

For the region D in Example 5, this becomes

$$\begin{aligned}
 I_z &= \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^4 \sin^3 \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{\rho^5}{5} \right]_0^1 \sin^3 \phi \, d\phi \, d\theta \\
 &= \frac{1}{5} \int_0^{2\pi} \int_0^{\pi/3} (1 - \cos^2 \phi) \sin \phi \, d\phi \, d\theta = \frac{1}{5} \int_0^{2\pi} \left[-\cos \phi + \frac{\cos^3 \phi}{3} \right]_0^{\pi/3} d\theta \\
 &= \frac{1}{5} \int_0^{2\pi} \left(-\frac{1}{2} + 1 + \frac{1}{24} - \frac{1}{3} \right) d\theta = \frac{1}{5} \int_0^{2\pi} \frac{5}{24} d\theta = \frac{1}{24} (2\pi) = \frac{\pi}{12}.
 \end{aligned}$$

Coordinate Conversion Formulas

CYLINDRICAL TO RECTANGULAR	SPHERICAL TO RECTANGULAR	SPHERICAL TO CYLINDRICAL
$x = r \cos \theta$	$x = \rho \sin \phi \cos \theta$	$r = \rho \sin \phi$
$y = r \sin \theta$	$y = \rho \sin \phi \sin \theta$	$z = \rho \cos \phi$
$z = z$	$z = \rho \cos \phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned}
 dV &= dx \, dy \, dz \\
 &= dz \, r \, dr \, d\theta \\
 &= \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
 \end{aligned}$$

In the next section we offer a more general procedure for determining dV in cylindrical and spherical coordinates. The results, of course, will be the same.

Exercises 15.7

Evaluating Integrals in Cylindrical Coordinates

Evaluate the cylindrical coordinate integrals in Exercises 1–6.

- $\int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} dz \, r \, dr \, d\theta$
- $\int_0^{2\pi} \int_0^3 \int_{r^2/3}^{\sqrt{18-r^2}} dz \, r \, dr \, d\theta$
- $\int_0^{2\pi} \int_0^{\theta/2\pi} \int_0^{3+24r^2} dz \, r \, dr \, d\theta$
- $\int_0^{\pi} \int_0^{\theta/\pi} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} z \, dz \, r \, dr \, d\theta$
- $\int_0^{2\pi} \int_0^1 \int_r^{1/\sqrt{2-r^2}} 3 \, dz \, r \, dr \, d\theta$
- $\int_0^{2\pi} \int_0^1 \int_{-1/2}^{1/2} (r^2 \sin^2 \theta + z^2) \, dz \, r \, dr \, d\theta$

Changing the Order of Integration in Cylindrical Coordinates

The integrals we have seen so far suggest that there are preferred orders of integration for cylindrical coordinates, but other orders usually work well and are occasionally easier to evaluate. Evaluate the integrals in Exercises 7–10.

- $\int_0^{2\pi} \int_0^3 \int_0^{z/3} r^3 \, dr \, dz \, d\theta$
- $\int_{-1}^1 \int_0^{2\pi} \int_0^{1+\cos \theta} 4r \, dr \, d\theta \, dz$

- $\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} (r^2 \cos^2 \theta + z^2) r \, d\theta \, dr \, dz$
- $\int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \int_0^{2\pi} (r \sin \theta + 1) r \, d\theta \, dz \, dr$

- Let D be the region bounded below by the plane $z = 0$, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.
 - $dz \, dr \, d\theta$
 - $dr \, dz \, d\theta$
 - $d\theta \, dz \, dr$
- Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.
 - $dz \, dr \, d\theta$
 - $dr \, dz \, d\theta$
 - $d\theta \, dz \, dr$

Finding Iterated Integrals in Cylindrical Coordinates

- Give the limits of integration for evaluating the integral

$$\iiint f(r, \theta, z) \, dz \, r \, dr \, d\theta$$

as an iterated integral over the region that is bounded below by the plane $z = 0$, on the side by the cylinder $r = \cos \theta$, and on top by the paraboloid $z = 3r^2$.

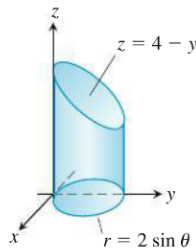
14. Convert the integral

$$\int_{-1}^1 \int_0^{\sqrt{1-y^2}} \int_0^x (x^2 + y^2) dz dx dy$$

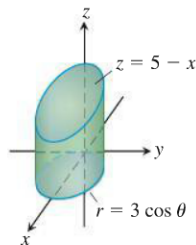
to an equivalent integral in cylindrical coordinates and evaluate the result.

