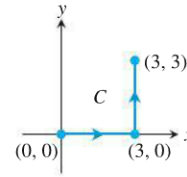


Exercises 16.2

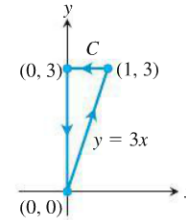
Vector Fields

Find the gradient fields of the functions in Exercises 1–4.

- $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$
- $f(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2}$
- $g(x, y, z) = e^z - \ln(x^2 + y^2)$
- $g(x, y, z) = xy + yz + xz$
- Give a formula $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ for the vector field in the plane that has the property that \mathbf{F} points toward the origin with magnitude inversely proportional to the square of the distance from (x, y) to the origin. (The field is not defined at $(0, 0)$.)
- Give a formula $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ for the vector field in the plane that has the properties that $\mathbf{F} = \mathbf{0}$ at $(0, 0)$ and that at any other point (a, b) , \mathbf{F} is tangent to the circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction with magnitude $|\mathbf{F}| = \sqrt{a^2 + b^2}$.



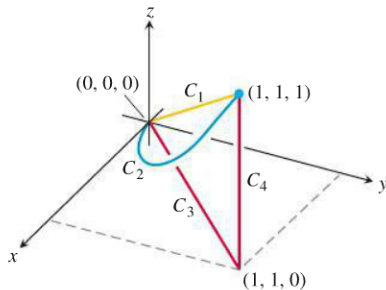
16. $\int_C \sqrt{x+y} \, dx$, where C is given in the accompanying figure



Line Integrals of Vector Fields

In Exercises 7–12, find the line integrals of \mathbf{F} from $(0, 0, 0)$ to $(1, 1, 1)$ over each of the following paths in the accompanying figure.

- The straight-line path C_1 : $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq 1$
 - The curved path C_2 : $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}$, $0 \leq t \leq 1$
 - The path $C_3 \cup C_4$ consisting of the line segment from $(0, 0, 0)$ to $(1, 1, 0)$ followed by the segment from $(1, 1, 0)$ to $(1, 1, 1)$
- $\mathbf{F} = 3y\mathbf{i} + 2x\mathbf{j} + 4z\mathbf{k}$ 8. $\mathbf{F} = [1/(x^2 + 1)]\mathbf{j}$
 - $\mathbf{F} = \sqrt{z}\mathbf{i} - 2x\mathbf{j} + \sqrt{y}\mathbf{k}$ 10. $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$
 - $\mathbf{F} = (3x^2 - 3x)\mathbf{i} + 3z\mathbf{j} + \mathbf{k}$
 - $\mathbf{F} = (y + z)\mathbf{i} + (z + x)\mathbf{j} + (x + y)\mathbf{k}$



Line Integrals with Respect to x , y , and z

In Exercises 13–16, find the line integrals along the given path C .

- $\int_C (x - y) \, dx$, where $C: x = t, y = 2t + 1$, for $0 \leq t \leq 3$
- $\int_C \frac{x}{y} \, dy$, where $C: x = t, y = t^2$, for $1 \leq t \leq 2$
- $\int_C (x^2 + y^2) \, dy$, where C is given in the accompanying figure

17. Along the curve $\mathbf{r}(t) = t\mathbf{i} - \mathbf{j} + t^2\mathbf{k}$, $0 \leq t \leq 1$, evaluate each of the following integrals.

a. $\int_C (x + y - z) \, dx$ b. $\int_C (x + y - z) \, dy$

c. $\int_C (x + y - z) \, dz$

18. Along the curve $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - (\cos t)\mathbf{k}$, $0 \leq t \leq \pi$, evaluate each of the following integrals.

a. $\int_C xz \, dx$ b. $\int_C xz \, dy$ c. $\int_C xyz \, dz$

Work

In Exercises 19–22, find the work done by \mathbf{F} over the curve in the direction of increasing t .

