y^2 z \right]_{y=0}^{y=2} dz = \int_0^2 4z \, dz = \left[2z^2 \right]_0^2 = 8. \end{aligned}$$

With these values, Equation (2) gives

$$\text{Average value of } xyz \text{ over the cube} = \frac{1}{\text{volume}} \iiint_{\text{cube}} xyz \, dV = \left(\frac{1}{8} \right) (8) = 1.$$

In evaluating the integral, we chose the order $dx \, dy \, dz$, but any of the other five possible orders would have done as well. ■

Properties of Triple Integrals

Triple integrals have the same algebraic properties as double and single integrals. Simply replace the double integrals in the four properties given in Section 15.2, page 892, with triple integrals.

Exercises 15.5

Triple Integrals in Different Iteration Orders

- Evaluate the integral in Example 2 taking $F(x, y, z) = 1$ to find the volume of the tetrahedron in the order $dz \, dx \, dy$.
- Volume of rectangular solid** Write six different iterated triple integrals for the volume of the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 2$, and $z = 3$. Evaluate one of the integrals.
- Volume of tetrahedron** Write six different iterated triple integrals for the volume of the tetrahedron cut from the first octant by the plane $6x + 3y + 2z = 6$. Evaluate one of the integrals.
- Volume of solid** Write six different iterated triple integrals for the volume of the region in the first octant enclosed by the cylinder $x^2 + z^2 = 4$ and the plane $y = 3$. Evaluate one of the integrals.
- Volume enclosed by paraboloids** Let D be the region bounded by the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$. Write six different triple iterated integrals for the volume of D . Evaluate one of the integrals.
- Volume inside paraboloid beneath a plane** Let D be the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2y$. Write triple iterated integrals in the order $dz \, dx \, dy$ and $dz \, dy \, dx$ that give the volume of D . Do not evaluate either integral.

Evaluating Triple Iterated Integrals

Evaluate the integrals in Exercises 7–20.

- $\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2) \, dz \, dy \, dx$
- $\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz \, dx \, dy$
- $\int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz} \, dx \, dy \, dz$
- $\int_0^1 \int_0^{3-3x} \int_0^{3-3x-y} dz \, dy \, dx$
- $\int_0^{\pi/6} \int_0^1 \int_{-2}^3 y \sin z \, dx \, dy \, dz$
- $\int_{-1}^1 \int_0^1 \int_0^2 (x + y + z) \, dy \, dx \, dz$
- $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dz \, dy \, dx$
- $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{2x+y} dz \, dx \, dy$
- $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz \, dy \, dx$
- $\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x \, dz \, dy \, dx$
- $\int_0^\pi \int_0^\pi \int_0^\pi \cos(u + v + w) \, du \, dv \, dw$ (uvw -space)
- $\int_0^1 \int_1^{\sqrt{e}} \int_1^e se^s \ln r \frac{(\ln t)^2}{t} \, dt \, dr \, ds$ (rst -space)

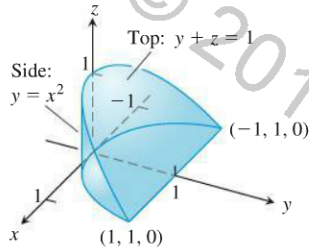
19. $\int_0^{\pi/4} \int_0^{\ln \sec v} \int_{-\infty}^{2t} e^x dx dt dv$ (tux-space)

20. $\int_0^7 \int_0^2 \int_0^{\sqrt{4-q^2}} \frac{q}{r+1} dp dq dr$ (pqr-space)

Finding Equivalent Iterated Integrals

21. Here is the region of integration of the integral

$$\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dz dy dx.$$

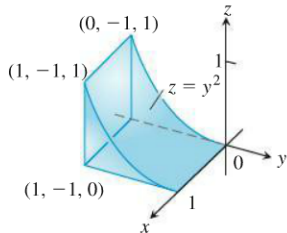


Rewrite the integral as an equivalent iterated integral in the order

- a. $dy dz dx$
- b. $dy dx dz$
- c. $dx dy dz$
- d. $dx dz dy$
- e. $dz dx dy$.

22. Here is the region of integration of the integral

$$\int_0^1 \int_{-1}^0 \int_0^{y^2} dz dy dx.$$



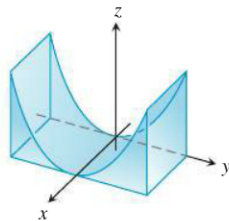
Rewrite the integral as an equivalent iterated integral in the order

- a. $dy dz dx$
- b. $dy dx dz$
- c. $dx dy dz$
- d. $dx dz dy$
- e. $dz dx dy$.

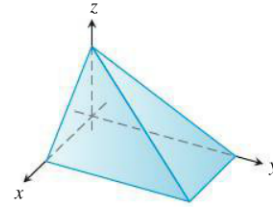
Finding Volumes Using Triple Integrals

Find the volumes of the regions in Exercises 23–36.