In Exercises 15–20, set up the iterated integral for evaluating $\iiint_D f(r, \theta, z) dz r dr d\theta$ over the given region D .

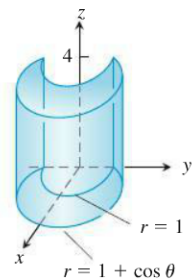
15. D is the right circular cylinder whose base is the circle $r = 2 \sin \theta$ in the xy -plane and whose top lies in the plane $z = 4 - y$.



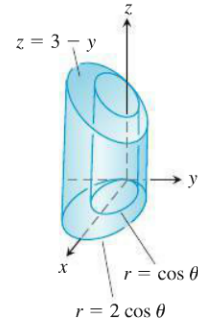
16. D is the right circular cylinder whose base is the circle $r = 3 \cos \theta$ and whose top lies in the plane $z = 5 - x$.



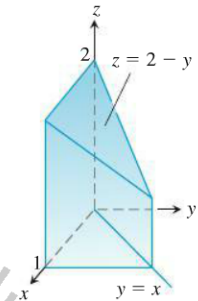
17. D is the solid right cylinder whose base is the region in the xy -plane that lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$ and whose top lies in the plane $z = 4$.



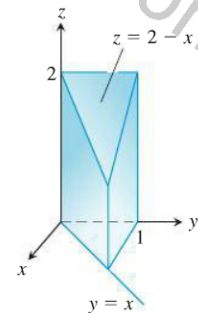
18. D is the solid right cylinder whose base is the region between the circles $r = \cos \theta$ and $r = 2 \cos \theta$ and whose top lies in the plane $z = 3 - y$.



19. D is the prism whose base is the triangle in the xy -plane bounded by the x -axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane $z = 2 - y$.



20. D is the prism whose base is the triangle in the xy -plane bounded by the y -axis and the lines $y = x$ and $y = 1$ and whose top lies in the plane $z = 2 - x$.



Evaluating Integrals in Spherical Coordinates

Evaluate the spherical coordinate integrals in Exercises 21–26.

21. $\int_0^\pi \int_0^\pi \int_0^{2 \sin \phi} \rho^2 \sin \phi d\rho d\phi d\theta$

22. $\int_0^{2\pi} \int_0^{\pi/4} \int_0^2 (\rho \cos \phi) \rho^2 \sin \phi d\rho d\phi d\theta$

23. $\int_0^{2\pi} \int_0^\pi \int_0^{(1-\cos \phi)/2} \rho^2 \sin \phi d\rho d\phi d\theta$

24. $\int_0^{3\pi/2} \int_0^\pi \int_0^1 5\rho^3 \sin^3 \phi d\rho d\phi d\theta$

25. $\int_0^{2\pi} \int_0^{\pi/3} \int_{\sec \phi}^2 3\rho^2 \sin \phi d\rho d\phi d\theta$

26.
$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec \phi} (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

Changing the Order of Integration in Spherical Coordinates

The previous integrals suggest there are preferred orders of integration for spherical coordinates, but other orders give the same value and are occasionally easier to evaluate. Evaluate the integrals in Exercises 27–30.

27.
$$\int_0^2 \int_{-\pi}^0 \int_{\pi/4}^{\pi/2} \rho^3 \sin 2\phi \, d\phi \, d\theta \, d\rho$$

28.
$$\int_{\pi/6}^{\pi/3} \int_{\csc \phi}^{2 \csc \phi} \int_0^{2\pi} \rho^2 \sin \phi \, d\theta \, d\rho \, d\phi$$

29.
$$\int_0^1 \int_0^{\pi} \int_0^{\pi/4} 12\rho \sin^3 \phi \, d\phi \, d\theta \, d\rho$$

30.
$$\int_{\pi/6}^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_{\csc \phi}^2 5\rho^4 \sin^3 \phi \, d\rho \, d\theta \, d\phi$$

31. Let D be the region in Exercise 11. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration.

- a. $d\rho \, d\phi \, d\theta$ b. $d\phi \, d\rho \, d\theta$

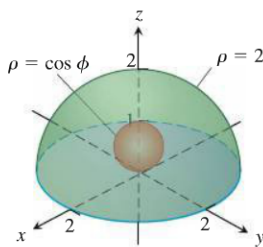
32. Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration.

- a. $d\rho \, d\phi \, d\theta$ b. $d\phi \, d\rho \, d\theta$

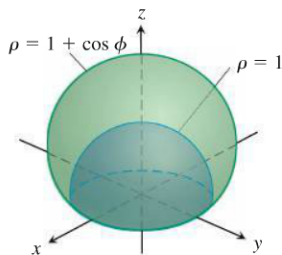
Finding Iterated Integrals in Spherical Coordinates

In Exercises 33–38, (a) find the spherical coordinate limits for the integral that calculates the volume of the given solid and then (b) evaluate the integral.