19. $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$

$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq 1$

20. $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + (x + y)\mathbf{k}$

$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (t/6)\mathbf{k}$, $0 \leq t \leq 2\pi$

21. $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$

$\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq 2\pi$

22. $\mathbf{F} = 6z\mathbf{i} + y^2\mathbf{j} + 12x\mathbf{k}$

$\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (t/6)\mathbf{k}$, $0 \leq t \leq 2\pi$

Line Integrals in the Plane

- Evaluate $\int_C xy \, dx + (x + y) \, dy$ along the curve $y = x^2$ from $(-1, 1)$ to $(2, 4)$.
- Evaluate $\int_C (x - y) \, dx + (x + y) \, dy$ counterclockwise around the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.
- Evaluate $\int_C \mathbf{F} \cdot \mathbf{T} \, ds$ for the vector field $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j}$ along the curve $x = y^2$ from $(4, 2)$ to $(1, -1)$.
- Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ counterclockwise along the unit circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$.

Work, Circulation, and Flux in the Plane

27. **Work** Find the work done by the force $\mathbf{F} = xy\mathbf{i} + (y - x)\mathbf{j}$ over the straight line from $(1, 1)$ to $(2, 3)$.
28. **Work** Find the work done by the gradient of $f(x, y) = (x + y)^2$ counterclockwise around the circle $x^2 + y^2 = 4$ from $(2, 0)$ to itself.
29. **Circulation and flux** Find the circulation and flux of the fields

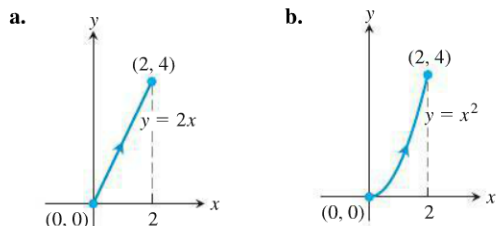
$$\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$$

around and across each of the following curves.

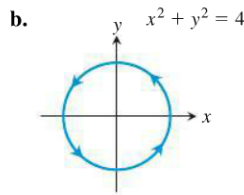
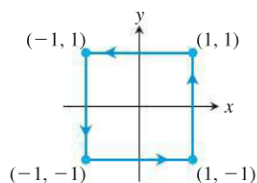
- a. The circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$
- b. The ellipse $\mathbf{r}(t) = (\cos t)\mathbf{i} + (4 \sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$
30. **Flux across a circle** Find the flux of the fields $\mathbf{F}_1 = 2x\mathbf{i} - 3y\mathbf{j}$ and $\mathbf{F}_2 = 2x\mathbf{i} + (x - y)\mathbf{j}$ across the circle $\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$.

In Exercises 31–34, find the circulation and flux of the field \mathbf{F} around and across the closed semicircular path that consists of the semicircular arch $\mathbf{r}_1(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$, $0 \leq t \leq \pi$, followed by the line segment $\mathbf{r}_2(t) = t\mathbf{i}$, $-a \leq t \leq a$.

31. $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ 32. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j}$
33. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ 34. $\mathbf{F} = -y^2\mathbf{i} + x^2\mathbf{j}$
35. **Flow integrals** Find the flow of the velocity field $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\mathbf{j}$ along each of the following paths from $(1, 0)$ to $(-1, 0)$ in the xy -plane.
- a. The upper half of the circle $x^2 + y^2 = 1$
- b. The line segment from $(1, 0)$ to $(-1, 0)$
- c. The line segment from $(1, 0)$ to $(0, -1)$ followed by the line segment from $(0, -1)$ to $(-1, 0)$
36. **Flux across a triangle** Find the flux of the field \mathbf{F} in Exercise 35 outward across the triangle with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$.
37. Find the flow of the velocity field $\mathbf{F} = y^2\mathbf{i} + 2xy\mathbf{j}$ along each of the following paths from $(0, 0)$ to $(2, 4)$.



- a. Use any path from $(0, 0)$ to $(2, 4)$ different from parts (a) and (b).
38. Find the circulation of the field $\mathbf{F} = y\mathbf{i} + (x + 2y)\mathbf{j}$ around each of the following closed paths.



- c. Use any closed path different from parts (a) and (b).