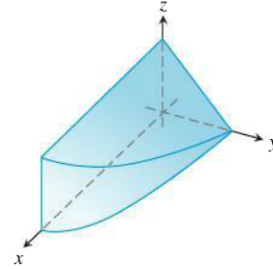
23. The region between the cylinder $z = y^2$ and the xy -plane that is bounded by the planes $x = 0, x = 1, y = -1, y = 1$



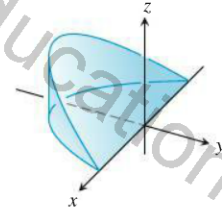
24. The region in the first octant bounded by the coordinate planes and the planes $x + z = 1, y + 2z = 2$



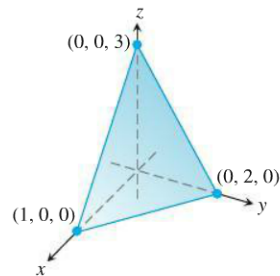
25. The region in the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder $x = 4 - y^2$



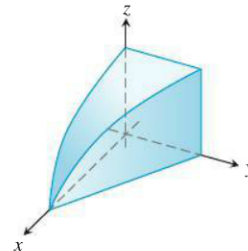
26. The wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y$ and $z = 0$



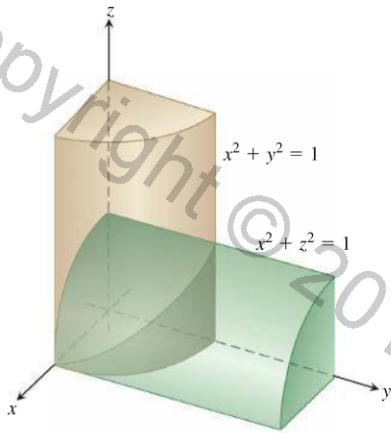
27. The tetrahedron in the first octant bounded by the coordinate planes and the plane passing through $(1, 0, 0), (0, 2, 0),$ and $(0, 0, 3)$



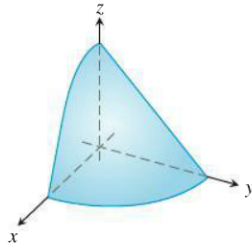
28. The region in the first octant bounded by the coordinate planes, the plane $y = 1 - x$, and the surface $z = \cos(\pi x/2), 0 \leq x \leq 1$



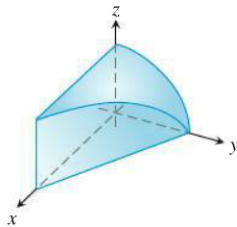
29. The region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is shown in the accompanying figure



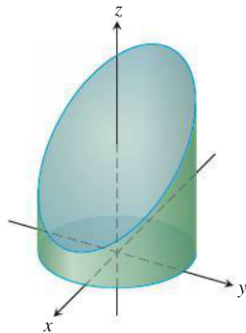
30. The region in the first octant bounded by the coordinate planes and the surface $z = 4 - x^2 - y^2$



31. The region in the first octant bounded by the coordinate planes, the plane $x + y = 4$, and the cylinder $y^2 + 4z^2 = 16$



32. The region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$



33. The region between the planes $x + y + 2z = 2$ and $2x + 2y + z = 4$ in the first octant
34. The finite region bounded by the planes $z = x$, $x + z = 8$, $z = y$, $y = 8$, and $z = 0$
35. The region cut from the solid elliptical cylinder $x^2 + 4y^2 \leq 4$ by the xy -plane and the plane $z = x + 2$
36. The region bounded in back by the plane $x = 0$, on the front and sides by the parabolic cylinder $x = 1 - y^2$, on the top by the paraboloid $z = x^2 + y^2$, and on the bottom by the xy -plane

Average Values

In Exercises 37–40, find the average value of $F(x, y, z)$ over the given region.

37. $F(x, y, z) = x^2 + 9$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$
38. $F(x, y, z) = x + y - z$ over the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 2$
39. $F(x, y, z) = x^2 + y^2 + z^2$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 1$
40. $F(x, y, z) = xyz$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$

Changing the Order of Integration

Evaluate the integrals in Exercises 41–44 by changing the order of integration in an appropriate way.

41. $\int_0^4 \int_0^1 \int_{2y}^2 \frac{4 \cos(x^2)}{2\sqrt{z}} dx dy dz$
42. $\int_0^1 \int_0^1 \int_{x^2}^1 12xz e^{zy^2} dy dx dz$
43. $\int_0^1 \int_{\sqrt{z}}^1 \int_0^{\ln 3} \frac{\pi e^{2x} \sin \pi y^2}{y^2} dx dy dz$
44. $\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4-z} dy dz dx$

Theory and Examples

45. **Finding an upper limit of an iterated integral** Solve for a :

$$\int_0^1 \int_0^{4-a-x^2} \int_a^{4-x^2-y} dz dy dx = \frac{4}{15}.$$

46. **Ellipsoid** For what value of c is the volume of the ellipsoid $x^2 + (y/2)^2 + (z/c)^2 = 1$ equal to 8π ?

47. **Minimizing a triple integral** What domain D in space minimizes the value of the integral

$$\iiint_D (4x^2 + 4y^2 + z^2 - 4) dV?$$

Give reasons for your answer.

48. **Maximizing a triple integral** What domain D in space maximizes the value of the integral

$$\iiint_D (1 - x^2 - y^2 - z^2) dV?$$

Give reasons for your answer.