33. The solid between the sphere $\rho = \cos \phi$ and the hemisphere $\rho = 2, z \geq 0$



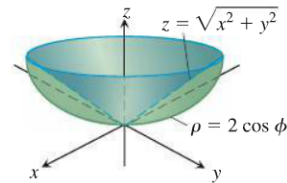
34. The solid bounded below by the hemisphere $\rho = 1, z \geq 0$, and above by the cardioid of revolution $\rho = 1 + \cos \phi$



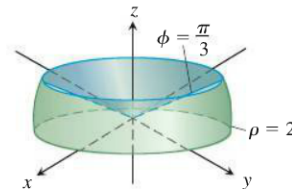
35. The solid enclosed by the cardioid of revolution $\rho = 1 - \cos \phi$

36. The upper portion cut from the solid in Exercise 35 by the xy -plane

37. The solid bounded below by the sphere $\rho = 2 \cos \phi$ and above by the cone $z = \sqrt{x^2 + y^2}$



38. The solid bounded below by the xy -plane, on the sides by the sphere $\rho = 2$, and above by the cone $\phi = \pi/3$



Finding Triple Integrals

39. Set up triple integrals for the volume of the sphere $\rho = 2$ in (a) spherical, (b) cylindrical, and (c) rectangular coordinates.

40. Let D be the region in the first octant that is bounded below by the cone $\phi = \pi/4$ and above by the sphere $\rho = 3$. Express the volume of D as an iterated triple integral in (a) cylindrical and (b) spherical coordinates. Then (c) find V .

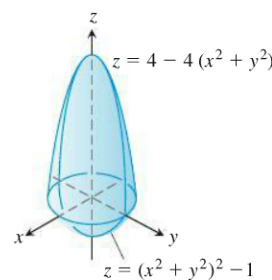
41. Let D be the smaller cap cut from a solid ball of radius 2 units by a plane 1 unit from the center of the sphere. Express the volume of D as an iterated triple integral in (a) spherical, (b) cylindrical, and (c) rectangular coordinates. Then (d) find the volume by evaluating one of the three triple integrals.

42. Express the moment of inertia I_z of the solid hemisphere $x^2 + y^2 + z^2 \leq 1, z \geq 0$, as an iterated integral in (a) cylindrical and (b) spherical coordinates. Then (c) find I_z .

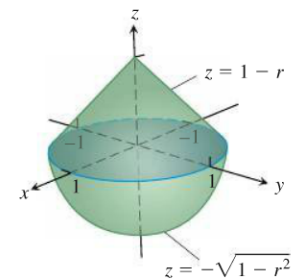
Volumes

Find the volumes of the solids in Exercises 43–48.

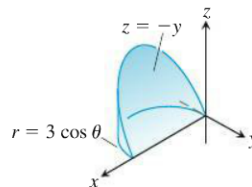
43.



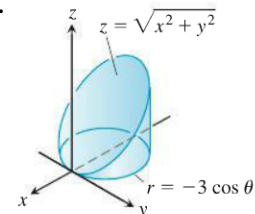
44.



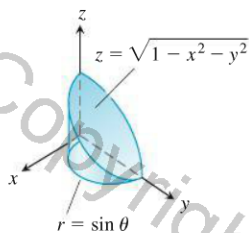
45.



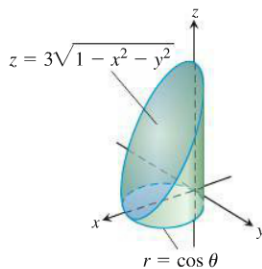
46.



47.



48.