Vector Fields in the Plane

39. **Spin field** Draw the spin field

$$\mathbf{F} = -\frac{y}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2 + y^2}}\mathbf{j}$$

(see Figure 16.12) along with its horizontal and vertical components at a representative assortment of points on the circle $x^2 + y^2 = 4$.

40. **Radial field** Draw the radial field

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j}$$

(see Figure 16.11) along with its horizontal and vertical components at a representative assortment of points on the circle $x^2 + y^2 = 1$.

41. **A field of tangent vectors**

- a. Find a field $\mathbf{G} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ in the xy -plane with the property that at any point $(a, b) \neq (0, 0)$, \mathbf{G} is a vector of magnitude $\sqrt{a^2 + b^2}$ tangent to the circle $x^2 + y^2 = a^2 + b^2$ and pointing in the counterclockwise direction. (The field is undefined at $(0, 0)$.)
- b. How is \mathbf{G} related to the spin field \mathbf{F} in Figure 16.12?

42. **A field of tangent vectors**

- a. Find a field $\mathbf{G} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ in the xy -plane with the property that at any point $(a, b) \neq (0, 0)$, \mathbf{G} is a unit vector tangent to the circle $x^2 + y^2 = a^2 + b^2$ and pointing in the clockwise direction.
- b. How is \mathbf{G} related to the spin field \mathbf{F} in Figure 16.12?

43. **Unit vectors pointing toward the origin** Find a field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the xy -plane with the property that at each point $(x, y) \neq (0, 0)$, \mathbf{F} is a unit vector pointing toward the origin. (The field is undefined at $(0, 0)$.)

44. **Two “central” fields** Find a field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the xy -plane with the property that at each point $(x, y) \neq (0, 0)$, \mathbf{F} points toward the origin and $|\mathbf{F}|$ is (a) the distance from (x, y) to the origin, (b) inversely proportional to the distance from (x, y) to the origin. (The field is undefined at $(0, 0)$.)

45. **Work and area** Suppose that $f(t)$ is differentiable and positive for $a \leq t \leq b$. Let C be the path $\mathbf{r}(t) = t\mathbf{i} + f(t)\mathbf{j}$, $a \leq t \leq b$, and $\mathbf{F} = y\mathbf{i}$. Is there any relation between the value of the work integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

and the area of the region bounded by the t -axis, the graph of f , and the lines $t = a$ and $t = b$? Give reasons for your answer.

46. **Work done by a radial force with constant magnitude** A particle moves along the smooth curve $y = f(x)$ from $(a, f(a))$ to

$(b, f(b))$. The force moving the particle has constant magnitude k and always points away from the origin. Show that the work done by the force is

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = k[(b^2 + (f(b))^2)^{1/2} - (a^2 + (f(a))^2)^{1/2}].$$

Flow Integrals in Space

In Exercises 47–50, \mathbf{F} is the velocity field of a fluid flowing through a region in space. Find the flow along the given curve in the direction of increasing t .

47. $\mathbf{F} = -4xy\mathbf{i} + 8y\mathbf{j} + 2\mathbf{k}$

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}, \quad 0 \leq t \leq 2$$

48. $\mathbf{F} = x^2\mathbf{i} + yz\mathbf{j} + y^2\mathbf{k}$

$$\mathbf{r}(t) = 3t\mathbf{j} + 4t\mathbf{k}, \quad 0 \leq t \leq 1$$

49. $\mathbf{F} = (x - z)\mathbf{i} + x\mathbf{k}$

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{k}, \quad 0 \leq t \leq \pi$$

50. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + 2\mathbf{k}$

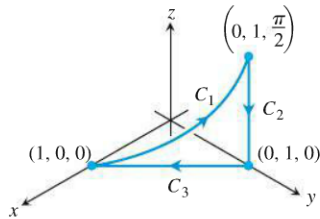
$$\mathbf{r}(t) = (-2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + 2t\mathbf{k}, \quad 0 \leq t \leq 2\pi$$