49. **Sphere and cones** Find the volume of the portion of the solid sphere $\rho \leq a$ that lies between the cones $\phi = \pi/3$ and $\phi = 2\pi/3$.
50. **Sphere and half-planes** Find the volume of the region cut from the solid sphere $\rho \leq a$ by the half-planes $\theta = 0$ and $\theta = \pi/6$ in the first octant.
51. **Sphere and plane** Find the volume of the smaller region cut from the solid sphere $\rho \leq 2$ by the plane $z = 1$.
52. **Cone and planes** Find the volume of the solid enclosed by the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$.
53. **Cylinder and paraboloid** Find the volume of the region bounded below by the plane $z = 0$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.
54. **Cylinder and paraboloids** Find the volume of the region bounded below by the paraboloid $z = x^2 + y^2$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2 + 1$.
55. **Cylinder and cones** Find the volume of the solid cut from the thick-walled cylinder $1 \leq x^2 + y^2 \leq 2$ by the cones $z = \pm \sqrt{x^2 + y^2}$.
56. **Sphere and cylinder** Find the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.
57. **Cylinder and planes** Find the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 4$.
58. **Cylinder and planes** Find the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $x + y + z = 4$.
59. **Region trapped by paraboloids** Find the volume of the region bounded above by the paraboloid $z = 5 - x^2 - y^2$ and below by the paraboloid $z = 4x^2 + 4y^2$.
60. **Paraboloid and cylinder** Find the volume of the region bounded above by the paraboloid $z = 9 - x^2 - y^2$, below by the xy -plane, and lying *outside* the cylinder $x^2 + y^2 = 1$.
61. **Cylinder and sphere** Find the volume of the region cut from the solid cylinder $x^2 + y^2 \leq 1$ by the sphere $x^2 + y^2 + z^2 = 4$.
62. **Sphere and paraboloid** Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

Average Values

63. Find the average value of the function $f(r, \theta, z) = r$ over the region bounded by the cylinder $r = 1$ between the planes $z = -1$ and $z = 1$.
64. Find the average value of the function $f(r, \theta, z) = r$ over the solid ball bounded by the sphere $r^2 + z^2 = 1$. (This is the sphere $x^2 + y^2 + z^2 = 1$.)
65. Find the average value of the function $f(\rho, \phi, \theta) = \rho$ over the solid ball $\rho \leq 1$.
66. Find the average value of the function $f(\rho, \phi, \theta) = \rho \cos \phi$ over the solid upper ball $\rho \leq 1, 0 \leq \phi \leq \pi/2$.

Masses, Moments, and Centroids

67. **Center of mass** A solid of constant density is bounded below by the plane $z = 0$, above by the cone $z = r, r \geq 0$, and on the sides by the cylinder $r = 1$. Find the center of mass.
68. **Centroid** Find the centroid of the region in the first octant that is bounded above by the cone $z = \sqrt{x^2 + y^2}$, below by the plane $z = 0$, and on the sides by the cylinder $x^2 + y^2 = 4$ and the planes $x = 0$ and $y = 0$.
69. **Centroid** Find the centroid of the solid in Exercise 38.
70. **Centroid** Find the centroid of the solid bounded above by the sphere $\rho = a$ and below by the cone $\phi = \pi/4$.
71. **Centroid** Find the centroid of the region that is bounded above by the surface $z = \sqrt{r}$, on the sides by the cylinder $r = 4$, and below by the xy -plane.
72. **Centroid** Find the centroid of the region cut from the solid ball $r^2 + z^2 \leq 1$ by the half-planes $\theta = -\pi/3, r \geq 0$, and $\theta = \pi/3, r \geq 0$.
73. **Moment of inertia of solid cone** Find the moment of inertia of a right circular cone of base radius 1 and height 1 about an axis through the vertex parallel to the base. (Take $\delta = 1$.)
74. **Moment of inertia of solid sphere** Find the moment of inertia of a solid sphere of radius a about a diameter. (Take $\delta = 1$.)
75. **Moment of inertia of solid cone** Find the moment of inertia of a right circular cone of base radius a and height h about its axis. (*Hint:* Place the cone with its vertex at the origin and its axis along the z -axis.)
76. **Variable density** A solid is bounded on the top by the paraboloid $z = r^2$, on the bottom by the plane $z = 0$, and on the sides by the cylinder $r = 1$. Find the center of mass and the moment of inertia about the z -axis if the density is
- a. $\delta(r, \theta, z) = z$ b. $\delta(r, \theta, z) = r$.
77. **Variable density** A solid is bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. Find the center of mass and the moment of inertia about the z -axis if the density is
- a. $\delta(r, \theta, z) = z$ b. $\delta(r, \theta, z) = z^2$.
78. **Variable density** A solid ball is bounded by the sphere $\rho = a$. Find the moment of inertia about the z -axis if the density is
- a. $\delta(\rho, \phi, \theta) = \rho^2$ b. $\delta(\rho, \phi, \theta) = r = \rho \sin \phi$.
79. **Centroid of solid semiellipsoid** Show that the centroid of the solid semiellipsoid of revolution $(r^2/a^2) + (z^2/h^2) \leq 1, z \geq 0$, lies on the z -axis three-eighths of the way from the base to the top. The special case $h = a$ gives a solid hemisphere. Thus, the centroid of a solid hemisphere lies on the axis of symmetry three-eighths of the way from the base to the top.
80. **Centroid of solid cone** Show that the centroid of a solid right circular cone is one-fourth of the way from the base to the vertex. (In general, the centroid of a solid cone or pyramid is one-fourth of the way from the centroid of the base to the vertex.)
81. **Density of center of a planet** A planet is in the shape of a sphere of radius R and total mass M with spherically symmetric density distribution that increases linearly as one approaches its center.