51. **Circulation** Find the circulation of $\mathbf{F} = 2x\mathbf{i} + 2z\mathbf{j} + 2y\mathbf{k}$ around the closed path consisting of the following three curves traversed in the direction of increasing t .

$$C_1: \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq \pi/2$$

$$C_2: \mathbf{r}(t) = \mathbf{j} + (\pi/2)(1 - t)\mathbf{k}, \quad 0 \leq t \leq 1$$

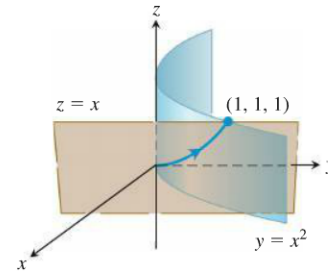
$$C_3: \mathbf{r}(t) = t\mathbf{i} + (1 - t)\mathbf{j}, \quad 0 \leq t \leq 1$$



52. **Zero circulation** Let C be the ellipse in which the plane $2x + 3y - z = 0$ meets the cylinder $x^2 + y^2 = 12$. Show, without evaluating either line integral directly, that the circulation of the field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ around C in either direction is zero.

53. **Flow along a curve** The field $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$ is the velocity field of a flow in space. Find the flow from $(0, 0, 0)$ to

$(1, 1, 1)$ along the curve of intersection of the cylinder $y = x^2$ and the plane $z = x$. (*Hint:* Use $t = x$ as the parameter.)



54. **Flow of a gradient field** Find the flow of the field $\mathbf{F} = \nabla(xy^2z^3)$:

a. Once around the curve C in Exercise 52, clockwise as viewed from above

b. Along the line segment from $(1, 1, 1)$ to $(2, 1, -1)$.

COMPUTER EXPLORATIONS

In Exercises 55–60, use a CAS to perform the following steps for finding the work done by force \mathbf{F} over the given path:

a. Find $d\mathbf{r}$ for the path $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$.

b. Evaluate the force \mathbf{F} along the path.

c. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

55. $\mathbf{F} = xy^6\mathbf{i} + 3x(xy^5 + 2)\mathbf{j}$; $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$

56. $\mathbf{F} = \frac{3}{1+x^2}\mathbf{i} + \frac{2}{1+y^2}\mathbf{j}$; $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \leq t \leq \pi$

57. $\mathbf{F} = (y + yz \cos xyz)\mathbf{i} + (x^2 + xz \cos xyz)\mathbf{j} + (z + xy \cos xyz)\mathbf{k}$; $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + \mathbf{k}$, $0 \leq t \leq 2\pi$

58. $\mathbf{F} = 2xy\mathbf{i} - y^2\mathbf{j} + ze^x\mathbf{k}$; $\mathbf{r}(t) = -t\mathbf{i} + \sqrt{t}\mathbf{j} + 3t\mathbf{k}$, $1 \leq t \leq 4$

59. $\mathbf{F} = (2y + \sin x)\mathbf{i} + (z^2 + (1/3)\cos y)\mathbf{j} + x^4\mathbf{k}$; $\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (\sin 2t)\mathbf{k}$, $-\pi/2 \leq t \leq \pi/2$

60. $\mathbf{F} = (x^2y)\mathbf{i} + \frac{1}{3}x^3\mathbf{j} + xy\mathbf{k}$; $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (2\sin^2 t - 1)\mathbf{k}$, $0 \leq t \leq 2\pi$

16.3 Path Independence, Conservative Fields, and Potential Functions

A **gravitational field** \mathbf{G} is a vector field that represents the effect of gravity at a point in space due to the presence of a massive object. The gravitational force on a body of mass m placed in the field is given by $\mathbf{F} = m\mathbf{G}$. Similarly, an **electric field** \mathbf{E} is a vector field in space that represents the effect of electric forces on a charged particle placed within it. The force on a body of charge q placed in the field is given by $\mathbf{F} = q\mathbf{E}$. In gravitational and electric fields, the amount of work it takes to move a mass or charge from one point to another depends on the initial and final positions of the object—not on which path is taken between these positions. In this section we study vector fields with this property and the calculation of work integrals associated with them.