What is the density at the center of this planet if the density at its edge (surface) is taken to be zero?

82. Mass of planet's atmosphere A spherical planet of radius R has an atmosphere whose density is $\mu = \mu_0 e^{-ch}$, where h is the altitude above the surface of the planet, μ_0 is the density at sea level, and c is a positive constant. Find the mass of the planet's atmosphere.

Theory and Examples

83. Vertical planes in cylindrical coordinates

a. Show that planes perpendicular to the x -axis have equations of the form $r = a \sec \theta$ in cylindrical coordinates.

b. Show that planes perpendicular to the y -axis have equations of the form $r = b \csc \theta$.

84. (Continuation of Exercise 83.) Find an equation of the form $r = f(\theta)$ in cylindrical coordinates for the plane $ax + by = c$, $c \neq 0$.

85. Symmetry What symmetry will you find in a surface that has an equation of the form $r = f(z)$ in cylindrical coordinates? Give reasons for your answer.

86. Symmetry What symmetry will you find in a surface that has an equation of the form $\rho = f(\phi)$ in spherical coordinates? Give reasons for your answer.

15.8 Substitutions in Multiple Integrals

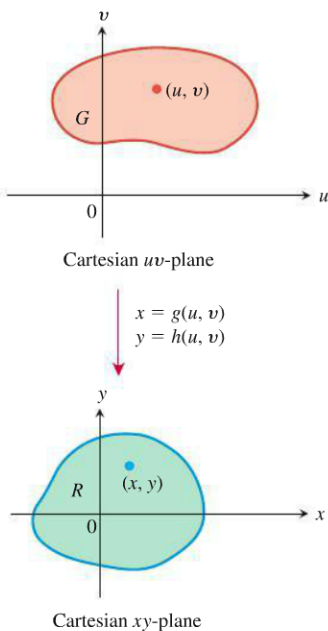


FIGURE 15.54 The equations $x = g(u, v)$ and $y = h(u, v)$ allow us to change an integral over a region R in the xy -plane into an integral over a region G in the uv -plane.

This section introduces the ideas involved in coordinate transformations to evaluate multiple integrals by substitution. The method replaces complicated integrals by ones that are easier to evaluate. Substitutions accomplish this by simplifying the integrand, the limits of integration, or both. A thorough discussion of multivariable transformations and substitutions is best left to a more advanced course, but our introduction here shows how the substitutions just studied reflect the general idea derived for single integral calculus.

Substitutions in Double Integrals

The polar coordinate substitution of Section 15.4 is a special case of a more general substitution method for double integrals, a method that pictures changes in variables as transformations of regions.

Suppose that a region G in the uv -plane is transformed into the region R in the xy -plane by equations of the form

$$x = g(u, v), \quad y = h(u, v),$$

as suggested in Figure 15.54. We assume the transformation is one-to-one on the interior of G . We call R the **image** of G under the transformation, and G the **preimage** of R . Any function $f(x, y)$ defined on R can be thought of as a function $f(g(u, v), h(u, v))$ defined on G as well. How is the integral of $f(x, y)$ over R related to the integral of $f(g(u, v), h(u, v))$ over G ?

To gain some insight into the question, let's look again at the single variable case. To be consistent with how we are using them now, we interchange the variables x and u used in the substitution method for single integrals in Chapter 5, so the equation is stated here as

$$\int_{g(a)}^{g(b)} f(x) dx = \int_a^b f(g(u)) g'(u) du. \quad x = g(u), \quad dx = g'(u) du$$

To propose an analogue for substitution in a double integral $\iint_R f(x, y) dx dy$, we need a derivative factor like $g'(u)$ as a multiplier that transforms the area element $du dv$ in the region G to its corresponding area element $dx dy$ in the region R . Let's denote this factor by J . In continuing with our analogy, it is reasonable to assume that J is a function of both variables u and v , just as g' is a function of the single variable u . Moreover, J should register instantaneous change, so partial derivatives are going to be involved in its expression. Since four partial derivatives are associated with the transforming equations $x = g(u, v)$ and $y = h(u, v)$, it is also reasonable to assume that the factor $J(u, v)$ we seek includes them all. These features are captured in the following definition constructed from the partial derivatives, and named after the German mathematician Carl Jacobi.

HISTORICAL BIOGRAPHY

Carl Gustav Jacob Jacobi
(1804–1851)